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Duro Basic, François Malrait, Pierre Rouchon

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Euler-Lagrange models with complex currents of three-phase electrical machines and observability issues

D. Basic F. Malrait P. Rouchon [*][†]

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Abstract

A new Lagrangian formulation with complex currents is developed and yields a direct and simple method for modeling three-phase permanent-magnet and induction machines. The Lagrangian is the sum a mechanical one and of a magnetic one. This magnetic Lagrangian is expressed in terms of rotor angle, complex stator and rotor currents. A complexification procedure widely used in quantum electrodynamic is applied here in order to derive the Euler-Lagrange equations with complex stator and rotor currents. Such complexification process avoids the usual separation into real and imaginary parts and simplifies notably the calculations. Via simple modifications of such magnetic Lagrangians we derive new dynamical models describing permanent-magnet machines with both saturation and saliency, and induction machines with both magnetic saturation and space harmonics. For each model we also provide its Hamiltonian thus its magnetic energy. This energy is also expressed with complex currents and can be directly used in Lyapunov and/or passivity based control. Further, we briefly investigate the observability of this class of Euler-Lagrange models, in the so-called sensorless case when the measured output is the stator current and the load torque is constant but unknown. For all the dynamical models obtained via such variational principles, we prove that their linear tangent systems are unobservable around a one-dimensional family of steady-states attached to the same constant stator voltage and current. This negative result explains why sensorless control of three-phase electrical machines around zero stator frequency remains yet a difficult control problem.

Keywords Lagrangian with complex coordinates, space-harmonics, magnetic saturation, induction machine, permanent-magnet machine, sensorless control.

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[*] Duro Basic and François Malrait are with Schneider Electric, STIE, 33, rue André Blanchet, 27120 Pacy-sur Eure. E-mail: duro.basic@schneider-electric.com, francois.malrait@schneider-electric.com

[†] Pierre Rouchon is with Mines ParisTech, Centre Automatique et Systèmes, Mathématiques et Systèmes, 60 Bd Saint-Michel, 75272 Paris cedex 06, France. E-mail: pierre.rouchon@mines-paristech.fr

1 Introduction

Modeling electrical machines with magnetic-saturation and space-harmonics effects is not a straightforward task and could lead to complicated developments when a detailed physical description is included (see, e.g.,[3, 1]). Even if such effects are not dominant they could play an important role for sensorless control (no rotor position or velocity sensor). For a permanent-magnet machine, the rotor position will be unobservable without saliency. For standard models of induction machines (no magnetic saturation, no space-harmonics) the rotor velocity is always unobservable at zero stator frequency [2, 6]. A global observability analysis based on such standard models is given in [7]. In this note we develop a systematic method to include into such standard control models magnetic-saturation, saliency and/or space-harmonics effects. Our initial motivation was to see whether such non-observability is destroyed by such modelling changes or not. It appears that any physical-consistent model admits the same kind of observability deficiency at zero stator frequency.

By physically consistent models we mean Lagrangian-based models. We propose here an extension of Lagrangian modeling of three-phase machines with real variables (see, e.g.[9]) to complex electrical variables. It is directly inspired from quantum electro-dynamics where Lagrangian with complex generalized positions and velocities are widely used (see, e.g., [4], page 87). We obtain, from such Lagrangian functions, physically consistent and synthetic Euler-Lagrange models directly expressed with complex stator and rotor currents. Such modeling method by-passes the usual detailed physical descriptions that are not easily accessible to the control community. Here we propose a much more direct way: it just consists in modifying the magnetic part of the Lagrangian directly expressed with complex currents and then in deriving the dynamic equations from the Euler-Lagrange with complex variables. We obtain automatically the dynamics of the electrical part as a set of complex differential equations. We suggest here simple Lagrangians modeling simultaneously magnetic-saturation, saliency and space-harmonic effects. The obtained dynamics extend directly the ones used in almost all control-theoretic papers and include also more elaborate ones that can be found in specialized books such as [1].

For permanent-magnet three-phase machines, the general structure of any physically consistent model including magnetic-saturation, saliency and other conservative effects is given by equations (4) with magnetic Lagrangian $\mathcal{L}_m$ and Hamiltonian $\mathcal{H}_m$ related by (5). For induction three-phase machines the physically consistent models are given by (15) where the magnetic Lagrangian $\mathcal{L}_m$ is related to the magnetic energy $\mathcal{H}_m$ by (16). Such synthetic formulation of the dynamical equations is new and constitutes the first contribution of this note. We propose here natural modifications of the standard Lagrangian to include magnetic-saturation, saliency and space-harmonics, derive the corresponding dynamical equations and magnetic energies that could be used in the future to construct controlled Lyapunov functions and/or storage functions for passivity-based feedback laws.

From a control theoretic point of view we just prove here that the severe observability difficulties encountered in sensorless control and well explained in [7] for the standard model resulting from the quadratic Lagrangian (13), remain present for models (15) where the magnetic Lagrangian is any function of the rotor angle, stator and rotor currents. Consequently, addition of magnetic saturations, saliency and harmonics effects, do not remove observability issues at zero stator frequency in the sensorless case (see proposition 1). This observability obstruction has nether been stated for models with magnetic-saturation and space-harmonics of three-phase machines and constitutes the second contribution of this notes. Contrarily to observability, non-observability is not a generic property and could be destroyed by generic and small changes in the equations. Since proposition 1 is based on the class of models derived from (1) or (15) with arbitrary magnetic Lagrangian $\mathcal{L}_m$, we prove here that any physically consistent model of three-phase machines where the non-conservative effects result only from voltage supply and Ohmic losses, such non-observability holds true around zero stator frequency. This means that non-observability around zero stator frequency is robust to generic and physically consistent modifications of the equations. As far as we know this negative and physically robust result is new. It indicates that sensorless control of three-phase electrical machines around zero stator frequency cannot be just addressed via refined physical models but also requires advanced and nonlinear control techniques.

