Non-linear Symmetry-preserving Observer on Lie Groups

Silvère Bonnabel, Philippe Martin, Pierre Rouchon

Non-linear Symmetry-preserving Observers on Lie Groups

Silvere Bonnabel, Philippe Martin and Pierre Rouchon

Abstract

In this paper we give a geometrical framework for the design of observers on finite-dimensional Lie groups for systems which possess some specific symmetries. The design and the error (between true and estimated state) equation are explicit and intrinsic. We consider also a particular case: left-invariant systems on Lie groups with right equivariant output. The theory yields a class of observers such that error equation is autonomous. The observers converge locally around *any* trajectory, and the global behavior is independent from the trajectory, which reminds the linear stationary case.

I. Introduction

Symmetries (invariances) have been used to design controllers and for optimal control theory ([6], [9], [7], [15], [12], [13]), but far less for the design of observers. [4] develops a theory of symmetry-preserving observers and presents three non-linear observers for three examples of engineering interest: a chemical reactor, a non-holonomic car, and an inertial navigation system. In the two latter examples the state space and the group of symmetry have the same dimension and (since the action is free) the state space can be identified with the group (up to some discrete group). Applying the general theory to the Lie group case, we develop here a proper theory of symmetry-preserving observers on Lie groups. The advantage over [4] is that the observer design is explicit (the implicit function theorem is not needed) and intrinsic, the error equation and its first-order approximation can be computed explicitly, and are intrinsic, and all the formulas are globally defined. Moreover, this paper is a step further in the symmetry-preserving observers theory since [4] does not deal at all with convergence issues in the general case. Here using

Centre Automatique et Systèmes, Ecole des Mines de Paris, 60 boulevard Saint-Michel,75272 Paris CEDEX 06, FRANCE
bonnabel@montefiore.ulg.ac.be, philippe.martin@ensmp.fr, pierre.rouchon@ensmp.fr

the explicit error equation we introduce a new class of trajectories around which we build convergent observers. In the case of section III a class of first-order convergent observers around any trajectory is given. The theory applies to various systems of engineering interest modeled as invariant systems on Lie groups, such as cart-like vehicles and rigid bodies in space. In particular it is well suited to attitude estimation and some inertial navigation examples.

The paper is organized as follows: in section II we give a general framework for symmetry-preserving observers on Lie groups. It explains the general form of the observers [10], [8], [5] and [4] based on the group structure of $\mathrm{SO}(3)$ and (resp.) $\mathrm{SE}(2)$, without considering the convergence issues. The design, the error equation and its first-order approximation are given explicitly. It is theoretically explained why the error equation in the car example of [4] does not depend on the trajectory (although it depends on the inputs). Then we introduce a new class of trajectories called permanent trajectories which extend the notion of equilibrium point for systems with symmetries: making a symmetry-preserving observer around such a trajectory boils down to make a Luenberger observer around an equilibrium point. We characterize permanent trajectories geometrically and give a locally convergent observer around any permanent trajectory.

In section III we consider the special case of a left-invariant system with right equivariant output. It can be looked at as the motion of a generalized rigid body in space with measurements expressed in the body-fixed frame, as it will be explained in section III-A.1. Thus it applies to some inertial navigation examples. In particular it allows to explain theoretically why the error equation in the inertial navigation example of [4] is autonomous. A class of first-order convergent observers such that the error equation is autonomous is derived. This property reminds much of the linear stationary case. We also explore the links between right equivariance of the output map and observability.

Preliminary versions of section III can be found in [2], [3].

II. Symmetry-preserving observers on Lie groups

A. *Invariant observer and error equation*

Consider the following system :

$$\frac{d}{dt}x(t) = f(x,u) \tag{1}$$

$$y = h(x,u) \tag{2}$$

where $x \in G$, $u \in \mathcal{U} = \mathbb{R}^m$, $y \in \mathcal{Y} = \mathbb{R}^p$ (the whole theory can be easily adapted to the case where $\mathcal{U}$ and $\mathcal{Y}$ are smooth m and p-dimensional manifolds, for instance Lie groups), and f is a smooth vector field on G. $u \in \mathcal{U}$ is a known input (control, measured perturbation, constant parameter, time t etc.).

