Invariant Extended Kalman Filter: theory and application to a velocity-aided attitude estimation problem

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***Abstract*— A new version of the Extended Kalman Filter (EKF) is proposed for nonlinear systems possessing symmetries. Instead of using a linear correction term based on a linear output error, it uses a geometrically adapted correction term based on an invariant output error; in the same way the gain matrix is not updated from of a linear state error, but from an invariant state error. The benefit is that the gain and covariance equations converge to constant values on a much bigger set of trajectories than equilibrium points as is the case for the EKF, which should result in a better convergence of the estimation.**

This filter is applied to the practically relevant problem of estimating the velocity and attitude of a moving rigid body, e.g. an aircraft, from GPS velocity, inertial and magnetic measurements. In this context it can be seen as an extension of the "Multiplicative EKF" often used for quaternion estimation.

I. INTRODUCTION

Estimating the state of a nonlinear system from the knowledge of its input and output is an ubiquitous problem in system theory. A widely used approach to design such an estimator (also termed filter or observer) is the so-called Extended Kalman Filter (EKF). The system is seen as a stochastic differential equation,

$$\dot{x} = f(x,u) + Mw \tag{1}$$

$$y = h(x,u) + Nv, \tag{2}$$

where x, u, y belong to an open subset of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$; w, v are independent white gaussian noises of size n and p, and M, N are square matrices. The input u and output y are known signals, and the state x must be estimated. An estimation $\hat{x}(t)$ of $x(t)$ is then computed by the EKF

$$\dot{\hat{x}} = f(\hat{x},u) + K\cdot\big(y - h(\hat{x},u)\big)$$
$$\dot{P} = AP + PA^T + MM^T - PC^T(NN^T)^{-1}CP,$$

with $K = PC^T(NN^T)^{-1}$, $A = \partial_1 f(\hat{x},u)$ and $C = \partial_1 h(\hat{x},u)$ (∂_i means the partial derivative with respect to the i^{th} argument). The rationale is to compute the gain K as in a linear Kalman filter since the estimation error $\Delta x = \hat{x} - x$ satisfies up to higher order terms the linear equation

$$\Delta\dot{x} = (A - KC)\Delta x - Mw + KNv. \tag{3}$$

Of course the convergence of the EKF is not guaranteed in general as in the linear case, see e.g. [1] for some (local) convergence results.

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Another drawback of this "linear" approach is that it does not respect the geometry when (part of) the state space is a manifold. This situation frequently arises e.g. in the context of aerospace engineering, where the attitude of an aircraft is usually represented by a unit quaternion rather than Euler angles; ad hoc modifications of the EKF are then used, in particular the so-called Multiplicative EKF (MEKF) introduced in [2], [3], [4].

In this paper we propose a modification of the EKF for nonlinear systems possessing symmetries. Instead of using a linear correction term based on a linear output error, it uses a geometrically adapted correction term based on an invariant output error; in the same way the gain matrix is not updated from of a linear state error, but from an invariant state error. For that to make sense from a stochastic point of view, we assume the driving and observation noise enter the system in an invariant way. This "Invariant EKF" (IEKF) builds on the ideas developed in [5], [6], [7], see also [8], [9], [10] for related approaches. More generally it adds to the several attempts to introduce geometry in the problem of nonlinear filtering, see e.g. [11], [12], [13].

The main benefit of the IEKF is that the matrices A and C are constant on a much bigger set of trajectories (so-called "permanent trajectories" [6]) than equilibrium points as is the case for the EKF. Near such trajectories, we are back to the "true", i.e. linear, Kalman filter where convergence is guaranteed. Informally, this means the IEKF should in general converge at least around any slowly-varying permanent trajectory, rather than just around any slowly-varying equilibrium point for the EKF.

We then apply the IEKF to the practically relevant problem of estimating the velocity and attitude of a moving rigid body, e.g. an aircraft, from velocity, inertial and magnetic measurements. We design two different versions (Left and Right IEKF), which can be seen as extensions of the MEKF.

