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OUTPUT STABILIZATION VIA NONLINEAR LUENBERGER OBSERVERS*

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Abstract. The present paper addresses the problem of the existence of an (output) feedback law that asymptotically steers to zero prescribed outputs, while keeping all state variables bounded, for any initial conditions in a given compact set. The problem can be viewed as an extension of the classical problem of semiglobally stabilizing the trajectories of a controlled system to a compact set. The problem also encompasses a version of the classical problem of output regulation. Under only a *weak minimum phase* assumption, it is shown that there exists a controller solving the problem at hand. The paper is deliberately focused on theoretical results regarding the existence of such a controller. Practical aspects involving the design and the implementation of the controller are left to a forthcoming work.

Key words. output stabilization, nonlinear output regulation, nonlinear observers, Lyapunov functions, nonminimum-phase systems, robust control

AMS subject classifications. 93D15, 93D21, 93C10, 93B52, 93C15, 93D05

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1. Introduction.

The problem of controlling a system in such a way that some prescribed outputs converge to zero while all other state variables remain bounded is a relevant problem in control theory. It includes, as special cases, the problem of asymptotic stabilization of a fixed equilibrium point and the problem of asymptotic stabilization of a fixed invariant set. It also includes design problems in which some selected variables are required to asymptotically track (or to asymptotically reject) certain signals generated by an independent autonomous system. Problems of this kind, usually referred to as problems of "output regulation," have been extensively studied in the past for linear systems (see [10, 18, 17]) as well as, beginning with the seminal work [23], for nonlinear systems. As a matter of fact, these problems can be viewed as problems in which a "regulated" output of an "augmented system" (a system consisting of the controlled plant and the exogenous system generator) must be asymptotically steered to zero while all other state variables are kept bounded. As pointed out in [23], for instance, the basic challenges in a problem of this type are to create an invariant set on which the desired regulated output vanishes, and to render this set asymptotically attractive.

Even though paper [23] is limited in scope (the design method suggested therein being only meant to secure local, and nonrobust, regulation about an equilibrium point) it has the merit of highlighting a few basic concepts and ideas which shaped all subsequent developments in this area of research. These ideas include the fundamental link between the problem in question and the notion of "zero dynamics" (a concept introduced and studied earlier by the same authors), the necessity of the existence of

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a (controlled) invariant set on which the desired regulated output vanishes, and an embryo of design philosophy based on the idea of making this invariant set locally (and exponentially) attractive.

In the past 15 years, the design philosophy of [23] was extended in several directions. One clear need was to move from "local" to "nonlocal" convergence, a goal which was pursued, for instance, in [25, 21, 30, 5], where different approaches (at increasing levels of generality) have been proposed. Another concern was to obtain design methods which are insensitive, or even robust, with respect to model uncertainties. This issue was originally addressed in [20], where it was shown how, under appropriate hypotheses, the property of (local) asymptotic regulation can be made robust with respect to plant parameter variations, extending in this way a celebrated property of linear regulators.

In the presence of plant parameter variations, the challenge is to design a (parameter-independent) controller in such a way that the closed-loop system possesses a (possibly parameter *dependent*) attractive invariant set on which the regulated output vanishes. The two issues of forcing the existence of such an invariant set and of making the invariant set (locally or nonlocally) attractive are of course interlaced, and this is precisely what, in the past, has determined the various scenarios under which different solutions to the problem have been proposed. In the paper [20], for instance, a solution was achieved by assuming that the set of all feed-forward controls which force the regulated output to be identically zero had to be generated by a single (parameter-independent) *linear* system. This assumption was weakened in [12], in [11], and subsequently in [6], where it was replaced with the assumption that the controls in question are generated by a single (parameter-independent) *nonlinear* system, uniformly observable in the sense of [19].

The crucial observation that made the advances in [6] and [11] possible was the realization that the two issues of forcing the existence of an invariant set (on which the regulated variable vanishes) and of making the invariant set attractive are intimately related to, and actually can be cast as, the problem of designing a (nonlinear) *observer*. As a matter of fact, the design method suggested in [6] was based almost entirely on the construction of a nonlinear "high-gain" observer following the methods of Gauthier and Kupka [19], while the design method suggested in [11] was based almost entirely on the construction of a nonlinear adaptive observer following the methods of Bastin and Gevers [3] and Marino and Tomei [24].

Having realized that the design of observers is instrumental in the design of controllers which solve the problem in question, researchers came to the idea of examining whether alternative options, in the design of observers, could be of some help in weakening the assumptions even further. This turns out to be true, as shown in the present paper, in the case when we adopt the approach to the design of nonlinear observers outlined by Kazantis and Kravaris [27] and further pursued by Kreisselmeier and Engel [28], Krener and Xiao [26], and Andrieu and Praly [1].

While in all earlier contributions it was assumed that the controls which force the regulated output to be identically zero could be interpreted as outputs of a (in general, nonlinear) system having special observability properties (which eventually became part of the controller), a crucial property highlighted in the proof of Theorem 3 of [1] shows that no assumption of this kind is actually needed. The controls in question can *always* be generated by means of a system of appropriate dimension whose dynamics are linear but whose output map is a nonlinear (and, in general, only continuous but not necessarily locally Lipschitzian) map. Once this system is embedded in the controller, boundedness of all closed-loop trajectories and convergence to the desired

invariant set can be guaranteed, as in the earlier contributions [6] and [11], by a somewhat standard paradigm which blends practical stabilization with a small-gain property for feedback interconnection of systems which are input-to-state stable (with restrictions).

The purpose of this paper is to provide a complete proof of how the results of [1] can be exploited for the design of a controller solving the problem, and also to show how some technical hypotheses used in the asymptotic analysis of [7] can be totally removed, yielding in this way a general theory cast only on a very simple and meaningful assumption. This paper is deliberately meant to present only those theoretical results needed to show the *existence* of the solution of the problem in question. Issues related to practical aspects involving constructive design and implementation will be dealt with in a forthcoming work.

The paper is organized as follows. In the next section the main framework under which the problem is solved is presented and discussed. Then section 3 presents an outline of the main results concerning the existence of the output feedback regulator. Section 4 concludes the paper with some with final remarks. Technical proofs of the results in section 3 are postponed to Appendices A and B.

Notation. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm, and for $\mathcal{C}$ a closed subset of $\mathbb{R}^n$, $|x|_{\mathcal{C}} = \min_{y\in\mathcal{C}} |x-y|$ denotes the distance of x from $\mathcal{C}$. For $\mathcal{S}$ a subset of $\mathbb{R}^n$, $\operatorname{cl}\mathcal{S}$ and $\operatorname{int}\mathcal{S}$ are the closure of $\mathcal{S}$ and the interior of $\mathcal{S}$, respectively, and $\partial\mathcal{S}$ its boundary. For the smooth[1] dynamical system $\dot{x} = f(x)$, the value at time t of the solution passing through x_0 at time $t = 0$ will be written as $x(t, x_0)$. The more compact notation $x(t)$ will be used instead of $x(t, x_0)$, when the initial condition is clear from the context. A set $\mathcal{S}$ is said to be forward (backward) invariant for $\dot{x} = f(x)$ if each $x_0 \in \mathcal{S}$, $x(t, x_0) \in \mathcal{S}$ for all $t \geq 0$ ($t \leq 0$). The set is invariant if it is backward and forward invariant. For a locally Lipschitz function $V(t)$ we define Dini's derivative of V at t as

$$D^+V(t) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(t+h) - V(t)].$$

By extension, when $V(t)$ is obtained by evaluating V along a solution $x(t, x_0)$, we denote also

$$D^+V(x_0) = \lim_{h\to 0^+} \sup \frac{1}{h}[V(x(h, x_0)) - V(x_0)]. \tag{1}$$

Note that if $\limsup = \lim$, this is simply $L_fV(x_0)$, the Lie derivative at x_0 of V along f.

2. The framework.

2.1. The problem of output stabilization and the main result.

We consider in what follows a nonlinear *single input–single output* smooth system described by[2]

$$\begin{aligned} \dot{z} &= f(z, y), \\ \dot{y} &= q(z, y) + u \end{aligned} \tag{2}$$

[1] In this paper, "smooth" means "differentiable a sufficiently large number of times" so that all of what we write makes sense.

[2] System (2) is described in the well-known *normal form* with relative degree 1 (see [22]). As discussed in section 2.2, the case of systems in normal form with a higher relative degree can be dealt with in the proposed framework.

with state $(z, y) \in \mathbb{R}^n \times \mathbb{R}$, *measured output* y, and *control input* $u \in \mathbb{R}$, and with initial conditions $(z(0), y(0))$ ranging in a known arbitrary compact set $Z \times \Xi \subset \mathbb{R}^n \times \mathbb{R}$. Associated with (2) there is a *controlled* output $e \in \mathbb{R}^p$ expressed as

$$e = h(z, y) \tag{3}$$

in which $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^p$ is a smooth function.