Definition 1: Let G be a Lie Group with identity e and Σ an open set (or more generally a manifold). A *left* group action $(\phi_g)_{g\in G}$ on Σ is a smooth map

$$(g, \xi) \in G \times \Sigma \mapsto \phi_g(\xi) \in \Sigma$$

such that:

- $\phi_e(\xi) = \xi$ for all ξ
- $\phi_{g_2}\big(\phi_{g_1}(\xi)\big) = \phi_{g_2 g_1}(\xi)$ for all g_1, g_2, ξ.

In analogy one defines a right group action the same way except that $\phi_{g_2}\big(\phi_{g_1}(\xi)\big) = \phi_{g_1 g_2}(\xi)$ for all g_1, g_2, ξ. Suppose G acts on the left on $\mathcal{U}$ and $\mathcal{Y}$ via $\psi_g : \mathcal{U} \to \mathcal{U}$ and $\rho_g : \mathcal{Y} \to \mathcal{Y}$. Suppose the dynamics (1) is invariant in the sense of [4] where the group action on the state space (the group itself) is made of left multiplication: for any $g \in G$, $DL_g f(x, u) = f(gx, \psi_g(u))$, i.e:

$$\forall\ x, g \in G \quad f(L_g(x), \psi_g(u)) = DL_g f(x, u)$$

where $L_g : x \mapsto gx$ is the left multiplication on G, and DL_g the induced map on the tangent space. DL_g maps the tangent space $TG|_x$ to $TG|_{gx}$. Let $R_g : x \mapsto xg$ denote the right multiplication and DR_g its induced map on the tangent space. As in [4], we suppose that the output $y = h(x, u)$ is equivariant, i.e, $h\big(\varphi_g(x), \psi_g(u)\big) = \rho_g\big(h(x, u)\big)$ for all g, x, u.

Definition 2: Consider the change of variables $X = gx$, $U = \psi_g(u)$ and $Y = \rho_g(y)$. The system (1)-(2) is left-invariant with equivariant output if for all $g \in G$ it is unaffected by the latter transformation: $\frac{d}{dt}X(t) = f(X, U),\ Y = h(X, U)$.

We are going to build observers which respect the symmetries (left-invariance under the group action) adapting the constructive method of [4] to the Lie group case.

1) Invariant pre-observers: Following [14] (or [4]) consider the action $(\phi_g)_{g\in G}$ of G on $\Sigma = \mathbb{R}^s$ where s is any positive integer. Let $(x, z) \in G \times \mathbb{R}^s$, one can compute (at most) s functionally independent scalar invariants of the variables (x, z) the following way: $I(x, z) = \phi_{x^{-1}}(z) \in \mathbb{R}^s$. It has the property that any invariant real-valued function $J(x, z)$ which verifies $J(gx, \phi_g(z)) = J(x, z)$ for all g, x, z is a function of the components of $I(x, z)$: $J(x, z) = \mathcal{H}(I(x, z))$. Applying this general method we find a complete set of invariants of $(x,u) \in G \times \mathcal{U}$:

$$I(x,u) = \psi_{x^{-1}}(u) \in \mathcal{U}. \tag{3}$$

Take n linearly independent vectors $(W_1, \ldots, W_n)$ in $TG|_e = \mathfrak{g}$, the Lie algebra of the group G. Define n vector fields by the invariance relation $w_i(x) = DL_x W_i \in TG|_x, i = 1...n, x \in G$. The vector fields form an invariant frame [14]. According to [4]

Definition 3 (pre-observer): The system $\frac{d}{dt}\hat{x} = F(\hat{x}, u, y)$ is a *pre-observer* of (1)-(2) if $F\big(x, u, h(x,u)\big) = f(x,u)$ for all $(x,u) \in G \times \mathcal{U}$.

The definition does not deal with convergence; if moreover $x(t)^{-1}\hat{x}(t) \to e$ as $t \to +\infty$ for every (close) initial conditions, the pre-observer is an (asymptotic) *observer*. It is said to be is *G-invariant* if $F\big(g\hat{x}, \psi_g(u), \varrho_g(y)\big) = DL_g F(\hat{x}, u, y)$ for all $(g, \hat{x}, u, y) \in G \times G \times \mathcal{U} \times \mathcal{Y}$.