In section 2 we recall the simplest model of a permanent-magnet machine and its Euler-Lagrange formulation based on the two scalar components of the complex stator current. Using the complexification procedure detailed in appendix, we show how to use complex representation of stator-current in Lagrangian formulation of the dynamics. This leads us to the general form of physically consistent models (equation (4)). Finally we obtain, just by simple modifications of the magnetic Lagrangian, physically consistent models with magnetic saturation and saliency effects (equation (10)). Section 3 deals with induction machines and admits the same progression as the previous one: we start with the usual (α, β) model, describe its complex Lagrangian formulation, derive physically consistent models (equations (15)) and specialize them to saturation and space-harmonics effects (equation (21)). In section 4, we prove proposition 1 that states the main observability issues of these Euler-Lagrangian models at zero stator frequency. In conclusion we show how to transpose this modelling based on complex currents associated to a Lagrangian formulation to complex fluxes associated to a Hamiltionian formulation with complex generalized positions and momentums. The appendix details the complexification procedure. It explains how to derive the Euler-Lagrange equations when some generalized positions and velocities are treated as complex quantities. Throughout the paper, we define models in (α, β) frame, using the standard transformation from three phases frame (see, e.g., [8]).

2 Permanent-magnet three-phase machines

2.1 The usual model and its magnetic energy

In the (α, β) frame (total power invariant transformation), the dynamic equations read (see, e.g., [8]):

$$\left\{\begin{array}{l}\frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left(\left(\bar{\phi}e^{\jmath n_p \theta}\right)^* \imath_s\right) - \tau_L \\ \frac{d}{dt}\left(\lambda \imath_s + \bar{\phi}e^{\jmath n_p \theta}\right) = u_s - R_s \imath_s\end{array}\right. \tag{1}$$

where

- * stands for complex-conjugation, $\Im$ means imaginary part, $\jmath = \sqrt{-1}$ and n_p is the number of pairs of poles.
- θ is the rotor mechanical angle, J and τ_L are the inertia and load torque, respectively.
- $\imath_s \in \mathbb{C}$ is the stator current, $u_s \in \mathbb{C}$ the stator voltage.
- $\lambda = (L_d + L_q)/2$ with inductances $L_d = L_q > 0$ (no saliency here).
- The stator flux is $\phi_s = \lambda \imath_s + \bar{\phi}e^{\jmath n_p \theta}$ with the constant $\bar{\phi} > 0$ representing to the rotor flux due to permanent magnets.

The Lagrangian associated to this system is the sum of the mechanical one $\mathcal{L}_c$ and magnetic one $\mathcal{L}_m$ defined as follows:

$$\mathcal{L}_c = \frac{J}{2}\dot{\theta}^2, \quad \mathcal{L}_m = \frac{\lambda}{2}\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|^2 \tag{2}$$

where $\bar{\imath} = \bar{\phi}/\lambda > 0$ is the permanent magnetizing current.

It is well known that (1) derives from a variational principle (see, e.g.,[9]) and thus can be written as Euler-Lagrange equations with source terms corresponding to energy exchange with the environment. Consider the additional complex variable $q_s \in \mathbb{C}$ defined by $\frac{d}{dt}q_s = \imath_s$. Take the Lagrangian $\mathcal{L} = \mathcal{L}_c + \mathcal{L}_m$ as a real function of the generalized coordinates $q = (\theta, q_{s\alpha}, q_{s\beta})$ and generalized velocities $\dot{q} = (\dot{\theta}, \imath_{s\alpha}, \imath_{s\beta})$:

$$\mathcal{L}(q, \dot{q}) = \frac{J}{2}\dot{\theta}^2 + \frac{\lambda}{2}\left((\dot{q}_{s\alpha} + \bar{\imath}\cos n_p\theta)^2 + (\dot{q}_{s\beta} + \bar{\imath}\sin n_p\theta)^2\right) \tag{3}$$

Then the mechanical equation in (1) reads

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = -\tau_L$$

where $-\tau_L$ corresponds to the energy exchange through the mechanical load torque. Similarly, the real part of complex and electrical equation in (1) reads

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{s\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial q_{s\alpha}} = u_{s\alpha} - R_s \imath_{s\alpha}$$

and its imaginary part

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_{s\beta}}\right) - \frac{\partial \mathcal{L}}{\partial q_{s\beta}} = u_{s\beta} - R_s \imath_{s\beta}$$

since $\frac{\partial \mathcal{L}}{\partial q_{s\alpha}} = \frac{\partial \mathcal{L}}{\partial q_{s\beta}} = 0$ and $\dot{q}_s = \imath_s$. The energy exchanges here are due to the power supply through the voltage u_s and also to dissipation and irreversible phenomena due to stator resistance represented by the Ohm law $-R_s\imath_s$.

2.2 Euler-Lagrange equation with complex current

The drawback of such Lagrangian formulation is that we have to split into real and imaginary parts the generalized complex coordinates with $q_s = q_{s\alpha} + \jmath q_{s\beta}$, ($q_{s\alpha}$ and $q_{s\beta}$ real) and velocities $\dot{q}_s = \imath_s = \dot{q}_{s\alpha} + \jmath \dot{q}_{s\beta}$, ($\dot{q}_{s\alpha}$ and $\dot{q}_{s\beta}$ real). We do not preserve the elegant formulation of the electrical part through complex variables and equations.