For system (2)–(3) the problem of semiglobal (with respect to $Z \times \Xi$) output stabilization is defined as follows. Find, if possible, a dynamic output feedback controller of the form

$$\begin{aligned} \dot{\eta} &= \varphi(\eta, y), \\ u &= \varrho(\eta, y) \end{aligned} \tag{4}$$

with state $\eta \in \mathbb{R}^\nu$ and a compact set $M \subset \mathbb{R}^\nu$ such that, in the associated closed-loop system

$$\begin{aligned} \dot{z} &= f(z, y), \\ \dot{y} &= q(z, y) + \varrho(\eta, y), \\ \dot{\eta} &= \varphi(\eta, y), \\ e &= h(z, y), \end{aligned} \tag{5}$$

the positive orbit of $Z \times \Xi \times M$ is bounded and, for each $(z(0), y(0), \eta(0)) \in Z \times \Xi \times M$,

$$\lim_{t \to \infty} e(t) = 0.$$

The problem at issue will be solved only under the following *weak minimum-phase* assumption which requires that system

$$\dot{z} = f(z, 0), \tag{6}$$

representing the zero dynamics of (2) associated with the input u and output y, has a compact attractor which is asymptotically stable. In more precise terms the assumption in question is formulated as follows.

Assumption. There exists a compact set $\mathcal{A} \subset \mathbb{R}^n$ such that

(a_1) the set $\mathcal{A}$ is locally asymptotically stable[3] for system (6) with a domain of attraction $\mathcal{D} \supset Z$;

(a_2) $h(z, 0) = 0$ for all $z \in \mathcal{A}$.

Comments on this assumption and on the proposed framework are postponed until after the next theorem, which presents the main result of the paper.

THEOREM 1. *There exists an $m > 0$, a controllable pair $(F, G) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{m \times 1}$, a continuous function $\gamma : \mathbb{R}^m \to \mathbb{R}$, and for any compact set $M \subset \mathbb{R}^m$, a continuous function $\kappa : \mathbb{R}^p \to \mathbb{R}$, such that the controller*

$$\begin{aligned} \dot{\eta} &= F\eta + Gu, \qquad \eta(0) \in M, \\ u &= \gamma(\eta) + v, \\ v &= \kappa(y) \end{aligned} \tag{7}$$

solves the problem of semiglobal (with respect to $Z \times \Xi$) output stabilization.

[3]Refer to Appendix A for precise definitions regarding the notion of asymptotic stability.

2.2. Remarks on the framework and the result. The framework presented above deals with problems which can be viewed as problems of "external stabilization" of nonlinear systems, namely, problems in which the goal is to steer to zero only a few selected *external* variables, represented in the current framework by the controlled outputs e, while all other variables are simply kept bounded. In this respect the method can be used to handle systems in which there are uncontrollable internal motions that are not necessarily converging to an equilibrium but remain otherwise bounded.

The main idea pursued in the present paper to solve the problem at hand is to extend "high-gain" control paradigms, conventionally used to stabilize a minimum-phase system with respect to an equilibrium point (see [8, 33]), to the case of compact attractors. In this respect assumption (a_1) can be interpreted as a "weak" version of classical minimum-phase hypotheses for the case in which the asymptotic behavior of the zero dynamics (6) is not constrained to being an equilibrium but rather is allowed to be governed by complex bounded dynamics. Clearly, to take advantage of the fact that the trajectories of (6) with initial conditions in Z, as required in part (a_1) of the assumption, are attracted by a compact set $\mathcal{A}$, it would be desirable to have, asymptotically, y converging to zero. Of course since the controlled variable e is required to asymptotically decay to 0, it is also appropriate to assume, as done in (a_2), that $h(z,0)$ vanishes on $\mathcal{A}$. If this were to occur, in fact, then also the controlled variable e would converge to zero, and the problem would be solved. To make y converge to zero, one might wish to appeal to (somewhat standard) "high-gain" arguments and design a control law of the form $u = -ky$. However, it is well known (see, e.g., [33]) that to have y asymptotically converging to zero in a "high-gain" scheme, it is somewhat necessary that the "coupling" term $q(z,y)$ between the upper and the lower subsystem of (2) asymptotically vanishes. More specifically, it is necessary that $q(z,0)$ vanish on the set $\mathcal{A}$ to which the state z of the upper subsystem converges if y decays to zero. Now, in general, there is no guarantee that $q(z,0)$ would vanish on $\mathcal{A}$, and this is why a more elaborate controller has to be synthesized. As a matter of fact, the main result of the paper is that a suitable *dynamic* controller ensures that a property of this kind is achieved.

For the sake of simplicity, we have chosen to illustrate our theory in the case of a system of the form (2), which is rather special for a number of reasons. As a matter of fact, this system has relative degree 1 between the control input u and the measured output y, its "high-frequency gain" is equal to 1, and the dynamics (6) is assumed to possess a compact attractor which is asymptotically stable. However, the main result of the paper lends itself to the synthesis of regulators for more general classes of systems. A natural way in which assumption (a_1) can be weakened consists of viewing y as a "virtual control" of

$$\dot{z} = f(z,y)$$

and assuming the existence of a map $\alpha : \mathbb{R}^n \to \mathbb{R}$ and a compact set $\mathcal{A}$ such that

(a_1') the set $\mathcal{A}$ is locally asymptotically stable for

$$\dot{z} = f(z,\alpha(z))$$

with a domain of attraction $\mathcal{D} \supset Z$,

(a_2') $h(z,\alpha(z)) = 0$ for all $z \in \mathcal{A}$.

If this is the case, in fact, the change of variable $\tilde{y} = y - \alpha(z)$ transforms system (2) into a system in which assumptions (a_1) and (a_2) are fulfilled. The result of Theorem 1 can therefore be used, provided that the variable $\tilde{y}$ is available for feedback.

An extension of this kind is useful in handling cases of systems having relative degree $r > 1$ between the control input u and the measured output y. Consider, to this end, the case of a plant modeled by equations of the form

$$\begin{aligned} \dot{z}_1 &= f(z_1, Cz_2), \\ \dot{z}_2 &= Az_2 + B\zeta, \\ \dot{\zeta} &= q(z_1, z_2, \zeta) + u, \end{aligned} \tag{8}$$

in which $z_1 \in \mathbb{R}^{n-r+1}$, $z_2 \in \mathbb{R}^{r-1}$, $y \in \mathbb{R}$, and A, B, C is a triplet in "prime" form, with measured output

$$y = Cz_2 \tag{9}$$

and controlled output

$$e = h(z_1, z_2, \zeta). \tag{10}$$

Let $Z = Z_1 \times Z_2$ be the prescribed compact set of initial conditions for (z_1, z_2) and suppose there exists a compact set $\mathcal{A}_1$ which is locally asymptotically stable for

$$\dot{z}_1 = f(z_1, 0),$$

with a domain of attraction $\mathcal{D}_1 \supset Z_1$, and such that

$$h(z_1, 0, 0) = 0 \qquad \forall z_1 \in \mathcal{A}_1. \tag{11}$$

If this is the case, the problem of steering e to zero while keeping all internal states bounded can be easily handled in the following way. Standard backstepping arguments (see, e.g., [4] and also [11]) show the existence of matrix K such that, in the system

$$\begin{aligned} \dot{z}_1 &= f(z_1, Cz_2), \\ \dot{z}_2 &= Az_2 + BKz_2, \end{aligned} \tag{12}$$

the compact set $\mathcal{A} = \{(z_1, z_2) : z_1 \in \mathcal{A}_1,\ z_2 = 0\}$ is locally asymptotically stable for (12) with a domain of attraction $\mathcal{D} \supset Z_1 \times Z_2$. Thus, by letting $z = \mathrm{col}(z_1, z_2)$ and $\alpha(z) = Kz_2$, by virtue of (11), it turns out that assumptions (a'_1) and (a'_2) above are fulfilled by system (12) with controlled output (10). This, in view of the previous discussion, guarantees the existence of a controller solving the problem at hand provided that the variable $\tilde{y} := \zeta - Kz_2$ is available for feedback. In particular, since $\mathrm{col}(z_2, \zeta) = (y, \dot{y}, \ldots, y^{(r-1)})$, Theorem 1 guarantees the existence of a *partial state feedback* controller, namely, a controller relying not only on the knowledge of the measured output (9) but also on all its first $r-1$ derivatives with respect to time. This is not a restriction, though, because—as shown, for instance, in [14] and [33]—as long as convergence from a compact set of initial conditions is sought, all components of $\mathrm{col}(z_2, \zeta)$ can always be estimated by means of an "approximate" observer driven only by its first component Cz_2, namely, the actual measured output y.

Finally, it is worth noting that all the results presented in the paper could be generalized to treat the case in which the "high-frequency gain," which is assumed to be unitary in (2), is a generic sign-definite function of the state, namely, the case in which the dynamics of y in (2) is described by

$$\dot{y} = q(z, y) + b(z, y)u \tag{13}$$

with $b(z, y) \geq \underline{b}$ for some known $\underline{b} > 0$. Details on how this can be accomplished are rather straightforward and are not deliberately presented here, as they would only add notational complications without any extra conceptual value.