Lemma 1: Any invariant pre-observer reads

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u) + DL_{\hat{x}} \left(\sum_{i=1}^{n} \mathcal{L}_i \left(\psi_{\hat{x}^{-1}}(u), \rho_{\hat{x}^{-1}}(y) \right) W_i \right) \tag{4}$$

where the $\mathcal{L}_i$ are any smooth functions of their arguments such that $\mathcal{L}_i \left(\psi_{\hat{x}^{-1}}(u), h(e, \psi_{\hat{x}^{-1}}(u)) \right) = 0$. The proof is analogous to [4]: one can write $DL_{\hat{x}^{-1}}(\frac{d}{dt}\hat{x} - f(\hat{x},u)) = \sum_{i=1}^{n} \mathcal{F}_i \left(\hat{x}, u, y \right) W_i \in \mathfrak{g}$, where the $\mathcal{F}_i' s$ are invariant scalar functions of their arguments. But a complete set of invariants of $\hat{x}, u, y$ is made of the components of $(\psi_{\hat{x}^{-1}}(u), \rho_{\hat{x}^{-1}}(y))$, thus $\mathcal{F}_i \left(\hat{x}, u, y \right) = \mathcal{L}_i \left(\psi_{\hat{x}^{-1}}(u), \rho_{\hat{x}^{-1}}(y) \right)$. And when $\hat{x} = x$ we have $\rho_{\hat{x}^{-1}}(y) = h(\hat{x}^{-1}x, \psi_{\hat{x}^{-1}}(u)) = h(e, \psi_{\hat{x}^{-1}}(u))$ and the $\mathcal{L}_i$'s cancel.

2) Invariant state-error dynamics: Consider the invariant state-error $\eta = x^{-1}\hat{x} \in G$. It is invariant by left multiplication : $\eta = (gx)^{-1}(g\hat{x})$ for any $g \in G$. Notice that a small error corresponds to η close to e. Contrarily to [4], the time derivative of η can be computed explicitly. We recall R_g denotes the right multiplication map on G. Since we have

- for any $g_1, g_2 \in G$, $DL_{g_1} DL_{g_2} = DL_{g_1 g_2}$, $DR_{g_1} DR_{g_2} = DR_{g_2 g_1}$, $DL_{g_1} DR_{g_2} = DR_{g_2} DL_{g_1}$
- $I(\hat{x}, u) = \psi_{\hat{x}^{-1}}(u) = \psi_{(x\eta)^{-1}}(u)$
- $\rho_{\hat{x}^{-1}}(h(x,u)) = h(\hat{x}^{-1}x, \psi_{\hat{x}^{-1}}(u))$ writes $\rho_{\hat{x}^{-1}}(y) = h(\eta^{-1}, \psi_{(x\eta)^{-1}}(u))$
- $\frac{d}{dt}\eta = \frac{d}{dt}(x^{-1}\hat{x}) = DL_{x^{-1}} \frac{d}{dt}\hat{x} - DR_{\hat{x}} \frac{d}{dt} x^{-1}$ with $\frac{d}{dt} x^{-1} = -DL_{x^{-1}} DR_{x^{-1}} \frac{d}{dt} x$

the error dynamics reads

$$\begin{aligned} \frac{d}{dt}\eta = {} & DL_\eta f(e, \psi_{(x\eta)^{-1}}(u)) - DR_\eta f(e, \psi_{x^{-1}}(u)) \\ & + DL_\eta \left(\sum_{i=1}^{n} \mathcal{L}_i \left(\psi_{(x\eta)^{-1}}(u), h(\eta^{-1}, \psi_{(x\eta)^{-1}}(u)) \right) W_i \right). \end{aligned} \tag{5}$$

The invariant error η obeys a differential equation that is coupled to the system trajectory $t \mapsto (x(t), u(t))$ only via the invariant term $I(x,u) = \psi_{x^{-1}}(u)$. Note that when $\psi_g(u) \equiv u$ the invariant error dynamics is independent of the state trajectory $x(t)$! This the reason why we have this property in the non-holonomic car example of [4].