Let us apply the complexification procedure detailed in appendix to the Lagrangian $\mathcal{L}(\theta, q_{s\alpha}, q_{s\beta}, \dot{\theta}, \dot{q}_{s\alpha}, \dot{q}_{s\beta})$ defined in (3). The complexification process only focuses on q_s and $\dot{q}_s = \imath_s$ by considering $\mathcal{L}$ as a function of $(\theta, q_s, q_s^*, \dot{\theta}, \imath_s, \imath_s^*)$:

$$\mathcal{L}(\theta, \dot{\theta}, \imath_s, \imath_s^*) \qquad = \qquad \frac{J}{2}\dot{\theta}^2 \quad + \quad \frac{\lambda}{2}\left(\imath_s + \bar{\imath} e^{\jmath n_p \theta}\right)\left(\imath_s^* + \bar{\imath} e^{-\jmath n_p \theta}\right).$$

Following the notations in appendix, $n^c = 1$ with $q^c = q_s$, $n^r = 1$ with $q^r = \theta$, $S^r = -\tau_L$ and $S^c = u_s - R_s \imath_s$. Then according to (25) the usual equations (1) read

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} - \tau_L, \quad 2\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \imath_s^*}\right) = u_s - R_s \imath_s$$

since $\frac{\partial \mathcal{L}}{\partial q_s^*} = 0$ and $\frac{\partial \mathcal{L}}{\partial \dot{q}_s^*} = \frac{\partial \mathcal{L}}{\partial \imath_s^*}$.

More generally, for any magnetic Lagrangian $\mathcal{L}_m$ that is a real value function of θ, $\imath_s$ and $\imath_s^*$ and that is $\frac{2\pi}{n_p}$ periodic versus θ, we get the general model (with saliency, saturation, space-harmonics, ...) of three-phase permanent-magnet machine:

$$\frac{d}{dt}\left(J\dot{\theta}\right) = \frac{\partial \mathcal{L}_m}{\partial \theta} - \tau_L, \quad 2\frac{d}{dt}\left(\frac{\partial \mathcal{L}_m}{\partial \imath_s^*}\right) = u_s - R_s \imath_s \tag{4}$$

We recover the usual equation with $\phi_s = 2\frac{\partial \mathcal{L}_m}{\partial \imath_s^*}$ corresponding to the stator flux and to the conjugate momentum p^c of q^c as shown in appendix. According to (28), the Hamiltonian $\mathcal{H}$ is the sum of two energy: $\mathcal{H} = \mathcal{H}_c + \mathcal{H}_m$. The mechanical kinetic energy $\mathcal{H}_c = \frac{J}{2}\dot{\theta}^2$ and the magnetic energy

$$\mathcal{H}_m\left(\theta, \imath_s, \imath_s^*\right) = \frac{\partial \mathcal{L}_m}{\partial \imath_s}\imath_s + \frac{\partial \mathcal{L}_m}{\partial \imath_s^*}\imath_s^* - \mathcal{L}_m. \tag{5}$$

The standard model (1) derives from a magnetic Lagrangian of the form $\mathcal{L}_m = \frac{\lambda}{2}\left|\imath_s + \bar{\imath} e^{\jmath n_p \theta}\right|^2$ with λ and $\bar{\imath}$ are two positive parameters. Its corresponding magnetic energy reads $\mathcal{H}_m = \frac{\lambda}{2}\left(|\imath_s|^2 - \bar{\imath}^2\right)$. We recover the usual magnetic energy $\frac{\lambda}{2}|\imath_s|^2$ up to the constant magnetizing energy $\frac{\lambda}{2}\bar{\imath}^2$.

In (4), many other formulations of $\mathcal{L}_m$ are possible and depend on particular modeling issues. Usually, the dominant part of $\mathcal{L}_m$ will be of the form $\frac{\bar{\lambda}}{2}\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|^2$ ($\bar{\lambda}$, $\bar{\imath}$ positive constants) to which correction terms that are "small" scalar functions of $(\theta, \imath_s, \imath_s^*)$ are added.

2.3 Saliency models

Adding to $\mathcal{L}_m$ the correction $-\frac{\mu}{2}\Re\left(\imath_s^2 e^{-2\jmath n_p \theta}\right)$ with $|\mu| < \lambda$ ($\Re$ means real part) provides a simple way to represent saliency phenomena while the dominant part of the magnetic Lagrangian (and thus of the dynamics) remains attached to $\frac{\lambda}{2}\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|^2$. With a magnetic Lagrangian of the form

$$\mathcal{L}_m = \frac{\lambda}{2}\left(\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right)\left(\imath_s^* + \bar{\imath}e^{-\jmath n_p \theta}\right) - \frac{\mu}{4}\left(\left(\imath_s^* e^{\jmath n_p \theta}\right)^2 + \left(\imath_s e^{-\jmath n_p \theta}\right)^2\right) \quad (6)$$

where $\lambda = (L_d + L_q)/2$ and $\mu = (L_q - L_d)/2$ (inductances $L_d > 0$ and $L_q > 0$), equations (4) become ($\lambda\bar{\imath} = \bar{\phi}$)

$$\begin{cases} \dfrac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left((\lambda\imath_s^* + \bar{\phi}e^{-\jmath n_p \theta} - \mu\imath_s e^{-2\jmath n_p \theta})\imath_s\right) - \tau_L \\ \dfrac{d}{dt}\left(\lambda\imath_s + \bar{\phi}e^{\jmath n_p \theta} - \mu\imath_s^* e^{2\jmath n_p \theta}\right) = u_s - R_s\imath_s \end{cases} \quad (7)$$

and we recover the usual model with saliency effect. In this case the magnetic energy is given by:

$$\mathcal{H}_m = \frac{\lambda}{2}\left(|\imath_s|^2 - \bar{\imath}^2\right) - \frac{\mu}{4}\left(\left(\imath_s^* e^{\jmath n_p \theta}\right)^2 + \left(\imath_s e^{-\jmath n_p \theta}\right)^2\right). \quad (8)$$

2.4 Saturation and saliency models

We can also take into account magnetic saturation effects, i.e., the fact that inductances depend on the currents. Let us assume first that only the inductances λ and μ in (6) depend on the modulus $\rho = \left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|$. The magnetic Lagrangian now reads

$$\mathcal{L}_m = \frac{\lambda\left(\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|\right)}{2}\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|^2 - \frac{\mu\left(\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|\right)}{4}\left(\left(\imath_s^* e^{\jmath n_p \theta}\right)^2 + \left(\imath_s e^{-\jmath n_p \theta}\right)^2\right). \quad (9)$$