Finally we present experimental and simulation results.

II. INVARIANT EXTENDED KALMAN FILTER

A. Symmetry-preserving observers

We briefly recall here the main ideas of [5], [6]. The theory is constructive and is directly applicable to the system considered in this paper.

Definition 1: Let G be a Lie Group with identity e and Σ an open set (or more generally a manifold). A *transformation group* $(\phi_g)_{g\in G}$ on Σ is a smooth map

$$(g,\xi) \in G\times\Sigma \mapsto \phi_g(\xi) \in \Sigma$$

such that:

- $\phi_e(\xi) = \xi$ for all ξ
- $\phi_{g_2} \circ \phi_{g_1}(\xi) = \phi_{g_2 g_1}(\xi)$ for all g_1, g_2, ξ.

By construction ϕ_g is a diffeomorphism on Σ for all g. The transformation group is *local* if $\phi_g(\xi)$ is defined only for g around e. In this case the transformation law $\phi_{g_2} \circ \phi_{g_1}(\xi) = \phi_{g_2 g_1}(\xi)$ is imposed only when it makes sense; "for all g" accordingly means "for all g around e, and "for all ξ" means "for all ξ in some neighborhood".

Consider now the smooth output system

$$\dot{x} = f(x,u) \tag{4}$$

$$y = h(x,u) \tag{5}$$

where x belongs to an open subset $\mathscr{X} \subset \mathbb{R}^n$, u to an open subset $\mathscr{U} \subset \mathbb{R}^m$ and y to an open subset $\mathscr{Y} \subset \mathbb{R}^p$, $p \leq n$.

Consider also the local group of transformations on $\mathscr{X} \times \mathscr{U}$ defined by $(X,U) = \big(\varphi_g(x), \psi_g(u)\big)$, where φ_g and ψ_g are local diffeomorphisms.

Definition 2: The system $\dot{x} = f(x,u)$ is *invariant* if $f\big(\varphi_g(x), \psi_g(u)\big) = D\varphi_g(x) \cdot f(x,u)$ for all g, x, u.
The property also reads $\dot{X} = f(X,U)$, i.e., the system is left unchanged by the transformation.

Definition 3: The output $y = h(x,u)$ is *equivariant* if there exists a transformation group $(\rho_g)_{g \in G}$ on $\mathscr{Y}$ such that $h\big(\varphi_g(x), \psi_g(u)\big) = \rho_g\big(h(x,u)\big)$ for all g, x, u.
With $(X,U) = \big(\varphi_g(x), \psi_g(u)\big)$ and $Y = \rho_g(y)$, the definition reads $Y = h(X,U)$.

Definition 4: The observer $\dot{\hat{x}} = F(\hat{x}, u, y)$ of the system (4)-(5) is *symmetry-preserving* (or *invariant*) if $F\big(\varphi_g(\hat{x}), \psi_g(u), \rho_g(y)\big) = D\varphi_g(\hat{x}) \cdot F(\hat{x}, u, y)$ for all $g, \hat{x}, u, y$.
The property also reads $\dot{\hat{X}} = F(\hat{X}, U, Y)$, i.e., the system is left unchanged by the transformation.

We now state the two main results in the special case where $g \mapsto \varphi_g(x)$ is invertible (i.e. when G is of dimension n), see [5] for the general case. $\mathscr{X}$ can then be (locally) identified with G; if $\mathscr{X} = G$ from start with globally defined transformations (as in the example treated in this paper), all computations are moreover global. The group action coincides with left translations L_g, i.e. $\varphi_g(x) = L_g(x)$; right translations R_g write $R_g(x) = xg = \varphi_x(g)$. See [6] for details.

Theorem 1: A symmetry-preserving observer reads

$$\dot{\hat{x}} = f(\hat{x},u) + DL_{\hat{x}}(e) \cdot K \cdot \Big(\rho_{\hat{x}^{-1}}(y) - \rho_{\hat{x}^{-1}}\big(h(\hat{x},u)\big)\Big),$$

where the matrix gain K may depend only on the invariant quantity $\hat{I} := \psi_{\hat{x}^{-1}}(u)$ and of the invariant output error $\rho_{\hat{x}^{-1}}(y) - \rho_{\hat{x}^{-1}}\big(h(\hat{x},u)\big)$.