3) Invariant first order approximation: For η close to e, one can set in (5) $\eta = \exp(\epsilon\xi)$ where ξ is an element of the Lie algebra $\mathfrak{g}$ and $\epsilon \in \mathbb{R}$ is small. The linearized invariant state error equation can always be written in the same tangent space $\mathfrak{g}$: up to order second terms in ϵ

$$\frac{d}{dt}\xi = [\xi, f(e, \psi_{x^{-1}}(u))] - \frac{\partial f}{\partial u}(e, \psi_{x^{-1}}(u))\frac{\partial \psi}{\partial g}(e, \psi_{x^{-1}}(u))\xi - \sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}_i}{\partial h}(\psi_{x^{-1}}(u), h(e, \psi_{x^{-1}}(u))\frac{\partial h}{\partial x}(e, \psi_{x^{-1}}(u))\xi\right) W_i \quad (6)$$

where [,] denotes the Lie bracket of $\mathfrak{g}$, ψ is viewed as a function of (g, u), and $\frac{\partial \mathcal{L}_i}{\partial h}$ denotes the partial derivative of $\mathcal{L}_i$ with respect to its second argument. The gains $\frac{\partial \mathcal{L}_i}{\partial h}(\psi_{x^{-1}}(u), h(e, \psi_{x^{-1}}(u))$ can be tuned via linear techniques to achieve local convergence.

B. Local convergence around permanent trajectories

The aim of this paragraph is to extend local convergence results around an equilibrium point to a class of trajectories we call permanent trajectories.

Definition 4: A trajectory of (1) is permanent if $I(x(t), u(t)) = \bar{I}$ is independent of t.

Note that adapting this definition to the general case of symmetry-preserving observers [4] is straightforward. Any trajectory of the system verifies $\frac{d}{dt}x(t) = DL_{x(t)}f(e, \psi_{x(t)^{-1}}(u(t)))$ thanks to the invariance of the dynamics. It is permanent if $I(x(t), u(t)) = \psi_{x^{-1}(t)}(u(t)) = \bar{u}$ is independent of t. The permanent trajectory $x(t)$ is then given by $x(0)\exp(t\bar{w})$ where $\bar{w}$ is the left invariant vector field associated to $f(e, \bar{u})$. Thus $x(t)$ corresponds, up to a left translation defined by the initial condition, to a one-parameter sub-group.

Let us make an observer around an arbitrary permanent trajectory: denote by $(x_r(t), u_r(t))$ a permanent trajectory associated to $\bar{u} = \psi_{\bar{x}_r^{-1}(t)}u_r(t)$. Let us suppose we made an invariant observer following (4). Then the error equation (5) writes

$$\frac{d}{dt}\eta = DL_\eta f(e, \psi_{\eta^{-1}}(\bar{u})) - DR_\eta f(e, \bar{u}) + DL_\eta\left(\sum_{i=1}^{n}\mathcal{L}_i\left(\psi_{\eta^{-1}}(\bar{u}), h(\eta^{-1}, \psi_{\eta^{-1}}(\bar{u}))\right) W_i\right). \quad (7)$$

since $\psi_{(x_r\eta)^{-1}}(u) = \psi_{\eta^{-1}}(\psi_{x_r^{-1}}(u)) = \psi_{\eta^{-1}}(\bar{u})$. The first order approximation (6) is now a time invariant system:

$$\frac{d}{dt}\xi = [\xi, f(e,\bar{u})] - \frac{\partial f}{\partial u}(e,\bar{u})\frac{\partial \psi}{\partial g}(e,\bar{u})\xi - \sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}_i}{\partial h}(\bar{u}, h(e,\bar{u}))\frac{\partial h}{\partial x}(e,\bar{u})\xi\right)W_i$$

Let us write ξ and $f(e,u)$ in the frame defined by the W_i's: $\xi = \sum_{k=1}^n \xi^k W_k$ and $f(e,\bar{u}) = \sum_{k=1}^n \bar{f}^k W_k$. Denote by C_{ij}^k the structure constants associated with the Lie algebra of G: $[W_i, W_j] = \sum_{k=1}^n C_{ij}^k W_k$. The above system reads:

$$\frac{d}{dt}\xi = (A + \bar{\mathcal{L}}C)\xi \tag{8}$$

where

$$A = \left(\sum_{k=1}^{n} C_{jk}^i \bar{f}^k - \left[\frac{\partial f}{\partial u}(e,\bar{u})\frac{\partial \psi}{\partial g}(e,\bar{u})\right]_{i,j}\right)_{1\leq i,j\leq n},$$

$$\bar{\mathcal{L}} = \left(-\frac{\partial \mathcal{L}_i}{\partial h_k}(\bar{u}, h(e,\bar{u}))\right)_{\substack{1\leq i\leq n\\ 1\leq k\leq p}}, \quad C = \left(\frac{\partial h_k}{\partial x_j}(e,\bar{u})\right)_{\substack{1\leq k\leq p\\ 1\leq j\leq n}}$$

where $(x_1, \ldots, x_n)$ are the local coordinates around e defined by the exponential map: $x = \exp(\sum_{i=1}^n x_i W_i)$. If we assume that the pair (A, C) is observable we can choose the poles of $A + \bar{\mathcal{L}}C$ to get an invariant and locally convergent observer around any permanent trajectory associated to $\bar{u}$. Let $W(x) = [W_1(x), .., W_n(x)]$. It suffices to take:

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u(t)) + W(\hat{x})\bar{\mathcal{L}}\rho_{\hat{x}^{-1}}(y(t)) \tag{9}$$

Examples: In the non-holonomic car example of [4], permanent trajectories are made of lines and circle with constant speed. In the inertial navigation example of [4], $\psi_{x^{-1}}(u) = \begin{pmatrix} q*\omega*q^{-1} \\ q*(a+v\times\omega)*q^{-1}\end{pmatrix}$, a trajectory is permanent if $q*\omega*q^{-1}$ and $q*(a+v\times\omega)*q^{-1}$ are independent of t. Some computations show that any permanent trajectory reads:

$$q(t) = \exp\left(\frac{\Omega}{2}t\right) * q_0$$

$$v(t) = q_0^{-1} * \left((\lambda\Omega t + \Upsilon + \exp\left(-\frac{\Omega}{2}t\right) * \Gamma * \exp\left(\frac{\Omega}{2}t\right)\right) * q_0$$

where Ω, Υ and Γ are constant vectors of $\mathbb{R}^3$, λ is a constant scalar and q_0 is a unit-norm quaternion. Theses constants can be arbitrarily chosen. Hence, the general permanent trajectory corresponds, up to a Galilean transformation, to an helicoidal motion uniformly accelerated along the rotation axis when $\lambda \neq 0$; when λ tends to infinity and Ω to 0, we recover as a degenerate case a uniformly accelerated line. When $\lambda = 0$ and $\Gamma = 0$ we recover a coordinated turn.

III. LEFT INVARIANT DYNAMICS AND RIGHT EQUIVARIANT OUTPUT

A. Invariant observer and error equation

1) Left invariant dynamics and right equivariant output: Consider the following system:

$$\frac{d}{dt}x(t) = f(x,t) \tag{10}$$

$$y = h(x) \tag{11}$$

where we still have $x \in G$, $y \in \mathcal{Y}$, and f is a smooth vector field on G. Let us suppose the dynamics (10) is *left*-invariant (see e.g [1]), i.e: $\forall g, x \in G \quad f(L_g(x), t) = DL_g f(x,t)$. For all $g \in G$, the transformation $X(t) = gx(t)$ leaves the dynamics equations unchanged: $\frac{d}{dt}X(t) = f(X(t), t)$. As in [1] let $\omega_s = DL_{x^{-1}} \frac{d}{dt}x \in \mathfrak{g}$. Indeed one can look at any left invariant dynamics on G as a motion of a "generalized rigid body" with configuration space G. Thus one can look at $\omega_s(t) = f(e,t)$ as the "angular velocity in the body", where e is the group identity element (whereas $DR_{x^{-1}} \frac{d}{dt}x$ is the "angular velocity in space"). We will systematically write the left-invariant dynamics (10)

$$\frac{d}{dt}x(t) = DL_x \omega_s(t) \tag{12}$$

Let us suppose that $h : G \to Y$ is a *right* equivariant smooth output map. The group action on itself by right multiplication corresponds to the transformations $(\rho_g)_{g \in G}$ on the output space $\mathcal{Y}$: for all $x, g \in G$, $h(xg) = \rho_g(h(x))$ i.e

$$h(R_g(x)) = \rho_g(h(x))$$

Left multiplication corresponds then for the generalized body to a change of space-fixed frame, and right multiplication to a change of body-fixed frame. If all the measurements correspond to a part of the state x expressed in the body-fixed frame, they are affected by a change of body-fixed frame, and the output map is right equivariant. Thus the theory allows to build non-linear observers such that the error equation is *autonomous*, in particular for *cart-like vehicles* and *rigid bodies in space* (according to the Eulerian motion) with *measurements in the body-fixed frame* (see the example below).