The dynamics is given by (4) with such $\mathcal{L}_m$. Denote $\lambda' = \frac{d\lambda}{d\rho}$. With $\frac{\partial\lambda}{\partial\theta} = n_p \frac{\Im\left(\bar{\imath}e^{-\jmath n_p \theta}\imath_s\right)}{\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|}\lambda'$ and $\frac{\partial\lambda}{\partial\imath_s^*} = \frac{\imath_s + \bar{\imath}e^{\jmath n_p \theta}}{2\left|\imath_s + \bar{\imath}e^{\jmath n_p \theta}\right|}\lambda'$, we get the following dynamical model with both saliency and magnetic-saturation effects:

$$\begin{aligned}
&\frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left(\left(\Lambda\left(\imath_s^* + \bar{\imath}e^{-\jmath n_p\theta}\right) - M\imath_s e^{-2\jmath n_p\theta}\right)\imath_s\right) - \tau_L \\
&\frac{d}{dt}\left(\Lambda\left(\imath_s + \bar{\imath}e^{\jmath n_p\theta}\right) - M\imath_s^* e^{2\jmath n_p\theta}\right) = u_s - R_s\imath_s
\end{aligned} \tag{10}$$

with $\Lambda = \lambda + \frac{\left|\imath_s + \bar{\imath}e^{\jmath n_p\theta}\right|}{2}\lambda'$ and $M = \mu + \frac{\left|\imath_s + \bar{\imath}e^{\jmath n_p\theta}\right|}{2}\mu'$. It is interesting to compute the magnetic energy $\mathcal{H}_m$ from general formula (5):

$$\begin{aligned}
\mathcal{H}_m = {} & \frac{\lambda + \left|\imath_s + \bar{\imath}e^{\jmath n_p\theta}\right|\lambda'}{2}\left|\imath_s\right|^2 - \frac{\lambda}{2}\bar{\imath}^2 \\
& + \frac{\left|\imath_s + \bar{\imath}e^{\jmath n_p\theta}\right|\lambda'}{4}\bar{\imath}\left(\imath_s e^{-\jmath n_p\theta} + \imath_s^* e^{\jmath n_p\theta}\right) \\
& - \frac{\mu + \mu'\frac{\imath_s^*\left(\imath_s + \bar{\imath}e^{\jmath n_p\theta}\right) + \imath_s\left(\imath_s^* + \bar{\imath}e^{-\jmath n_p\theta}\right)}{2\left|\imath_s + \bar{\imath}e^{\jmath n_p\theta}\right|}}{4}\left(\left(\imath_s^* e^{\jmath n_p\theta}\right)^2 + \left(\imath_s e^{-\jmath n_p\theta}\right)^2\right).
\end{aligned} \tag{11}$$

Such magnetic energy formulae are not straightforward. They are not obtained by replacing λ and μ in the standard magnetic energy (8) by Λ and M respectively.

3 Induction three-phase machines

We will now proceed as for permanent-magnet machines. Let us recall first the usual dynamical equations of an induction machine with complex stator and rotor currents. They admit the following form

$$\begin{cases}
\frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left(L_m \imath_r^* e^{-\jmath n_p\theta}\imath_s\right) - \tau_L \\
\frac{d}{dt}\left(L_m\left(\imath_r + \imath_s e^{-\jmath n_p\theta}\right) + L_{fr}\imath_r\right) = -R_r\imath_r \\
\frac{d}{dt}\left(L_m\left(\imath_s + \imath_r e^{\jmath n_p\theta}\right) + L_{fs}\imath_s\right) = u_s - R_s\imath_s
\end{cases} \tag{12}$$

where

- n_p is the number of pairs of poles, θ is the rotor mechanical angle, J and τ_L are the inertia and load torque, respectively.
- $\imath_r \in \mathbb{C}$ is the rotor current (in the rotor frame, different from the (d,q) frame) , $\imath_s \in \mathbb{C}$ the stator current (in the stator frame, identical to the (α,β) frame) and $u_s \in \mathbb{C}$ the stator voltage (in the stator frame). The stator and rotor resistances are $R_s > 0$ and $R_r > 0$.
- The inductances L_m, L_{fr} and L_{fs} are positive parameters with $L_{fr}, L_{fs} \ll L_m$.
- The stator (resp. rotor) flux is $\phi_s = L_m\left(\imath_s + \imath_r e^{\jmath n_p\theta}\right) + L_{fs}\imath_s$ (resp. $\phi_r = L_m\left(\imath_r + \imath_s e^{-\jmath n_p\theta}\right) + L_{fr}\imath_r$).

3.1 Euler-Lagrange equation with complex current

With notations of appendix, $n^c = 2$ with $q^c = (\imath_r, \imath_s)$, $n^r = 1$ with $q^r = \theta$, $S^c = (-R_r \imath_r, u_s - R_s \imath_s)$ and $S^r = -\tau_L$. The Lagrangian associated to (12), expressed with complex currents $\imath_r$ and $\imath_s$, reads:

$$\mathcal{L}(\theta, \dot{\theta}, \imath_r, \imath_r^*, \imath_s, \imath_s^*) = \frac{J}{2}\dot{\theta}^2 + \frac{L_m}{2}\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|^2 + \frac{L_{fr}}{2}|\imath_r|^2 + \frac{L_{fs}}{2}|\imath_s|^2.$$

The first term $\frac{J}{2}\dot{\theta}^2$ represents the mechanical Lagrangian and the remaining sum the magnetic Lagrangian $\mathcal{L}_m$. The dynamics (12) read