Instead of the usual "linear" error $\hat{x} - x$, we can now use an invariant error, with the remarkable following property.

Theorem 2: The error system for the invariant state error $\eta := x^{-1}\hat{x}$ reads $\dot{\eta} = \Upsilon(\eta, \hat{I})$.
Explicitly, after using repeatedly invariance properties,

$$\begin{aligned}
\dot{\eta} &= DL_{x^{-1}}(\hat{x}) \cdot \dot{\hat{x}} + DR_{\hat{x}}(x^{-1}) \cdot \overbrace{x^{-1}}^{\cdot} \\
&= DL_{x^{-1}}(\hat{x}) \cdot \dot{\hat{x}} - DR_{\hat{x}}(x^{-1}) \cdot \big(DR_{x^{-1}}(e) \cdot DL_{x^{-1}}(x) \cdot \dot{x}\big) \\
&= DL_{\eta}(e) \cdot f(e,\hat{I}) - DR_{\eta}(e) \cdot f\big(e, \psi_{\eta}(\hat{I})\big) \\
&\quad + DL_{\eta}(e) \cdot K \cdot \big(h(\eta^{-1}, \hat{I}) - h(e, \hat{I})\big).
\end{aligned} \tag{6}$$

This result greatly simplifies the convergence analysis, since the error equation is autonomous but for the "free" known invariant $\hat{I}$. For a general (not symmetry-preserving) nonlinear observer the error equation depends on the trajectory $t \mapsto \big(x(t), u(t)\big)$ of the system, hence is in fact of dimension $2n$. In some sense this extends a result valid around equilibrium points (the linearization of the error equation around an equilibrium point is autonomous) to the much wider class of the so-called *permanent trajectories* [6] characterized by the fact that $\hat{I}$ is constant along them.

B. Invariant noises

Consider now the system

$$\dot{x} = f(x,u) + M(x)w \tag{7}$$

$$y = h(x,u) + N(x)v \tag{8}$$

where w, v are independent white gaussian noises. We also want the driving noise w and observation noise v to preserve invariance and extend the definitions of the previous section.

Definition 5: The system with noise (7)-(8) is *invariant* with *equivariant* output and *invariant* noises if for all g, x, u,

$$f\big(\varphi_g(x), \psi_g(u)\big) + M\big(\varphi_g(x)\big)w = D\varphi_g(x) \cdot \big(f(x,u) + M(x)w\big)$$
$$h\big(\varphi_g(x), \psi_g(u)\big) + N\big(\varphi_g(x)\big)v = \rho_g\big(h(x,u) + N(x)v\big).$$

We will also assume ρ_g is linear, i.e. $\rho_g(y_1 + y_2) = \rho_g(y_1) + \rho_g(y_2)$ for all g, y_1, y_2. This assumption is not used for the derivation of the IEKF, but only for a simple analysis of the stochastic error equations (9) and (10); it could be relaxed at the cost of a more complicated analysis.

C. Invariant Extended Kalman Filter

The Invariant EKF is given by

$$\begin{aligned}
\dot{\hat{x}} &= f(\hat{x},u) + DL_{\hat{x}}(e) \cdot K \cdot \big(\rho_{\hat{x}^{-1}}(y) - \rho_{\hat{x}^{-1}}(h(\hat{x},u))\big) \\
K &= PC^T\big(N(e)N^T(e)\big)^{-1} \\
\dot{P} &= AP + PA^T + M(e)M^T(e) - PC^T\big(N(e)N^T(e)\big)^{-1}CP,
\end{aligned}$$

where $C := \partial_1 h(e, \hat{I})$ and A is defined by

$$A\xi := \big[\xi, f(e,\hat{I})\big] - \partial_1 f(e,\hat{I}) \cdot \partial_1 \psi(e,\hat{I}) \cdot \xi$$

(where $[*,*]$ is the Lie bracket of the Lie algebra of G and ψ is seen as a function of two variables, see [6] for details).