2) Observability: If the dimension of the output space is strictly smaller than the dimension of the state space ($\dim y < \dim g$) the system is necessarily not observable. This comes from the fact that, in this case, there exists two distinct elements x_1 and x_2 of G such that $h(x_1) = h(x_2)$. If $x(t)$ is a trajectory of the system, we have $\frac{d}{dt}x(t) = DL_g\omega_s(t)$ and because of the left-invariance, $g_1x(t)$ and $g_2x(t)$ are also trajectories of the system:

$$\frac{d}{dt}(g_1x(t)) = DL_{g_1x}\omega_s(t), \quad \frac{d}{dt}(g_2x(t)) = DL_{g_2x}\omega_s(t).$$

But since h is right equivariant: $h(g_1x(t)) = \rho_{x(t)}h(g_1) = \rho_{x(t)}h(g_2) = h(g_2x(t))$. The trajectories $g_1x(t)$ and $g_2x(t)$ are distinct and for all t they correspond to the same output. The system is unobservable.

3) Applying the general theory of section II: There are two ways to apply the theory of section II. i) The most natural (respecting left-invariance) does not yield the most interesting properties: let $\mathcal{U} = \mathbb{R} \times \mathcal{Y}$ and let us look at $(u_1, u_2) = (t, h(e))$ as inputs. For all $g \in G$ let $\psi_g(t, h(e)) = (t, \rho_{g^{-1}}(h(e)))$. Define a new output map $H(x, u) = h(x) = \rho_x(h(e)) = \rho_x(u_2)$. It is unchanged by the transformation introduced in definition 2 since $H(X, U) = \rho_{gx}\big(\rho_{g^{-1}}(u_2)\big) = H(x, u)$ for all $g \in G$. (10)-(11) is then a left-invariant system in the sense of definition 2, when the output map is $H(x, u)$. ii) Let us rather look at $\omega_s(t)$ as an input : $u(t) = \omega_s(t) \in \mathcal{U}$, where $\mathcal{U} = \mathfrak{g} \equiv \mathbb{R}^n$ is the input space. Let us define for all g the map $\psi_g : G \to \mathcal{U}$ the following way

$$\psi_g = DL_{g^{-1}}DR_g$$

It means ψ_g is the differential of the interior automorphism of G. And the dynamics (10) writes $\frac{d}{dt}x = F(x, u) = DL_xu$ and can be viewed as a right-invariant dynamics. For all x, g we have indeed:

$$\frac{d}{dt}R_g(x) = DR_gDL_x\omega_s(t) = DL_xDL_gDL_{g^{-1}}DR_g\omega_s(t) = DL_{R_g(x)}\psi_g(\omega_s(t)) = F(R_g(x), \psi_g(u))$$

$(\psi_g)_{g\in G}$ and $(\rho_g)_{g\in G}$ are *right* group actions since for all $g_1, g_2 \in G$ we have $\psi_{g_1} \circ \psi_{g_2} = \psi_{g_2g_1}$ and $\rho_{g_1} \circ \rho_{g_2} = \rho_{g_2g_1}$. Thus we strictly apply the general theory of II, exchanging the roles of left and right multiplication.

4) Construction of the observers: Take n linearly independent vectors $(W_1, \ldots, W_n)$ in $TG|_e = \mathfrak{g}$. Consider the class of observers of the form

$$\frac{d}{dt}\hat{x} = DL_{\hat{x}}\omega_s(t) + DR_{\hat{x}}(\sum_{i=1}^{n}\mathcal{L}_i(\rho_{\hat{x}^{-1}}(y))W_i) \tag{13}$$

where the $\mathcal{L}_i$'s are smooth scalar functions such that $\mathcal{L}_i(h(e)) = 0$. They are invariant under the transformations defined above in section III-A.3-ii).

5) State-error dynamics: The error (invariant by right multiplication) is $G \ni \eta = (\hat{x}x^{-1}) = L_{\hat{x}}(x^{-1})$. The error equation is an *autonomous* differential equation (14) independent from the trajectory $t \mapsto x(t)$ (as in the linear stationary case):

$$\frac{d}{dt}\eta = DR_\eta(\sum_{i=1}^{n} \mathcal{L}_i(h(\eta^{-1}))W_i) \tag{14}$$

It can be deduced from (5) or directly computed using $\frac{d}{dt}\eta = DL_{\hat{x}}(\frac{d}{dt}x^{-1}) + D_x L_{\hat{x}}(x^{-1})\frac{d}{dt}\hat{x}$ and