2.3. Output regulation. A special case covered by the previous setup is the one in which system (2)–(3) is described by equations of the form

$$\begin{aligned} \dot{z}_1 &= f_1(z_1), \\ \dot{z}_2 &= f_2(z_1, z_2, y), \\ \dot{y} &= q(z_1, z_2, y) + u, \\ e &= h(z_1, z_2, y), \end{aligned} \tag{14}$$

i.e., a system with "triangular" zero dynamics. In this case, it is clear that the dynamics of z_1 is a totally *autonomous* dynamics, which can be viewed as an "exogenous" signal generator. This is the way in which the classical problem of *output regulation* is usually cast (see [23]). Depending on the control scenario, the variable z_1 may assume different meanings. It may represent exogenous disturbances to be rejected and/or references to be tracked. It may also contain a set of (constant or time-varying) uncertain parameters affecting the controlled plant.

In this context, it is important to note that the proposed framework encompasses a number of problems which have been recently addressed (see, among others, [25, 31, 11, 5, 4, 12]), all relying upon various versions of the so-called "minimum-phase" and "immersion" assumptions.

More specifically, as far as the assumption of "minimum-phase" is concerned, all the aforementioned works require that the dynamics

$$\begin{aligned} \dot{z}_1 &= f_1(z_1), \\ \dot{z}_2 &= f_2(z_1, z_2, 0), \end{aligned} \tag{15}$$

with $z_1 \in \mathbb{R}^s$, $z_2 \in \mathbb{R}^{n-s}$, possess some stability property. For instance, in [31] the assumption in question is the existence of a differentiable map $\pi : \mathbb{R}^s \to \mathbb{R}^{n-r}$ whose graph

$$\mathcal{A} = \{(z_1, z_2) \in \mathbb{R}^s \times \mathbb{R}^{n-r} \ : \ z_2 = \pi(z_1)\}$$

is invariant and locally exponentially stable for (15), uniformly with respect to the exogenous variable z_1, with a domain of attraction containing the assigned compact set of initial conditions and such that

$$h(z_1, z_2, 0) = 0 \qquad \forall\, (z_1,\, z_2) \in \mathcal{A}.$$

This assumption has been substantially weakened in [5] (see also [6] and [4]) by simply asking that the positive orbit of the set of initial conditions under the flow of (15) be bounded (which is, in turn, equivalent to the existence of a compact set $\mathcal{A}$, having the properties indicated above, which is the graph of a *set-valued* map π). Of course, it is apparent that all these cases fit into the framework presented in section 2.1.

Furthermore, the framework presented in the actual literature, where the problems of output regulation are usually tackled, requires an additional assumption commonly referred to as the "immersion" assumption. The latter refers to system (15) with output $q(z_1, z_2, 0)$ which is required to be immersed into a system with prescribed properties. To mention a few of these properties, typical assumptions required immersion into a linear observable system (see [20, 31, 30]), or into a nonlinear system in canonical observability form (see [6]), or into a nonlinear system in adaptive observability form (see [11]).

Remarkably, this additional assumption is not present in the framework proposed in this work. In this respect the important conceptual result proved in Theorem 1 is that no "immersion" assumption is, in principle, necessary for the problem of output regulation to be solvable.

3. Main results.

3.1. The basic approach. In this section we overview the main steps which will be followed to prove Theorem 1. Technical proofs of the results given here are presented in Appendix B.

We consider the closed-loop system (2), (7) which, after the change of coordinates

$$\eta \to x = \eta - Gy,$$

can be rewritten as

$$\begin{aligned} \dot{z} &= f_0(z) + f_1(z,y), \\ \dot{x} &= Fx - Gq_0(z) - Gq_1(z,y) + FGy, \\ \dot{y} &= q_0(z) + q_1(z,y) + \gamma(x+Gy) + v, \end{aligned} \tag{16}$$

in which

$$f_0(z) := f(z,0), \qquad q_0(z) := q(z,0) \tag{17}$$

and

$$\begin{aligned} f_1(z,y) &:= f(z,y) - f(z,0), \\ q_1(z,y) &:= q(z,y) - q(z,0). \end{aligned}$$

Observe that we have $f_1(z,0) \equiv 0$ and $q_1(z,0) \equiv 0$ for all $z \in \mathbb{R}^n$.

Remark. In the case when the y dynamic is described by (13) (namely, the system does not have unitary "high-frequency gain"), it turns out that the change of coordinates to be considered is of the form

$$\eta \to x = \eta - G \int_0^y \frac{1}{b(z,s)}\, ds,$$

which is well defined as $b(z,y) \geq \underline{b} > 0$. In this case the resulting system exhibits all the crucial properties of (16) on which the forthcoming stability analysis is based, with, in particular, the function $q_0(z)$ defined as $q_0(z) = q(z,0)/b(z,0)$.

In what follows, system (16) is seen as a system with input v, output y, and initial conditions contained in a set of the form $Z \times X \times C$ in which $X \subset \mathbb{R}^m$ and $C \subset \mathbb{R}$ are compact sets dependent on Ξ and M. A controller of the form (7) solves the problem at issue if, for some map $\kappa : \mathbb{R}^n \to \mathbb{R}^n$, the control law $v = \kappa(y)$ is such that all trajectories of (16) originating from $Z \times X \times C$ are bounded and

$$\lim_{t\to\infty} y(t) = 0, \qquad \lim_{t\to\infty} |z(t)|_{\mathcal{A}} = 0. \tag{18}$$

As a matter of fact, since systems (2), (7), and (16) are diffeomorphic, boundedness of the trajectories of (16) with initial conditions in $Z \times X \times C$ implies boundedness of the trajectories of (2), (7) originating from $Z \times \Xi \times M$. Furthermore, by virtue of assumption (a_2), since $h(\cdot)$ is a continuous function, condition (18) implies also that $\lim_{t\to\infty} e(t) = 0$, namely, that the problem of semiglobal output stabilization is solved.

By virtue of this fact, in the following we focus our attention on system (16), and we prove that (16) controlled by $v = \kappa(y)$ has bounded trajectories and that (18) holds. To this end, for notational convenience, denote

$$p = \mathrm{col}(z,x)$$

and rewrite system (16) in the more compact form

$$\begin{aligned} \dot{p} &= M(p) + N(p, y), \\ \dot{y} &= H(p) + K(p, y) + v, \end{aligned} \tag{19}$$

in which $M(\cdot)$ and $H(\cdot)$ are defined as

$$M(p) = \begin{pmatrix} f_0(z) \\ Fx - Gq_0(z) \end{pmatrix} \tag{20}$$

and

$$H(p) = q_0(z) + \gamma(x), \tag{21}$$

and $N(\cdot)$ and $K(\cdot)$ are therefore suitable smooth remainder functions which satisfy necessarily $N(p, 0) = 0$ and $K(p, 0) = 0$ for all p. Consistently set $P = Z \times X$ so that the initial conditions of (19) range in $P \times C$. System (19) is recognized as a system in normal form with relative degree one (with respect to input v and output y) and zero dynamics given by

$$\dot{p} = M(p). \tag{22}$$

Thus, following consolidated knowledge about stabilization of *minimum-phase* nonlinear systems (see [8, 2, 33]), the capability of stabilizing (19) by output feedback is expected to strongly rely upon asymptotic properties of the zero dynamics (22). This is confirmed by the next two results showing that the existence of an asymptotically stable attractor for system (22) is sufficient to achieve boundedness of trajectories and *practical* stabilization (Theorem 2), which becomes *asymptotic* if the function $H(\cdot)$ vanishes on the attractor (Theorem 3). These results, which in the context of this paper represent building blocks for proving Theorem 1, are interesting on their own, as they represent an extension of well-known stabilization paradigms for systems with equilibria (see [33]) to the case of systems of the form (19), with zero dynamics (22) possessing compact attractors. For precise definitions of asymptotic and exponential stability used in the statement of the theorems, the reader is referred to Appendix A.

THEOREM 2. *Consider system* (19) *with* $M(\cdot)$ *and* $N(\cdot)$ *at least locally Lipschitz functions and* $H(\cdot)$ *and* $K(\cdot)$ *at least continuous functions. Let the initial conditions be in* $P \times C$. *Assume that system* (22) *has a compact attractor* $\mathcal{B}$ *which is asymptotically stable with a domain of attraction* $\mathcal{D} \supset P$. *Then for all* $\epsilon > 0$ *there exists a* $\kappa^\star > 0$ *such that for all* $\kappa \geq \kappa^\star$ *the trajectories of* (19) *with* $v = -\kappa y$ *are bounded and* $\limsup_{t\to\infty} |y(t)| \leq \epsilon$ *and* $\limsup_{t\to\infty} |p(t)|_{\mathcal{B}} \leq \epsilon$.