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = -\tau_L, \qquad 2\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \imath_r^*}\right) = -R_r \imath_r, \quad 2\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \imath_s^*}\right) = u_s - R_s \imath_s,$$

The magnetic Lagrangian $\mathcal{L}_m$ has the following form that coincides here with the magnetic energy $\mathcal{H}_m$:

$$\mathcal{L}_m = \frac{L_m}{2}\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|^2 + \frac{L_{fr}}{2}|\imath_r|^2 + \frac{L_{fs}}{2}|\imath_s|^2. \tag{13}$$

More generally physically consistent model should be obtained with a Lagrangian of the form

$$\mathcal{L}_{\text{IM}} = \frac{J}{2}\dot{\theta}^2 + \mathcal{L}_m\left(\theta, \imath_r, \imath_r^*, \imath_s, \imath_s^*\right) \tag{14}$$

where $\mathcal{L}_m$ is the magnetic Lagrangian expressed with the rotor angle and currents. It is $\frac{2\pi}{n_p}$ periodic versus θ. Any physically admissible model of a three-phase induction machine reads

$$\frac{d}{dt}\left(J\dot{\theta}\right) = \frac{\partial \mathcal{L}_m}{\partial \theta} - \tau_L, \qquad 2\frac{d}{dt}\left(\frac{\partial \mathcal{L}_m}{\partial \imath_r^*}\right) = -R_r \imath_r, \quad 2\frac{d}{dt}\left(\frac{\partial \mathcal{L}_m}{\partial \imath_s^*}\right) = u_s - R_s \imath_s, \tag{15}$$

where the rotor and stator flux are given by

$$\phi_r = 2\frac{\partial \mathcal{L}_m}{\partial \imath_r^*}, \qquad \phi_s = 2\frac{\partial \mathcal{L}_m}{\partial \imath_s^*}.$$

In general the magnetic energy does not coincide with $\mathcal{L}_m$. It is given by (28) that yields:

$$\mathcal{H}_m\left(\theta, \imath_r, \imath_r^*, \imath_s, \imath_s^*\right) = \frac{\partial \mathcal{L}_m}{\partial \imath_r}\imath_r + \frac{\partial \mathcal{L}_m}{\partial \imath_r^*}\imath_r^* + \frac{\partial \mathcal{L}_m}{\partial \imath_s}\imath_s + \frac{\partial \mathcal{L}_m}{\partial \imath_s^*}\imath_s^* - \mathcal{L}_m. \tag{16}$$

3.2 Saturation models

A simple way to include saturation effects is to consider that the main inductances L_m appearing in (13) depends on the modulus $\rho = \left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|$. Thus we consider the following magnetic-saturation Lagrangian:

$$\mathcal{L}_m = \frac{L_m\left(\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|\right)}{2}\left(\imath_s + \imath_r e^{\jmath n_p \theta}\right)\left(\imath_s^* + \imath_r^* e^{-\jmath n_p \theta}\right) + \frac{L_{fr}}{2}\imath_r \imath_r^* + \frac{L_{fs}}{2}\imath_s \imath_s^*. \tag{17}$$

Since ($L'_m = \frac{dL_m}{d\rho}$) $\frac{\partial \mathcal{L}_m}{\partial \theta} = n_p \frac{\Im\left(\imath_r^* e^{-\jmath n_p \theta}\imath_s\right)}{\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|} L'_m$ and

$$\frac{\partial L_m}{\partial \imath_r^*} = \frac{\imath_s e^{-\jmath n_p \theta} + \imath_r}{2\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|} L'_m, \quad \frac{\partial L_m}{\partial \imath_s^*} = \frac{\imath_s + \imath_r e^{\jmath n_p \theta}}{2\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|} L'_m,$$

the saturation model (formula (15) with $\mathcal{L}_m$ given by (17)) reads

$$\left\{\begin{aligned} &\frac{d}{dt}\left(J\dot{\theta}\right) = n_p \Im\left(\Lambda_m \imath_r^* e^{-\jmath n_p \theta}\imath_s\right) - \tau_L \\ &\frac{d}{dt}\left(\Lambda_m\left(\imath_r + \imath_s e^{-\jmath n_p \theta}\right) + L_{fr}\imath_r\right) = -R_r \imath_r \\ &\frac{d}{dt}\left(\Lambda_m\left(\imath_s + \imath_r e^{\jmath n_p \theta}\right) + L_{fs}\imath_s\right) = u_s - R_s \imath_s \end{aligned}\right. \tag{18}$$

with $\Lambda_m = L_m + \frac{\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|}{2} L'_m$ function of $\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|$. We recover here usual saturation models (see, e.g., [1], page 428). Notice the similarity with permanent-magnet machines and (10). Following (16), the associated magnetic energy reads then

$$\mathcal{H}_m = \frac{L_m + \left|\imath_s + \imath_r e^{\jmath n_p \theta}\right| L'_m}{2}\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|^2 + \frac{L_{fr}}{2}|\imath_r|^2 + \frac{L_{fs}}{2}|\imath_s|^2.$$

Notice also that such similarity is not complete and could be misleading: contrarily to (18) that can be derived intuitively form (12) by replacing L_m by Λ_m, the above magnetic energy is not provided by (13) with L_m replaced by Λ_m.