- $D_x L_{\hat{x}}(x^{-1})\frac{d}{dt}\hat{x} = DR_{x^{-1}}(\frac{d}{dt}\hat{x}) = DR_{x^{-1}}DL_{\hat{x}}\ \omega_s(t) + DR_{x^{-1}}DR_{\hat{x}}\sum_{i=1}^{n}\mathcal{L}_i(\rho_{\hat{x}^{-1}}(y))W_i = DR_{x^{-1}}DL_{\hat{x}}\ \omega_s(t) + DR_\eta\sum_{i=1}^{n}\mathcal{L}_i(\rho_{\hat{x}^{-1}}(y))W_i$
- $DL_{\hat{x}}(\frac{d}{dt}x^{-1}) = -DL_{\hat{x}}DR_{x^{-1}}DL_{x^{-1}}\dot{x} = -DL_{\hat{x}}DR_{x^{-1}}\omega_s = -DR_{x^{-1}}DL_{\hat{x}}\omega_s(t)$
- $\mathcal{L}_i(\rho_{\hat{x}^{-1}}(y)) = \mathcal{L}_i(\rho_{\hat{x}^{-1}}(h(x))) = \mathcal{L}_i(h(\eta^{-1}))$.

6) First order approximation: We suppose that η is close to e. Let $\xi \in \mathfrak{g}$ such that $\eta = \exp(\epsilon\xi)$ with $\epsilon \in \mathbb{R}$ small. We have up to second order terms in ϵ

$$\frac{d}{dt}\xi = -\sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}_i}{\partial h}(h(e))\frac{\partial h}{\partial x}(e)\xi\right)W_i$$

Let us define a scalar product on the tangent space $\mathfrak{g}$ at e, and let us consider the adjoint operator of $Dh(e)$ in the sense of the metrics associated to the scalar product. The adjoint operator is denoted by $(Dh(e))^T$ and we take $\mathcal{L}(y) = K(Dh(e))^T(y - h(e))$. The first order approximation writes

$$\dot{\xi} = -K\ Dh^T\ Dh\ \xi \tag{15}$$

and for $K > 0$, admits as Lyapunov function $\|\xi\|^2$ which the length of ξ in the sense of the scalar product.

B. A class of non-linear first-order convergent observers

Consider for (10)-(11) the following observers : $\frac{d}{dt}\hat{x} = DL_{\hat{x}}\omega_s(t) + DR_{\hat{x}}[\sum_{i=1}^{n}[\mathcal{L}_i(\rho_{\hat{x}}^{-1}(h(x)))]W_i]$ where the $\mathcal{L}_i$'s are smooth scalar functions such that $\mathcal{L}_i(h(e)) = 0$. Using the first order approximation design, take $\mathcal{L}_1, ..., \mathcal{L}_n$ such that the symmetric part (in the sense of the scalar product chosen on $TG|_e$) of the linear map $\xi \mapsto -\sum_{i=1}^{n}\left(\frac{\partial \mathcal{L}_i}{\partial h}(h(e))\frac{\partial h}{\partial x}(e)\xi\right)W_i$ is negative. When it is negative definite, we get locally exponentially convergent non-linear observers around any system trajectory.

IV. Brief example: Magnetic-aided attitude estimation

To illustrate briefly the theory we give one of the simplest example: magnetic-aided inertial navigation as considered in [11], [3]. We just give the system equations, the application of the theory to this example being straightforward. It is necessary in order to pilot a flying body to have at least a good knowledge of its orientation. This holds for manual, or semi automatic or automatic piloting. In low-cost or "strap-down" navigation systems the measurements of angular velocity $\vec{\omega}$ and acceleration $\vec{a}$ by rather cheap gyrometers and accelerometers are completed by a measure of the earth magnetic field $\vec{B}$. These various measurements are fused (data fusion) according to the motion equations of the system. The estimation of the orientation is generally performed by an extended Kalman filter. But the use of extended Kalman filter requires much calculus capacity because of the matrix inversions. The orientation (attitude) can be described by an element of the group of rotations SO(3), which is the configuration space of a body fixed at a point. The motion equation are

$$\frac{d}{dt}R = R(\vec{\omega} \times \cdot) \tag{16}$$

where

- $R \in$ SO(3) is the quaternion of norm one which represents the rotation which maps the body frame to the earth frame,
- $\vec{\omega}$(t) is the instantaneous angular velocity vector measured by gyroscopes and $(\vec{\omega} \times \cdot)$ the skew-symmetric matrix corresponding to wedge product with $\vec{\omega}$.