THEOREM 3. *In addition to the hypotheses of the previous theorem, assume that* $H(p)|_{\mathcal{B}} = 0$. *Then there exists a continuous function* $\kappa : \mathbb{R} \to \mathbb{R}$ *such that the trajectories of* (19) *with* $v = \kappa(y)$ *are bounded and* $\lim_{t\to\infty} y(t) = 0$ *and* $\lim_{t\to\infty} |p(t)|_{\mathcal{B}} = 0$. *If, additionally,* $H(\cdot)$ *and* $K(\cdot)$ *are locally Lipschitz and the set* $\mathcal{B}$ *is also locally exponentially stable for* (22), *then there exists* $\kappa^\star > 0$ *such that for all* $\kappa \geq \kappa^\star$ *the same properties hold with* $v = -\kappa y$.

For the proofs of these theorems the reader is referred to sections B.1 and B.2, respectively.

Motivated by these results (and in particular by Theorem 3), we turn our attention to the study of the zero dynamics (22) (with $M(\cdot)$ as in (20)) and to the function $H(\cdot)$ in (21) by looking for the existence of a pair (F, G) and a function $\gamma(\cdot)$ which guarantee the basic requirements behind Theorem 3 with, in particular, $H(p)|_{\mathcal{B}} = 0$. Details in this direction are presented in the next subsection.

3.2. The properties of the "core subsystem" (22). The crucial result which will be proved in this part is that, under the assumption presented in section 2.1, there is a choice of the pair (F, G) and of the map $\gamma(\cdot)$ which guarantee the existence of an asymptotically stable compact attractor $\mathcal{B}$ for (22), on which the function $H(\cdot)$ in (21) vanishes. Moreover the projection of $\mathcal{B}$ on the z coordinates coincides with $\mathcal{A}$. In view of the arguments discussed in the previous subsection, this, along with an appropriate choice of $\kappa(\cdot)$ whose existence is claimed in Theorem 3, substantially proves Theorem 1.

The result in question is proved in the next three propositions. To this end, note that the core subsystem (22) in the original coordinates (z, x) is expressed as

$$\begin{aligned} \dot{z} &= f_0(z), \\ \dot{x} &= Fx - Gq_0(z) \end{aligned} \tag{23}$$

with an initial condition in $Z \times X$. The first proposition is related to the first basic requirement behind Theorem 2, namely, the existence of a locally asymptotically stable attractor for (23).

More precisely, under the only requirement, which is that F be a Hurwitz matrix, we show the existence of a set which is *forward invariant* and *locally asymptotically stable* for (23). The set in question is described by the graph of a map.

PROPOSITION 1. *Consider system* (23) *with the z-subsystem satisfying assumption* (a_1)*, and let* (F, G) *be any pair with* F *Hurwitz. Then*

(i) *there exists at least one continuous map* $\tau : \mathbb{R}^n \to \mathbb{R}^m$ *such that the set*

$$\text{graph}(\tau|_{\mathcal{A}}) := \{(z, x) \in \mathcal{A} \times \mathbb{R}^m \ : \ x = \tau(z)\} \tag{24}$$

is forward invariant for (23)*.*

(ii) *the set* $\text{graph}(\tau|_{\mathcal{A}})$ *is locally asymptotically stable for* (23) *with a domain of attraction containing* $Z \times X$*. Furthermore, the set in question is also locally exponentially stable for* (23) *if* $\mathcal{A}$ *is such for* (6)*.*

The proof of this proposition can be found in section B.3.

Remark. Indeed, there might be many different continuous maps τ having property (i) of Proposition 1. However, it turns out that if $\mathcal{A}_0$ is any compact *subset* of $\mathcal{A}$ which is also *backward invariant* for (6), then for each $z \in \mathcal{A}_0$ there is one and only one $x_z \in \mathbb{R}^m$ such that the set $\bigcup_{z \in \mathcal{A}_0} \{(z, x_z)\}$ is invariant for (23). In particular,

$$x_z = -\int_{-\infty}^{0} e^{-Fs} G q_0(z(s, z)) ds,$$

where $z(s, z)$ denotes the value at time $t = s$ of the solution of $\dot{z} = f_0(z)$ passing through $z \in \mathcal{A}_0$ at time $t = 0$ (see [11]).

The second crucial requirement imposed by Theorem 3 is that the function $H(\cdot)$ in (21) vanishes on the asymptotically stable attractor $\text{graph}(\tau|_{\mathcal{A}})$. Here is where the precise choices of the pair (F, G) and of the map $\gamma(\cdot)$ play a role. In particular note that, by definition of $H(\cdot)$ in (21) and of $\text{graph}(\tau|_{\mathcal{A}})$ in (24), it turns out that

$$H(p)|_{\text{graph}(\tau|_{\mathcal{A}})} = (q_0(z) + \gamma \circ \tau(z))|_{\mathcal{A}}, \tag{25}$$

from which it is apparent that $\gamma(\cdot)$ should be chosen to satisfy $\gamma \circ \tau(z) = -q_0(z)$ for all z in $\mathcal{A}$. It is easy to realize that the possibility of choosing $\gamma(\cdot)$ in this way

is intimately related to the fact that the map τ satisfies the *partial* (*with respect to* $q_0(\cdot)$) *injectivity condition*

$$\tau(z_1) = \tau(z_2) \qquad \Rightarrow \qquad q_0(z_1) = q_0(z_2) \qquad \forall\, z_1, z_2 \in \mathcal{A}. \tag{26}$$

In this respect it is interesting to note that what this condition says is that we need only to reconstruct the value of $q_0(z)$ as a function of z from the knowledge of $\tau(z)$ and not the whole state z (see also the remark at the end of the subsection).

As τ is dependent on the pair (F, G), the next natural point to be addressed is if there exists a choice of (F, G) yielding the desired property for $\tau(\cdot)$. To this end is devoted the next proposition which claims that, indeed, there exists a suitable choice of (F, G), with F Hurwitz, such that the associated map $\tau(\cdot)$ satisfies the required partial injectivity condition. Besides other technical constraints on the choice of F, which will be better detailed in the proof of the Proposition 2, the main requirement on F is given by its dimension, which is required to be sufficiently large with respect to the dimension of z.

PROPOSITION 2. *Set*

$$m = 2 + 2n.$$

Then there exist a controllable pair $(F, G) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times 1}$*, with* F *a Hurwitz matrix, and a class-*$\mathcal{K}$ *function* $\varrho : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$|q_0(z_1) - q_0(z_2)| \leq \varrho(|\tau(z_1) - \tau(z_2)|) \qquad \forall\, z_1, z_2 \in \mathcal{A} \tag{27}$$

in which $\tau(\cdot)$ *is a map (associated with* F*) having the properties indicated in Proposition* 1*.*

For the proof of this proposition the reader is referred to section B.4.

Remark. Going through the proof of the previous proposition, it turns out that the pair (F, G) can be chosen as any $(2n + 2)$-dimensional real representation of the $(n + 1)$-dimensional complex pair (F_c, G_c), with $F_c = \mathrm{diag}(\lambda_1, \ldots, \lambda_{r+1})$, $G_c = (g_1, \ldots, g_{r+1})^{\mathrm{T}}$ in which g_i are arbitrary nonzero real numbers and λ_i are $n + 1$ complex numbers taken arbitrarily outside a set of zero Lebesgue measure and with real part smaller than ℓ, a real number related to the Lipschitz constant of $f_0(\cdot)$ (see Proposition 4).

It turns out that the injectivity property (27) is a sufficient condition for the map $\gamma(\cdot)$ to exist. This is formalized in the following final proposition, proved in section B.5, which states that if (27) holds, then there exists a map $\gamma(\cdot)$ which makes $H(\cdot)$ vanish on the attractor $\mathrm{graph}(\tau|_{\mathcal{A}})$. The map $\gamma(\cdot)$ can be claimed, in general, to be only continuous. It is also Lipschitz in the special case in which the class-$\mathcal{K}$ function $\varrho(\cdot)$ in (27) is Lipschitz.

PROPOSITION 3. *Let* $\tau(\cdot)$ *be a continuous map satisfying* (27) *with* $\mathcal{A}$ *a closed set. Then there exists a continuous map* $\gamma : \mathbb{R}^m \to \mathbb{R}$ *such that*

$$q_0(z) + \gamma \circ \tau(z) = 0 \qquad \forall\, z \in \mathcal{A}. \tag{28}$$

If, in addition, the function $\varrho(\cdot)$ *in* (27) *is linearly bounded at the origin, then the map* γ *is Lipschitz.*

Combining the results of all the previous propositions, it appears that it is sufficient to choose the pair (F, G) of suitable dimension (with F Hurwitz) according to Proposition 2 and to choose $\gamma(\cdot)$ in order to satisfy relation (28). In fact, by doing

so we are guaranteed that the compact set $\mathcal{B} = \text{graph}(\tau|_{\mathcal{A}})$ is locally asymptotically stable for (23) with the map (21) which is vanishing on $\mathcal{B}$. This, indeed, makes it possible to apply Theorem 3 and to prove the existence of a continuous function $\kappa(\cdot)$, completing in this way the synthesis of the controller.