3.3 Space-harmonics with saturation

We can take into account space-harmonic effects by adding their contribution to the magnetic Lagrangian (17). According to [1], page 298, the iron path in general gets shorter as the harmonic order gets higher. Thus saturation effect has relatively smaller influence on the spatial harmonics. Following [5], the Lagrangian $\mathcal{L}_\nu$ of harmonic ν is

$$\mathcal{L}_\nu = \frac{L_\nu}{2}\left(\imath_s \imath_r^* e^{-\jmath \sigma_\nu \nu n_p \theta} + \imath_s^* \imath_r e^{\jmath \sigma_\nu \nu n_p \theta}\right) \tag{19}$$

with L_ν a small parameter ($|L_\nu| \ll L_m$) and with $\sigma_\nu = \pm 1$ depending on arithmetic conditions on ν (see [5], equations (25) to (29)). The total magnetic Lagrangian now reads

$$\begin{aligned}\mathcal{L}_m = {} & \frac{L_m\left(\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|\right)}{2}\left(\imath_s + \imath_r e^{\jmath n_p \theta}\right)\left(\imath_s^* + \imath_r^* e^{-\jmath n_p \theta}\right) \\ & + \frac{L_{fr}}{2}\imath_r \imath_r^* + \frac{L_{fs}}{2}\imath_s \imath_s^* + \frac{L_\nu}{2}\left(\imath_s \imath_r^* e^{-\jmath \sigma_\nu \nu n_p \theta} + \imath_s^* \imath_r e^{\jmath \sigma_\nu \nu n_p \theta}\right).\end{aligned} \tag{20}$$

Now the saturation model (18) is changed as follows:

$$\begin{aligned}\frac{d}{dt}\left(J\dot\theta\right) &= n_p \Im\left(\left(\Lambda_m e^{-\jmath n_p \theta} + L_\nu \sigma_\nu \nu e^{-\jmath \sigma_\nu \nu n_p \theta}\right)\imath_r^* \imath_s\right) - \tau_L \\ \frac{d}{dt}\left(\Lambda_m\left(\imath_r + \imath_s e^{-\jmath n_p \theta}\right) + L_{fr}\imath_r + L_\nu \imath_s e^{-\jmath \sigma_\nu \nu n_p \theta}\right) &= -R_r \imath_r \\ \frac{d}{dt}\left(\Lambda_m\left(\imath_s + \imath_r e^{\jmath n_p \theta}\right) + L_{fs}\imath_s + L_\nu \imath_r e^{\jmath \sigma_\nu \nu n_p \theta}\right) &= u_s - R_s \imath_s\end{aligned} \tag{21}$$

Following (16), the associated magnetic energy reads then

$$\begin{aligned}\mathcal{H}_m = {} & \frac{L_m + \left|\imath_s + \imath_r e^{\jmath n_p \theta}\right| L_m'}{2}\left|\imath_s + \imath_r e^{\jmath n_p \theta}\right|^2 + \frac{L_{fr}}{2}|\imath_r|^2 \\ & + \frac{L_{fs}}{2}|\imath_s|^2 + \frac{L_\nu}{2}\left(\imath_s \imath_r^* e^{-\jmath \sigma_\nu \nu n_p \theta} + \imath_s^* \imath_r e^{\jmath \sigma_\nu \nu n_p \theta}\right).\end{aligned}$$

Several space-harmonics can be included in a similar way. Moreover saturation of space-harmonics can be also tackled just by choosing L_ν as a function of $|\imath_s + \imath_r e^{-\jmath n_p \theta}|$. As far as we know such explicit models including magnetic saturation and space harmonics have never been given.

4 Observability issues at zero stator frequency

The sensorless control case is characterized by a load torque τ_L constant but unknown, control inputs u_s and measured outputs $\imath_s$. Models derived from (4) for permanent-magnet machines (resp. from (15) for inductions machines) can be always written in state-space form

$$\frac{d}{dt}X = f(X, U), \qquad Y = h(X) \tag{22}$$

where $X = (\tau_L, \theta, \dot\theta, \Re(\imath_s), \Im(\imath_s))$ (resp. $X = (\tau_L, \theta, \dot\theta, \Re(\imath_r), \Im(\imath_r), \Re(\imath_s), \Im(\imath_s))$) with $U = (\Re(u_s), \Im(u_s))$, $Y = (\Re(\imath_s), \Im(\imath_s))$ and $\frac{d}{dt}\tau_L = 0$. A stationary regime at zero stator frequency corresponds then to a steady state $(\bar X, \bar U, \bar Y)$ of (22) satisfying $f(\bar X, \bar U) = 0$ and $\bar Y = h(\bar X)$. The tangent linear system around this steady state is then

$$\frac{d}{dt}x = Ax + Bu, \qquad y = Cx \tag{23}$$

where $A = \frac{\partial f}{\partial X}(\bar{X}, \bar{U})$, $B = \frac{\partial f}{\partial U}(\bar{X}, \bar{U})$ and $C = \frac{\partial h}{\partial X}(\bar{X})$. If we assume that the linearized system (23) is observable, the Kalman criteria implies that the rank of the matrix $\begin{pmatrix} C \\ A \end{pmatrix}$ must be equal to $\dim(X)$. If it is the case, the mapping $X \mapsto (f(X, \bar{U}), h(X))$ is maximum rank around $\bar{X}$. This maximum rank condition just means that the set of algebraic equations characterizing the steady-state from the knowledge of $\bar{U}$ and $\bar{Y}$, $f(X, \bar{U}) = 0$ and $h(X) = \bar{Y}$ admits around $\bar{X}$ the maximum rank $\dim(X)$. Such rank is not changed by any invertible manipulations of this set of equations characterizing the steady-state from the knowledge of the input and output values, $\bar{U}$ and $\bar{Y}$. Putting the implicit Euler-Lagrange equations (4) and (15) into their explicit state-space forms (22) involves such invertible manipulations.