If the output is the earth magnetic field $\vec{B}$ measured by the magnetometers *in the body-fixed frame* $y = R^{-1}\vec{B}$ ([5]), the output is right equivariant. The output has dimension 2 (the norm of y is constant) and the state space has dimension 3. Thus the system is not observable according to section III-A.2. This is why we make an additional assumption as in [11], [3]. Indeed the accelerometers measure $\vec{a} = \frac{d}{dt}\vec{v} + R^{-1}\vec{G}$ where $\frac{d}{dt}\vec{v}$ is the acceleration of the center of mass of the body and $\vec{G}$ is the gravity vector. We suppose the acceleration of the center of mass is small with respect to $\parallel \vec{G} \parallel$ (quasi-stationary flight). The measured output is thus $y = (y_G, y_B) = (R^{-1}\vec{G}, R^{-1}\vec{B})$. One can apply the theory as described in section III-A.3-i) or III-A.3-ii).

V. CONCLUSION

In this paper we completed the theory of [4] giving a general framework to symmetry-preserving observers when the state space is a Lie group. The observers are intrinsically and globally defined. By the way, we explained the nice properties of the error equation in two examples of [4]. In particular we derived observers which converge around any trajectory and such that the global behavior is independent of the trajectory as well as of the time-varying inputs for a general class of systems.

REFERENCES

[1] V. Arnold. *Mathematical Methods of Classical Mechanics*. Mir Moscou, 1976.

[2] S. Bonnabel, Ph. Martin, and P. Rouchon. Groupe de lie et observateur non-linéaire. In *CIFA 2006 (Conference Internationale Francophone d'Automatique), Bordeaux, France.*, June 2006.

[3] S. Bonnabel, Ph. Martin, and P. Rouchon. Non-linear observer on lie group for left-invariant dynamics with right-left equivariant output. *IFAC08*, 2008.

[4] S. Bonnabel, Ph. Martin, and P. Rouchon. Symmetry-preserving observers. *http://arxiv.org/abs/math.OC/0612193, Accepted for publication in IEEE AC*, Dec 2006.

[5] S. Bonnabel and P. Rouchon. *Control and Observer Design for Nonlinear Finite and Infinite Dimensional Systems*, chapter On Invariant Observers, pages 53–66. Number 322 in Lecture Notes in Control and Information Sciences. Springer, 2005.

[6] F. Fagnani and J. Willems. Representations of symmetric linear dynamical systems. *SIAM J. Control and Optim.*, 31:1267–1293, 1993.

[7] J.W. Grizzle and S.I. Marcus. The structure of nonlinear systems possessing symmetries. *IEEE Trans. Automat. Control*, 30:248–258, 1985.

[8] T Hamel and R. Mahony. Attitude estimation on so(3) based on direct inertial measurements. In *International Conference on Robotics and Automation, ICRA2006*, 2006.

[9] W. S. Koon and J. E. Marsden. Optimal control for holonomic and nonholonomic mechanical systems with symmetry and lagrangian reduction. *SIAM J. Control and Optim.*, 35:901–929, 1997.

[10] R. Mahony, T. Hamel, and J-M Pflimlin. Non-linear complementary filters on the special orthogonal group. *Accepted for publication in IEEE-AC*.

[11] R. Mahony, T. Hamel, and J-M Pflimlin. Complimentary filter design on the special orthogonal group so(3). In *Proceedings of the IEEE Conference on Decision and Control, CDC05, Seville*, 2005.

[12] Ph. Martin, P. Rouchon, and J. Rudolph. Invariant tracking. *ESAIM: Control, Optimisation and Calculus of Variations*, 10:1–13, 2004.

[13] P. Morin and C. Samson. Practical stabilization of driftless systems on lie groups, the transverse function approach. *IEEE Trans. Automat. Control*, 48:1493–1508, 2003.

[14] P. J. Olver. *Classical Invariant Theory*. Cambridge University Press, 1999.

[15] W. Respondek and I.A. Tall. Nonlinearizable single-input control systems do not admit stationary symmetries. *Systems and Control Letters*, 46:1–16, 2002.