Remark. The reader who is familiar with recent developments in the theory of nonlinear state observers will find it interesting to compare the previous results with the design method proposed by Kazantzis and Kravaris in [27] and pursued in [28], [26], and [1]. In the framework of [27], system (23) can be identified with the cascade of an "observed" system $\dot{z} = f_0(z)$ with output $y_z = q_0(z)$ driving an "observer" $\dot{x} = Fx - Gy_z$. If the map $\tau(\cdot)$ has a left inverse $\tau_\ell^{-1}(\cdot)$, the observer in question provides a state estimate $\hat{z} = \tau_\ell^{-1}(x)$. Such a left-inverse, as shown in [1], always exists provided that the dimension of x is sufficiently large, if the pair (f_0, q_0) has appropriate *observability* properties. In the present context of output stabilization, though, left invertibility of $\tau(\cdot)$ is not needed. In fact, what the controller is expected to do is reproduce only the output $q_0(z(t))$ and not the full state $z(t)$ of the "observed system." This motivates the absence of observability hypotheses on the pair (f_0, q_0).

4. Conclusions. This paper is focused on the existence of an output feedback law that asymptotically steers to zero a given controlled variable, while keeping all state variables bounded, for any initial conditions in a fixed compact set. The proposed framework encompasses and extends a number of existing results in the fields of output feedback stabilization and output regulation of nonlinear systems. The main assumption under which the theory is developed is the existence of a state feedback control law able to achieve boundedness of the trajectories of the zero dynamics of the controlled plant. In this sense the result presented here is applicable for a wide class of nonminimum-phase nonlinear systems not tractable in existing frameworks. In the paper only results regarding the existence of the controller solving the problem at hand have been presented, while practical aspects involving its design and implementation are left to a forthcoming work.

Appendix A. Converse Lyapunov result. Consider a system of the form

$$\dot{p} = f(p), \qquad p \in \mathbb{R}^n, \tag{29}$$

in which $f(p)$ is a C^k (with k sufficiently large) function, with an initial condition ranging over a fixed *compact* set P. For system (29) assume the existence of a compact set $\mathcal{B} \subset \mathbb{R}^n$ which is asymptotically stable for (29), with a domain of attraction $\mathcal{D} \supset P$. More precisely, by setting

$$|p|_{\mathcal{B}/\mathcal{D}} = \left(1 + \frac{1}{|p|_{\partial\,\text{cl}\,\mathcal{D}}}\right) |p|_{\mathcal{B}},$$

we assume that the set $\mathcal{B}$ satisfies the following two properties:[4]

Uniform stability: There exists a class-$\mathcal{K}$ function φ such that for any $\alpha > 0$,

$$|p_0|_{\mathcal{B}/\mathcal{D}} \le \alpha \qquad \Rightarrow \qquad |p(t, p_0)|_{\mathcal{B}/\mathcal{D}} \le \varphi(\alpha) \quad \forall\, t \ge 0.$$

Uniform attractivity: There exists a continuous function $T : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $\alpha > 0$ and $\epsilon > 0$,

$$|p_0|_{\mathcal{B}/\mathcal{D}} \le \alpha \qquad \Rightarrow \qquad |p(t, p_0)|_{\mathcal{B}/\mathcal{D}} \le \epsilon \quad \forall\, t \ge T(\alpha, \epsilon).$$

[4] In these properties the adjective "uniform" refers to independence with respect to a particular point p in a compact set $|p|_{\mathcal{B}\setminus\mathcal{D}} \le c$.

We say that $\mathcal{B}$ is also *locally exponentially stable* for (29) if there exist $M \geq 1$, $\lambda > 0$, and $c_0 > 0$ such that

$$|p_0|_{\mathcal{B}/\mathcal{D}} \leq c_0 \qquad \Rightarrow \qquad |p(t, p_0)|_{\mathcal{B}/\mathcal{D}} \leq M e^{-\lambda t} |p_0|_{\mathcal{B}/\mathcal{D}}.$$

In this framework it is possible to formulate the following converse Lyapunov result which claims the existence of a *locally Lipschitz* Lyapunov function vanishing on the attractor. The result is not formally proved, as it can be easily deduced by the arguments presented in [34] (see in particular Theorem 22.5 and the related Theorems 22.1 and 19.2 in the quoted reference).

THEOREM 4. *Under the above uniform stability and uniform attractivity conditions, there exists a continuous function $V : \mathcal{D} \to \mathbb{R}$ with the following properties:*

(a) *There exist class-$\mathcal{K}_\infty$ functions $\underline{a}(\cdot), \overline{a}(\cdot)$ such that*

$$\underline{a}(|p|_{\mathcal{B}/\mathcal{D}}) \leq V(p) \leq \overline{a}(|p|_{\mathcal{B}/\mathcal{D}}) \qquad \forall\, p \in \mathcal{D};$$

(b) *there exists $c > 0$ such that*

$$D^+ V(p) \leq -cV(p) \qquad \forall\, p \in \mathcal{D};$$

(c) *for all $\alpha > 0$ there exists $L_\alpha > 0$ such that for all $p_1, p_2 \in \mathcal{D}$ such that $|p_1|_{\mathcal{B}/\mathcal{D}} \leq \alpha$, $|p_2|_{\mathcal{B}/\mathcal{D}} \leq \alpha$, the following holds:*

$$|V(p_1) - V(p_2)| \leq L_\alpha |p_1 - p_2|.$$

If $\mathcal{B}$ is also locally exponentially stable for (29)*, then property* (a) *holds with $\underline{a}(\cdot), \overline{a}(\cdot)$ linear near the origin.*

With this result at hand, it is also possible to formulate a local input-to-state stability result for system (29) forced by an external signal. This is formalized in the next lemma.

LEMMA 1. *Let $x : \mathbb{R}_+ \to \mathbb{R}^m$ be a C^0 function. Consider the system*

$$\dot{p} = f(p) + \ell(p, x(t)) \tag{30}$$

in which $p \in \mathbb{R}^n$ and $\ell(p, 0) = 0$ for all $p \in \mathbb{R}^n$. The functions $f(\cdot), \ell(\cdot)$ are C^1. Suppose that system (29) *satisfies the above uniform stability and uniform attractivity conditions. Then there exist functions $\beta(\cdot, \cdot)$ and $\gamma(\cdot)$, respectively, of class $\mathcal{KL}$ and $\mathcal{K}$, and a $d^* > 0$ such that if*

$$|p_0|_{\mathcal{B}} \leq d^* \qquad \text{and} \qquad |x(t)| \leq d^* \quad \forall\, t \geq 0, \tag{31}$$

then the right maximal interval of definition of $p(t, p_0)$ is $[0, +\infty)$, and we have

$$|p(t, p_0)|_{\mathcal{B}} \leq \max \left\{ \beta(|p_0|_{\mathcal{B}}, t)\,,\, \gamma\left(\max_{\tau \in [0,t]} |x(\tau)| \right) \right\} \qquad \forall\, t \geq 0. \tag{32}$$

If the set $\mathcal{B}$ is also locally exponentially stable for (29)*, then there exist $N > 1$, $k > 0$, and $\bar{\gamma} > 0$ such that* (32) *can be modified to read*

$$|p(t, p_0)|_{\mathcal{B}} \leq N e^{-kt} |p_0|_{\mathcal{B}} + \bar{\gamma} \max_{\tau \in [0,t]} |x(\tau)| \qquad \forall\, t \geq 0. \tag{33}$$

Proof. Pick $\beta > 0$ such that if $|p|_{\mathcal{B}} \leq \beta$, then $p \in \mathcal{D}$, and note that there exists $d_\beta > 1$ such that for all p satisfying $|p|_{\mathcal{B}} \leq \beta$,

$$|p|_{\mathcal{B}/\mathcal{D}} \leq d_\beta |p|_{\mathcal{B}}. \tag{34}$$

Pick $d^* > 0$ arbitrarily for the time being but more specifically later on. As $\ell(\cdot)$ is differentiable and $\ell(p,0) = 0$, there is an $\hat{\ell} > 0$ such that for all $|p|_{\mathcal{B}} \leq \beta$ and $|x| \leq d^*$,

$$|\ell(p,x)| \leq \hat{\ell}|x|.$$

So consider the Lyapunov function V given by Theorem 4. By using properties (b) and (c) of this theorem, setting $\alpha = d_\beta \beta$, we obtain for system (30), as long as $|p_1|_{\mathcal{B}} < \beta$ and $|x| \leq d^*$,

$$\begin{aligned} D^+V(p_1,x) &= \limsup_{h\to 0^+} \frac{1}{h}[V(p(h,p_1)) - V(p_1)] \\ &= \limsup_{h\to 0^+} \frac{1}{h}[V(p_1 + hf(p_1) + h\ell(p,x)) - V(p_1)] \\ &\leq \limsup_{h\to 0^+} \frac{1}{h}[V(p_1 + hf(p_1) + h\ell(p_1,x)) - V(p_1 + hf(p_1))] \\ &\qquad + \limsup_{h\to 0^+} \frac{1}{h}[V(p_1 + hf(p_1)) - V(p_1)] \\ &\leq \limsup_{h\to 0^+} \frac{1}{h} L_\alpha h\ell(p_1,x) - cV(p_1) \leq L_\alpha \hat{\ell}|x| - cV(p_1). \end{aligned} \tag{35}$$