For permanent-magnet machines described by (4), this set of equations yields the following mapping

$$(\tau_L, \theta, \dot{\theta}, \imath_s) \mapsto (0, \dot{\theta}, \frac{\partial \mathcal{L}_m}{\partial \theta} - \tau_L, \bar{U} - R_s \imath_s, \imath_s)$$

where $\mathcal{L}_m$ depends on θ and $\imath_s$. Its rank should be maximum, i.e., equal to 5. This is not the case since its rank is obviously equal to 4. For induction machines described by (15), the mapping is

$$(\tau_L, \theta, \dot{\theta}, \imath_r, \imath_s) \mapsto (0, \dot{\theta}, \frac{\partial \mathcal{L}_m}{\partial \theta} - \tau_L, -R_r \imath_r, \bar{U} - R_s \imath_s, \imath_s)$$

where $\mathcal{L}_m$ depends on θ, $\imath_s$ and $\imath_r$. Its rank is equal to 6 whereas the maximum rank is 7. The above arguments yield following proposition:

Proposition 1. *Any dynamical model of permanent-magnet machines* (4) *(resp. induction machines* (15)*) is unobservable around zero stator frequency regime when the measured output is the stator current $\imath_s$ and the load torque is constant but unknown. By unobservable we mean that:*

- *to any constant input and output $\bar{u}_s$ and $\bar{\imath}_s$ satisfying $\bar{u}_s = R_s \bar{\imath}_s$ correspond a one dimensional family of steady states parameterized by the scalar variable ξ with*
  - $\tau_L = \frac{\partial \mathcal{L}_m}{\partial \theta}(\xi, \bar{\imath}_s, \bar{\imath}_s^*)$, $\theta = \xi$, $\quad \imath_s = \bar{\imath}_s$ *for the permanent-magnet machines,*
  - $\tau_L = \frac{\partial \mathcal{L}_m}{\partial \theta}(\xi, 0, 0, \bar{\imath}_s, \bar{\imath}_s^*)$, $\theta = \xi$, $\quad \imath_r = 0$, $\imath_s = \bar{\imath}_s$ *for induction machines.*
- *the linear tangent systems around such steady-states are not observable;*

5 Conclusion

The models proposed in this note, (10) for permanent-magnet machine and (21) for induction drives, are based on variational principles and Lagrangian formulation of the dynamics. Such formulations are particularly efficient to preserve the physical insight while maintaining a synthetic view without describing all the technological and material details (see [10] for an excellent and tutorial overview of variational principles in physics). Extensions to network of machines and generators connected via long lines can also be developed with similar variational principles and Euler-Lagrange equations with complex currents and voltages.

In this note we have put the emphasis on currents and thus Lagrangian modelling. Since flux variables are conjugated to current variables, Hamiltonian modelling is also possible when fluxes are used instead of currents. For permanent-magnet machines, the Hamiltonian counterpart of Lagrangian models (4) reads:

$$\frac{d}{dt}\left(J\dot{\theta}\right) = -\frac{\partial \mathcal{H}_m}{\partial \theta} - \tau_L, \quad \frac{d}{dt}\phi_s = u_s - 2R_s\frac{\partial \mathcal{H}_m}{\partial \phi_s^*}$$

where the magnetic energy $\mathcal{H}_m$ is considered as a function of the rotor angle θ, the stator flux ϕ_s and its complex conjugate ϕ_s^*. The stator current $\imath_s$ corresponds then to $2\frac{\partial \mathcal{H}_m}{\partial \phi_s^*}$. For induction machines, the Hamiltonian formulation associated to (15) becomes

$$\frac{d}{dt}\left(J\dot{\theta}\right) = -\frac{\partial \mathcal{H}_m}{\partial \theta} - \tau_L,$$
$$\frac{d}{dt}\phi_r = -2R_r\frac{\partial \mathcal{H}_m}{\partial \phi_r^*}, \quad \frac{d}{dt}\phi_s = u_s - 2R_s\frac{\partial \mathcal{H}_m}{\partial \phi_s^*}.$$

The magnetic energy $\mathcal{H}_m$ now depends on θ, the rotor flux ϕ_r and its complex conjugate ϕ_r^*, the stator flux ϕ_s and its complex conjugate ϕ_s^*. The rotor (resp. stator) current is then given by $2\frac{\partial \mathcal{H}_m}{\partial \phi_r^*}$ (resp. $2\frac{\partial \mathcal{H}_m}{\partial \phi_s^*}$). As for Lagrangian modelling, one can modify the magnetic energies of the standard models (1) and (12) to include, for example, magnetic-saturation or space-harmonics effects. This yields new formula expressing $\mathcal{H}_m$ as function of angle and fluxes. The corresponding flux-based models are then given by the above equations.

[Lagrangian and Hamiltonian with complex variables] It is explained in [4, page 87] how to use complex coordinates for Lagrangian and Hamiltonian systems. Here we propose a straightforward extension where the complexification procedure is only partial. Such extension cannot be found directly in text-book. In the context of electrical drives, such complexification applies only on electrical quantities whereas mechanical ones remain untouched.

Assume we have a Lagrangian system with generalized positions $q \in \mathbb{R}^n$, $n \geq 3$ and an analytic Lagrangian $\mathcal{L}(q, \dot{q})$. Let us decompose q into two set of components:

- the first set $q^c = (q_1, \ldots, q_{2n^c})$ with $0 < 2n^c \leq n$ will be identified with n^c complex numbers $q_k^c = q_{2k-1} + \jmath q_{2k}$, $k = 1, \ldots, n^c$;
- the second set $q^r = (q_{2n^c+1}, \ldots, q_n)$ gathers the $n^r = n - 2n^c$ components that will remain untouched and real.