Now assume (31) holds. Let $p(t,p_0)$, abbreviated $p(t)$, be the corresponding solution of (30). Let $[0,T_0)$ be its right maximal interval of definition when restricted to take values in the open set $\{p : |p|_{\mathcal{B}} < \beta\}$. Equation (35) holds for $p_1 = p(t)$ and all t in $[0,T_0)$. This implies

$$V(p(t)) \leq e^{-c(t-t_0)}V(p_0) + \frac{L_\alpha \hat{\ell}}{c} \max_{\tau\in[0,t]} |x(\tau)| \qquad \forall t \in [0,T_0). \tag{36}$$

This, in view of property (a) in Theorem 4, yields

(37)

$$|p(t)|_{\mathcal{B}} \leq |p(t)|_{\mathcal{B}/\mathcal{D}} \leq \underline{a}^{-1}(2e^{-ct}\,\overline{a}(|p_0|_{\mathcal{B}/\mathcal{D}})) + \underline{a}^{-1}\left(2\frac{L_\alpha \hat{\ell}}{c} \max_{\tau\in[0,t]} |x(\tau)|\right) \quad \forall t \in [0,T_0).$$

By using (34), it follows that if d^* is chosen so that

$$d^\star \leq \min\left\{ \frac{c}{2L_\alpha \hat{\ell}}\,\underline{a}\left(\frac{\beta}{3}\right), \frac{1}{d_\beta}\,\overline{a}^{-1}\left(\frac{1}{2}\,\underline{a}\left(\frac{\beta}{3}\right)\right)\right\}$$

we have

$$|p(t)|_{\mathcal{B}} < \beta \qquad \forall t \in [0,T_0).$$

From the definition of T_0, it must be infinite. So we have established that (37) holds for all $t \geq 0$ if (31) is satisfied. This proves the first part of the result. The second part of the result, namely, that under exponential stability the bound (33) holds, follows immediately by (37) by using the fact that the functions $\underline{a}(\cdot)$ and $\overline{a}(\cdot)$ can be linear near the origin. ∎

Appendix B. Proofs.

B.1. Proof of Theorem 2. The feedback interconnection (19) can be studied by means of arguments which are quite similar to those used in [22] to prove some of the main stabilization results of [33]. In doing this, we take advantage of the converse Theorem 4 presented in Appendix A.

Let $V : \mathcal{D} \to \mathbb{R}$ be the function given by Theorem 4. Pick a number $a > 0$ such that $C \subset B_a := \{y \in \mathbb{R} : |y| \leq a\}$ and $P \subset V^{-1}([0, a])$ (which is possible because of property (a) in Theorem 4). Define

$$\hat{c} = \max_{(p,y)\in V^{-1}([0,a+1])\times B_{a+1}} |H(p) + K(p, y)|.$$

Also, since $N(p, y)$ is locally Lipschitz and vanishes at $y = 0$, there is a number $\hat{n}$ such that

$$|N(p, y)| \leq \hat{n}|y| \qquad \forall (p, y) \in V^{-1}([0, a + 1]) \times B_{a+1}.$$

Finally, by property (c) in Theorem 4, there is a number L_V such that

$$|V(p_1) - V(p_2)| \leq L_V|p_1 - p_2| \qquad \forall (p_1, p_2) \in V^{-1}([0, a + 1])^2.$$

Then, by choosing $v = -ky$ in the y-dynamics in (19), we get (see notation (1))

$$D^+|y| \leq -\kappa|y| + \hat{c} \qquad \forall (p, y) \in V^{-1}([0, a + 1]) \times B_{a+1}. \tag{38}$$

Also, by following along the same lines as in (35), we get

$$D^+V(p) \leq L_V\hat{n}|y| - cV(p) \qquad \forall (p, y) \in V^{-1}([0, a + 1]) \times B_{a+1}. \tag{39}$$

So now consider a solution $(p(t), y(t))$ issued from a point in $P \times C \subset V^{-1}([0, a]) \times B_a$. Let $[0, T_1)$ be its right maximal interval of definition when restricted to take values in the open set $\operatorname{int}\big(V^{-1}([0, a + 1]) \times B_{a+1}\big)$. It follows that both (38) and (39) hold for $(p(t), y(t))$ when t is in $[0, T_1)$. They give successively, for all t in $[0, T_1)$,

$$\begin{aligned}
|y(t)| &\leq e^{-\kappa t}\, a + \frac{\hat{c}}{\kappa}\left(1 - e^{-\kappa t}\right),\\
V(p(t)) &\leq e^{-ct}\, a + L_V\hat{n}\left(\frac{\hat{c}}{\kappa}\frac{1 - e^{-ct}}{c} + \frac{e^{-ct} - e^{-\kappa t}}{\kappa - c}\left[a - \frac{\hat{c}}{\kappa}\right]\right)\\
&\leq a + L_V\hat{n}\left(\frac{\hat{c}}{c\kappa} + \frac{a}{\kappa - c}\right).
\end{aligned}$$

Hence, by selecting κ to satisfy

$$\kappa > \max\left\{2\hat{c}, (c + 3aL_V\hat{n}), \frac{3L_V\hat{n}\hat{c}}{c}\right\},$$

we get, for all t in $[0, T_1)$,

$$|y(t)| \leq a + \frac{1}{2} \quad \text{and} \quad V(p(t)) \leq a + \frac{2}{3}.$$

This says that the solution remains in $V^{-1}([0, a + \frac{2}{3}]) \times B_{a+\frac{1}{2}}$. So from its definition, T_1 is infinite. Then, from (38), we get

$$\limsup_{t\to+\infty} |y(t)| \leq \frac{\hat{c}}{\kappa}.$$

With (39), this in turn implies

$$\limsup_{t\to+\infty} V(p(t)) \leq \frac{L_V\hat{n}\hat{c}}{\kappa}.$$

In view of property (a) in Theorem 4, the latter yields

$$\limsup_{t\to+\infty} |p(t)|_{\mathcal{B}} \le \underline{a}^{-1}\left(\frac{L_V \hat{n}\hat{c}}{\kappa}\right).$$

So the conclusion of Theorem 2 holds if we further impose that κ satisfies

$$\kappa > \max\left\{\frac{\hat{c}}{\epsilon}, \frac{L_V \hat{n}\hat{c}}{\underline{a}(\epsilon)}\right\}. \qquad \square$$

B.2. Proof of Theorem 3. The proof of this result follows by standard small-gain arguments. Let $\kappa(y) = -\alpha(y)$, where $\alpha(\cdot)$ is a continuous function such that $\alpha(0) = 0$ and $y\alpha(y) > 0$ for all $y \neq 0$. By mimicking the proof of Theorem 2 it is possible to show that for any $\epsilon > 0$ there exist a $\kappa^\star > 0$ and a $T > 0$ such that, if $|\alpha(|y|)| \ge \kappa^\star |y|$, then each trajectory of the closed-loop system issuing from the compact set $P \times C$ satisfies

$$|p(t)|_{\mathcal{B}} \le 2\epsilon \quad \text{and} \quad |y(t)| \le 2\epsilon \quad \forall\, t \ge T.$$

Observe that Lemma 1 applies to the p-component of the closed-loop solution. So let $d^\star$ be given by this lemma. By picking ϵ above satisfying $2\epsilon \le d^\star$, and by applying Lemma 1 to the p-component by picking the initial condition at time $t = T$, we obtain

$$|p(t)|_{\mathcal{B}} \le \max\left\{\beta(|p(T)|_{\mathcal{B}}, t-T)\,,\, \gamma\left(\max_{\tau\in[T,t]} |y(\tau)|\right)\right\} \qquad \forall\, t \ge T. \tag{40}$$

With the properties of the functions H and K, there exist class-$\mathcal{K}$ functions $\varrho_h(\cdot)$ and $\varrho_q(\cdot)$ such that

$$|H(p)| \le \varrho_h(|p|_{\mathcal{B}}), \qquad |K(p,y)| \le \varrho_k(|y|).$$

Clearly $\varrho_h(\cdot)$ and $\varrho_k(\cdot)$ can be taken linearly bounded at the origin if $H(\cdot)$ and $K(\cdot)$ are locally Lipschitz. We obtain, for all (p, y),

$$D^+|y| \;\le\; \varrho_h(|p|_{\mathcal{B}}) + \varrho_k(|y|) - |\alpha(y)|.$$

So let us choose $\alpha(\cdot)$ so that

$$|\alpha(y)| \ge 3\max\{\varrho_h(\bar{\gamma}^{-1}(|y|)),\, \varrho_k(|y|),\, \kappa^\star|y|\} + |y|, \tag{41}$$

where $\bar{\gamma}(\cdot)$ is a class-$\mathcal{K}$ function such that $\bar{\gamma}\circ\gamma(s) < s$ for all $s \in \mathbb{R}^+$ with γ given by Lemma 1 (see (32)). This gives

$$D^+|y| \;\le\; -|y| + \left[\varrho_h(|p|_{\mathcal{B}}) - \varrho_h(\bar{\gamma}^{-1}(|y|))\right].$$