Thus we can identified q with (q^c, q^r) where $q^c \in \mathbb{C}^{n^c}$ and $q^r \in \mathbb{R}^{n^r}$. Since the Lagrangian $\mathcal{L}$ is a real-value and analytic function, it can be seen as an analytic function of the complex variables q^c, $\dot{q}^c$, q^{c*}, $\dot{q}^{c*}$ and of the real variables q^r and $\dot{q}^r$ (q^{c*} corresponds to the complex conjugate of q^c). This function with be denoted by $\widetilde{\mathcal{L}}(q^c, q^{c*}, q^r, \dot{q}^c, \dot{q}^{c*}, \dot{q}^r)$ and is equal to $\mathcal{L}(q, \dot{q})$ where

$$q = \left(\frac{q_1^c + q_1^{c*}}{2}, \frac{q_1^c - q_1^{c*}}{2\jmath}, \ldots, \frac{q_{n^c}^c + q_{n^c}^{c*}}{2}, \frac{q_{n^c}^c - q_{n^c}^{c*}}{2\jmath}, q_1^r, \ldots, q_{n^r}^r \right)$$
$$\dot{q} = \left(\frac{\dot{q}_1^c + \dot{q}_1^{c*}}{2}, \frac{\dot{q}_1^c - \dot{q}_1^{c*}}{2\jmath}, \ldots, \frac{\dot{q}_{n^c}^c + \dot{q}_{n^c}^{c*}}{2}, \frac{\dot{q}_{n^c}^c - \dot{q}_{n^c}^{c*}}{2\jmath}, \dot{q}_1^r, \ldots, \dot{q}_{n^r}^r \right).$$

Let us consider the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) = \frac{\partial \mathcal{L}}{\partial q_k} + S_k, \quad k = 1, \ldots, n \tag{24}$$

where the S_k-terms correspond to non conservative energy exchanges with the environment. Similarly to q, we decompose $S = (S_1, \ldots, S_n)$ into $S^c \in \mathbb{C}^{n^c}$ and $S^r \in \mathbb{R}^{n^r}$: S is identified with (S^c, S^r). We will reformulate these equations with $\widetilde{\mathcal{L}}$ and its partial derivatives. For $k = 2n^c + k'$ with $k' = 1, \ldots, n^r$, they remain unchanged $\frac{d}{dt}\left(\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}_{k'}^r}\right) = \frac{\partial \widetilde{\mathcal{L}}}{\partial q_{k'}^r} + S_{k'}^r$. The two scalar equations corresponding to $k = 2k' - 1$ and $k = 2k'$ with $k' = 1, \ldots, n^c$ yield to the following single complex equation $\frac{d}{dt}\left(2\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}_{k'}^c}\right) = 2\frac{\partial \widetilde{\mathcal{L}}}{\partial q_{k'}^c} + S_{k'}^c$ since we have the identities

$$2\frac{\partial \widetilde{\mathcal{L}}}{\partial q_{k'}^c} = \frac{\partial \mathcal{L}}{\partial q_{2k'-1}} - \jmath\frac{\partial \mathcal{L}}{\partial q_{2k'}}, \quad 2\frac{\partial \widetilde{\mathcal{L}}}{\partial q_{k'}^{c*}} = \frac{\partial \mathcal{L}}{\partial q_{2k'-1}} + \jmath\frac{\partial \mathcal{L}}{\partial q_{2k'}}$$

and similarly

$$2\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}_{k'}^c} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{2k'-1}} - \jmath\frac{\partial \mathcal{L}}{\partial \dot{q}_{2k'}}, \quad 2\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}_{k'}^{c*}} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{2k'-1}} + \jmath\frac{\partial \mathcal{L}}{\partial \dot{q}_{2k'}}.$$

This provides the following complex formulation of the real Euler-Lagrange equations (24)

$$\frac{d}{dt}\left(2\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}_k^{c*}}\right) = 2\frac{\partial \widetilde{\mathcal{L}}}{\partial q_k^{c*}} + S_k^c, \quad k = 1, \ldots, n^c \tag{25}$$

$$\frac{d}{dt}\left(\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}_k^r}\right) = \frac{\partial \widetilde{\mathcal{L}}}{\partial q_k^r} + S_k^r, \quad k = 1, \ldots, n^r. \tag{26}$$

In the usual complexification procedure ([4, page 87]) the coefficient 2 appearing in the above equations does not appear. This is due to our special choice $q_k^c = q_{2k-1} + \jmath q_{2k}$ instead of the usual choice $q_k^c = \frac{q_{2k-1} + \jmath q_{2k}}{\sqrt{2}}$. This special choice preserves the correspondence, commonly used in electrical engineering, between complex and real electrical quantities

Let us assume that, for each q, the mapping $\dot{q} \mapsto \frac{\partial \mathcal{L}}{\partial \dot{q}}$ is a smooth bijection. Then the Hamiltonian formulation of (24) reads

$$\frac{d}{dt} q_k = \frac{\partial \mathcal{H}}{\partial p_k}, \quad \frac{d}{dt} p_k = -\frac{\partial \mathcal{H}}{\partial q_k} + S_k, \quad k = 1, \ldots, n \tag{27}$$

with $\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \dot{q} - \mathcal{L}$ and $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. Let us decompose p into $p^c \in \mathbb{C}^{n^c}$ and $p^r \in \mathbb{R}^{n^r}$. Then $p^r = \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^r}$ and $p^c = 2 \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^{c*}}$. Simple computations yield another derivation of the Hamiltonian from $\widetilde{\mathcal{L}}$ directly:

$$\widetilde{\mathcal{H}} = \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^c} \dot{q}^c + \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^{c*}} \dot{q}^{c*} + \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^r} \dot{q}^r - \widetilde{\mathcal{L}} \tag{28}$$

where $\widetilde{\mathcal{H}}$ denotes the Hamiltonian $\mathcal{H}$ when is a considered as a function of $(q^c, q^{c*}, q^r, p^c, p^{c*}, p^r)$. Then (27) becomes

$$\frac{d}{dt} q_k^c = 2 \frac{\partial \widetilde{\mathcal{H}}}{\partial p_k^{c*}}, \quad \frac{d}{dt} p_k^c = -2 \frac{\partial \widetilde{\mathcal{H}}}{\partial q_k^{c*}} + S_k^c, \quad k = 1, \ldots, n^c \tag{29}$$

$$\frac{d}{dt} q_k^r = \frac{\partial \widetilde{\mathcal{H}}}{\partial p_k^r}, \quad \frac{d}{dt} p_k^r = -\frac{\partial \widetilde{\mathcal{H}}}{\partial q_k^r} + S_k^r, \quad k = 1, \ldots, n^r \tag{30}$$

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