So for the closed-loop solution, we get

$$|y(t)| \;\le\; \max\left\{\exp(-(t-T))|y(T)|\,,\, \sup_{s\in[T,t)} \bar{\gamma}(|p(s)|_{\mathcal{B}})\right\}$$

for all $t \ge T$. From this and (40) the first claim of the theorem follows by small-gain arguments. To prove the second claim of the theorem note that, if $H(\cdot)$ and $K(\cdot)$ are locally Lipschitz, the functions $\varrho_h(\cdot)$ and $\varrho_k(\cdot)$ in (41) can be taken as linear. Furthermore, by (33) in Lemma 1, the function $\bar{\gamma}(\cdot)$ also can be taken as linear. From this the claim directly follows by the previous arguments. $\square$

B.3. Proof of Proposition 1. Let $\mathcal{O}(Z)$ denote the positive orbit of Z under the flow of

$$\dot{z} = f_0(z), \tag{42}$$

namely,

$$\mathcal{O}(Z) = \mathrm{cl}\left\{ \bigcup_{t\geq 0} z(t, z_0) \ : \ z_0 \in Z \right\}.$$

The set $\mathcal{O}(Z)$ is a bounded and forward invariant set for (42), such that $\mathcal{A} \subset \mathcal{O}(Z)$. Moreover let $\hat{\mathcal{O}}(Z)$ be a compact strict superset of $\mathcal{O}(Z)$ such that $\hat{\mathcal{O}}(Z) \subset \mathcal{D}$, and define the system

$$\dot{\hat{z}} = a_0(\hat{z}) f_0(\hat{z}) \tag{43}$$

in which $a(\hat{z}) : \mathbb{R}^n \to \mathbb{R}$ is any bounded smooth function such that

$$a_0(\hat{z}) = \begin{cases} 1, & \hat{z} \in \mathcal{O}(Z), \\ 0, & \hat{z} \in \mathbb{R}^n \setminus \hat{\mathcal{O}}(Z). \end{cases}$$

Let $\hat{z}(t, z_0)$ and $z(t, z_0)$ denote the flows of (43) and (42), respectively, and note that, as a consequence of the fact that $\mathcal{O}(Z)$ is forward invariant and that systems (42) and (43) agree on $\mathcal{O}(Z)$, it turns out that

$$\hat{z}(t, z_0) = z(t, z_0) \qquad \forall\ z_0 \in \mathcal{O}(Z) \text{ and } t \geq 0. \tag{44}$$

Moreover note that for any $\hat{z}_0 \in \mathbb{R}^n$, (43) has a unique solution $\hat{z}(t, \hat{z}_0)$ which is defined and bounded on $t \in (-\infty, \infty)$.

Define now

$$\begin{array}{rcccl} \tau & : & \mathbb{R}^n & \to & \mathbb{R}^m, \\ & & z & \mapsto & \displaystyle\int_{-\infty}^{0} e^{-Fs} G q_0(\hat{z}(s, z)) ds, \end{array} \tag{45}$$

which, as a consequence of the fact that F is Hurwitz and $q_0(\hat{z}(s, z))$ is bounded and continuous in z for any $s \in \mathbb{R}$, is a well-defined *continuous* map. We show now that $\mathrm{graph}(\tau|_{\mathcal{A}}) = \{(z, \xi) \in \mathcal{A} \times \mathbb{R}^m : x = \tau(z)\}$ is a forward invariant set for (23). Pick $z_0 \in \mathcal{A}$ and $x_0 \in \mathbb{R}^m$, let $(z(t, z_0), x(t, z_0, x_0))$ denote the value at time t of the solution of (23) passing through (z_0, x_0) at time $t = 0$, and note that for all $t \geq 0$ (using (44)),

$$\begin{array}{rcl} x(t, z_0, \tau(z_0)) & = & e^{Ft}\tau(z_0) + \displaystyle\int_0^t e^{F(t-s)} G q_0(z(s, z_0)) ds \\ & = & e^{Ft} \displaystyle\int_{-\infty}^{0} e^{-Fs} G q_0(\hat{z}(s, z_0)) ds + \int_0^t e^{F(t-s)} G q_0(z(s, z_0)) ds \\ & = & \displaystyle\int_{-\infty}^{t} e^{F(t-s)} G q_0(\hat{z}(s, z_0)) ds = \int_{-\infty}^{0} e^{-Fs} G q_0(\hat{z}(s+t, z_0)) ds \\ & = & \tau(\hat{z}(t, z_0)) = \tau(z(t, z_0)). \end{array} \tag{46}$$

This, along with the fact that $\mathcal{A}$ is forward invariant for (42) and is a subset of $\mathcal{O}(Z)$, proves that $\mathrm{graph}(\tau|_{\mathcal{A}})$ is *forward invariant* for (23).

We now prove item (ii) of the proposition. To this end note that, by (46), it follows that

$$L_{a_0 f_0}\tau(z) = F\tau(z) - Gq_0(z)$$

for all $z \in \mathcal{A}$. Defining $\tilde{x} := x - \tau(z)$, the previous relation yields that $\dot{\tilde{x}}(t) = F\tilde{x}(t)$ for all $t \geq 0$ and for all initial states $x_0 \in \mathbb{R}^m$ and $z_0 \in \mathcal{A}$. This, the fact that F is Hurwitz, and that $\mathcal{A}$ is locally asymptotically (exponentially) stable for (42) immediately yield the desired result. □

B.4. Proof of Proposition 2. The result will be proved by taking the "complex" pair

$$F = \mathrm{diag}(\lambda_1, \ldots, \lambda_{n+1}), \qquad G = (g, \ldots, g)^{\mathrm{T}} \tag{47}$$

in which $\lambda_i \in \mathbb{C}_\ell = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < -\ell\}$, $i = 1, \ldots, n+1$, $\ell > 0$, and $g \neq 0$. Once we prove the result for the $(n+1)$-dimensional pair in (47), the claim of the proposition follows by taking any $(2n+2)$-dimensional "real" representation of (47).

By bearing in mind the definition of the map τ in (45) note that, as a consequence of the choice of F and G in (47), it turns out that

$$\tau(z) = \begin{pmatrix} \tau_{\lambda_1}(z) & \tau_{\lambda_2}(z) & \cdots & \tau_{\lambda_{n+1}}(z) \end{pmatrix}^{\mathrm{T}}, \qquad \tau_{\lambda_i}(z) = \int_{-\infty}^{0} e^{-\lambda_i s}\, g\, q_0(\hat{z}(s,z))ds. \tag{48}$$

We will prove next that there exists an $\ell > 0$ such that by arbitrarily choosing λ_i, $i = 1, \ldots, n+1$, in $\mathbb{C}_\ell \setminus S$, where S is a set of zero Lebesgue measure, the map τ is such that

$$\tau(z_1) = \tau(z_2) \qquad \Rightarrow \qquad q_0(z_1) = q_0(z_2) \qquad \forall\, z_1, z_2 \in \mathbb{R}^n. \tag{49}$$

More precisely we will prove that, having defined

$$\Upsilon = \{(z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ q_0(z_1) \neq q_0(z_2)\},$$

the set

$$S = \{(\lambda_1, \ldots, \lambda_{n+1}) \in \mathbb{C}_\ell^{n+1} \ : \ \exists\, (z_1, z_2) \in \Upsilon \ : \ \tau_{\lambda_i}(z_1) = \tau_{\lambda_i}(z_2) \quad \forall\, i = 1, \ldots, n+1\} \tag{50}$$

has a zero Lebesgue measure in $\mathbb{C}^{n+1}$ for a proper choice of ℓ. To this end the following theorem, proved in a more general setting in [1] (see also [9]), plays a crucial role.

THEOREM 5. *Let Ω and Υ be open subsets of $\mathbb{C}$ and $\mathbb{R}^{2n}$, respectively. Let $(\varpi, \lambda) \in \Upsilon \times \Omega \mapsto \delta_\tau(\varpi, \lambda) \in \mathbb{C}$ be a function which is holomorphic in λ for each $\varpi \in \Upsilon$ and C^1 for each $\lambda \in \Omega$. If, for each pair $\varpi \in \Upsilon$, the function $\lambda \mapsto \delta_\tau(\varpi, \lambda)$ is not identically zero on Ω, then the set*

$$S = \bigcup_{\varpi \in \Upsilon} \{(\lambda_1, \ldots, \lambda_{n+1}) \in \Omega^{n+1} \quad : \quad \delta_\tau(\varpi, \lambda_1) = \cdots = \delta_\tau(\varpi, \lambda_{n+1}) = 0\}$$

has a zero Lebesgue measure in $\mathbb{C}^{n+1}$.

To apply this theorem to our context we first observe the following.

PROPOSITION 4. *There exists an $\ell > 0$ such that for all $\lambda_i \in \mathbb{C}_\ell$, $i = 1, \ldots, n+1$, the map $\tau(\cdot)$ in (48) is C^1.*

Proof of Proposition 4. The map $\tau(\cdot)$ in (48) is C^1 if functions $e^{-\lambda_i s}\, g\, \partial q_0(\hat{z}(s,z))/\partial z$, $i = 1, \ldots, n+1$, are integrable on $s \in (-\infty, 0]$ for all $z \in \mathbb{R}^n$ (see [16]). Consider the expansion

$$\frac{\partial q_0(\hat{z}(s,z))}{\partial z} = \left[\frac{\partial q_0}{\partial z}\right]_{[z=\hat{z}(s,z)]} \frac{\partial \hat{z}(s,z)}{\partial z}.$$

By definition, there is a number M such that $|\hat{z}(s,z)| \leq M$ for all $s \leq 0$ and all $z \in \mathbb{R}^n$. This, along with the fact that $q_0(z)$ is C^1, shows that the first factor is bounded on $(-\infty, 0] \times \mathbb{R}^n$. As for the second factor, bearing in mind the notation introduced in section B.3, observe that

$$\frac{d}{ds}\frac{\partial \hat{z}(s,z)}{\partial z} = \left[\frac{\partial a_0(z) f_0(z)}{\partial z}\right]_{[z=\hat{z}(s,z)]} \frac{\partial \hat{z}(s,z)}{\partial z}.$$

Letting

$$\bar{f} = \max_{z \in \mathbb{R}^n} \frac{\partial a_0(z) f_0(z)}{\partial z},$$

we obtain

$$\left|\frac{\partial \hat{z}(s,z)}{\partial z}\right| \leq e^{\bar{f}|s|}$$

for all s and for all $z \in \mathbb{R}^n$. From this, the result immediately follows with $\ell = \bar{f}$. ∎

Now set $\varpi := (z_1, z_2)$ and

$$\delta_\tau(\varpi, \lambda) = \int_{-\infty}^{0} e^{-\lambda s}\, g\, [q_0(\hat{z}(s,z_1)) - q_0(\hat{z}(s,z_2))]\, ds = \tau_\lambda(z_1) - \tau_\lambda(z_2).$$

This function is C^1 in $\varpi \in \mathbb{R}^n \times \mathbb{R}^n$ and is holomorphic in $\lambda \in \mathbb{C}_\ell$ for every $\varpi \in \mathbb{R}^n \times \mathbb{R}^n$ (see [29, Chap. 19, p. 367]). Moreover as

$$\int_{-\infty}^{0} e^{-as}\, |g\, q_0(\hat{z}(s,z_1)) - g\, q_0(\hat{z}(s,z_2))\,|^2\, ds < +\infty$$

for all $\varpi \in \mathbb{R}^n \times \mathbb{R}^n$ and for all $a < 0$, the Plancherel theorem can be invoked to obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\delta_\tau(\varpi, a+is)|^2 ds = \int_{-\infty}^{0} e^{-2as}\, |g\, q_0(\hat{z}(s,z_1)) - g\, q_0(\hat{z}(s,z_2))\,|^2\, ds \tag{51}$$

for all $a < 0$ and for all $\varpi \in \mathbb{R}^n \times \mathbb{R}^n$.

Now note that, for $\varpi = (z_1, z_2) \in \Upsilon$, we have $q_0(z_1) \neq q_0(z_2)$, and by continuity of flow with respect to time, there exists a time $t_1 < 0$ such that

$$|g\, q_0(\hat{z}(s,z_1)) - g\, q_0(\hat{z}(s,z_2))| > 0 \qquad \forall\ s \in (t_1, 0]$$

which, combined with (51), yields

$$\int_{-\infty}^{\infty} |\delta_\tau(\varpi, a+is)|^2 ds > 0.$$

This implies that, for each $\varpi \in \Upsilon$, the function $\lambda \mapsto \delta_\tau(\varpi, \lambda)$ is not identically zero on $\mathbb{C}_\ell$. Hence Theorem 5 can be applied to obtain the desired result, namely, that the set (50) has a zero Lebesgue measure.

By this result we are guaranteed that by arbitrarily picking $n+1$ complex eigenvalues in $\mathbb{C}_\ell \setminus S$ (with ℓ dictated by Proposition 4) of F defined in (47), condition (49) is satisfied. From this it is easy to show that there exists a class-$\mathcal{K}$ function satisfying (27). Define

$$\varphi(s) = \sup_{\substack{|\tau(z_1)-\tau(z_2)|\le s \\ z_1,z_2\in\mathcal{A}}} |q_0(z_1) - q_0(z_2)|.$$

This function is increasing and, as a consequence of (49), $\varphi(0) = 0$. Moreover it is possible to prove that $\varphi(s)$ is continuous at $s = 0$. Suppose that it is not; namely, as $\varphi(\cdot)$ is increasing and $\varphi(0) = 0$, suppose that there exists a $\varphi^\star > 0$ such that $\lim_{s\to 0^+} \varphi(s) = \varphi^\star$. This implies that there exist sequences $\{z_{1n}\}, \{z_{2n}\}$ in $\mathcal{A}$, such that $|q_0(z_{1n}) - q_0(z_{2n})| \ge \varphi^\star/2$ and $|\tau(z_{1n}) - \tau(z_{2n})| < 1/n$ for any $n \in \mathbb{N}$. But, as $\mathcal{A}$ is bounded, there are subsequences of $\{z_{1n}\}$, $\{z_{2n}\}$ which, for $n \to \infty$, converge to $z_1^\star$, $z_2^\star$, respectively. As $\tau(\cdot)$ and $q_0(\cdot)$ are continuous, $\tau(z_1^\star) - \tau(z_2^\star) = 0$ and $|q_0(z_1^\star) - q_0(z_2^\star)| \ge \varphi^\star/2$, which contradict (49). Hence, $\varphi(s)$ is continuous at $s = 0$. With this result at hand, define the candidate class-$\mathcal{K}$ function

$$\varrho(s) = \frac{1}{s}\int_s^{2s} \varphi(\sigma)d\sigma \; + \; s$$

which satisfies

$$\varphi(s) \le \varrho(s). \tag{52}$$

By construction this function is continuous for all $s > 0$ and, as $\varphi(s)$ is continuous at $s = 0$ and by (52), it is also continuous at $s = 0$. Moreover since, by definition of $\varphi(\cdot)$,

$$\varphi(|\tau(z_1) - \tau(z_2)|) \ge |q_0(z_1) - q_0(z_2)| \qquad \forall\, z_1, z_2 \in \mathcal{A},$$

it turns out that (52) yields

$$|q_0(z_1) - q_0(z_2)| \le \varphi(|\tau(z_1) - \tau(z_2)|) \le \varrho(|\tau(z_1) - \tau(z_2)|) \qquad \forall\, z_1, z_2 \in \mathcal{A},$$

namely, (27) is satisfied. This concludes the proof of Proposition 2. □

B.5. Proof of Proposition 3. By the result of the previous proposition we know that

$$\tau(z_1) = \tau(z_2) \qquad \Rightarrow \qquad q_0(z_1) = q_0(z_2) \qquad \forall\, z_1, z_2 \in \mathcal{A}.$$

For any $x \in \tau(\mathcal{A})$, let $[x] = \{z \in \mathcal{A} : \tau(z) = x\}$. The previous property shows that the map $q_0(\cdot)$ is constant on $[x]$. As a consequence, there is a well-defined function $\gamma_0 : \tau(\mathcal{A}) \to \mathbb{R}$ such that

$$\gamma_0(\tau(z)) = -q_0(z) \qquad \forall z \in \mathcal{A}.$$

In fact, the value $\gamma_0(x)$ at any $x \in \tau(\mathcal{A})$ is simply defined by taking any $z \in [x]$ and setting $\gamma_0(x) := -q_0(z)$. Moreover by (27), the map in question is also continuous.

Now note that, as $\mathcal{A}$ is compact and $\tau(\cdot)$ and $q_0(\cdot)$ are continuous maps, $\tau(\mathcal{A}) \subset \mathbb{R}^m$ and $q_0(\mathcal{A}) \subset \mathbb{R}$ are compact sets. From this, Tietze's extension theorem (see, for instance, Theorem VII.5.1 in [13]) can be invoked to claim the existence of a continuous map $\gamma : \mathbb{R}^m \to \mathbb{R}$ which agrees with γ_0 on $\tau(\mathcal{A})$. This implies that $q_0(z) + \gamma \circ \tau(z) = 0$ for all $z \in \mathcal{A}$ and proves the first claim of the proposition.

Furthermore, if $\varrho(\cdot)$ is linearly bounded at the origin, by compactness arguments it is possible to claim the existence of a positive $\bar{\varrho}$ such that $|q_0(z_1) - q_0(z_2)| \leq \bar{\varrho}|\tau(z_1) - \tau(z_2)|$. It follows that γ_0 is a Lipschitz function on $\tau(\mathcal{A})$. From this, the Kirszbraun theorem (see, for instance, Theorem 2.10.43 in [15]) yields the existence of a Lipschitz map $\gamma : \mathbb{R}^m \to \mathbb{R}$, with Lipschitz constant ϱ, which agrees with γ_0 on $\tau(\mathcal{A})$. This completes the proof of the proposition. □

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