The idea is the same as for the usual EKF, but instead of using a linear correction term based on a linear output error, it uses a geometrically adapted correction term based on an invariant output error; in the same way the gain matrix is not updated from of a linear state error, but from an invariant state error. Indeed the error equation (6) becomes

$$\begin{aligned}
\dot{\eta} &= DL_{\eta}(e) \cdot f(e,\hat{I}) - DR_{\eta}(e) \cdot \Big(f\big(e, \psi_{\eta}(\hat{I})\big) + M(e)w\Big) \\
&\quad + DL_{\eta}(e) \cdot K \cdot \Big(h(\eta^{-1}, \hat{I}) - h(e,\hat{I}) + N(\eta^{-1})v\Big),
\end{aligned} \tag{9}$$

hence for η close to e, the linearized error equation is

$$\dot{\xi} = A - KC + Q_1\big(\xi, M(e)w\big)\big) + Q_2\big(\xi, KN(e)v\big)\big), \tag{10}$$

where Q_1, Q_2 are quadratic functions of their arguments. Notice A and C depend only on $\hat{I}$, rather than on $(\hat{x}, u)$ as in (3); this much simpler dependence and its consequences is the main interest of the IEKF.

There is nevertheless a slight problem with the linearized error equation (10) because of the quadratic terms Q_1, Q_2, which do not appear in the usual linearized equation (3). Rather than a linear inhomogeneous equation of the form

$$\dot{\zeta} = A_0(t)\zeta + \alpha W_1(t),$$

it is a multiplicative inhomogeneous equation of the form

$$\dot{\xi} = A_0(t)\xi + \alpha W_1(t) + \alpha W_2(t)\xi. \tag{11}$$

Here W_1 and W_2 are time-varying matrices with entries linear functions of the mutually independent white noises w and v, hence have zero average $\langle W_1 \rangle$ and $\langle W_2 \rangle$; and α is a constant encoding the noises magnitudes. Following [14, page 403ff.] it can be proved that whereas the average $\langle \zeta \rangle$ satisfies

$$\frac{d}{dt}\langle \zeta \rangle = A_0(t)\langle \zeta \rangle + \alpha \langle W_1 \rangle = A_0(t)\langle \zeta \rangle, \tag{12}$$

the average $\langle \xi \rangle$ satisfies

$$\begin{aligned} \frac{d}{dt}\langle \xi \rangle &= \Big(A_0(t) + \alpha \langle W_2 \rangle + \alpha^2 \langle W_2^2 \rangle\Big)\langle \xi \rangle \\ &\quad + \alpha^2 \langle W_1 W_2 \rangle + O(\alpha^3) \\ &= A_0(t)\langle \xi \rangle + O(\alpha^2) \end{aligned}$$

This means that whereas $\langle \zeta \rangle$ tends to zero (provided the EKF works, i.e. its linearized error equation (3) without noise converges), $\langle \xi \rangle$ will be biased by $O(\alpha^2)$ terms.

In a similar way and still following [14, page 403ff.], the covariance of ξ obeys the same equation as the covariance of ζ up to α^2 terms.

III. The "low-cost" inertial navigation problem

We now apply the IEKF to the practically relevant problem of estimating the velocity and attitude of a moving rigid body, e.g. an aircraft, from Global Positioning System (GPS) velocity, inertial and magnetic measurements. We consider cheap strapdown inertial sensors, hence cannot rely on the Schuler effect due to as in "true" inertial navigation. A model of a flat non-rotating Earth is then sufficient; for more details about inertial navigation with or without GPS aiding, see e.g. [15].

See also [16] for a discussion of the specific problem setting used here, as well as a more detailed approach on the design of a symmetry-preserving observer.

A. Motion equations

The motion of a flying rigid body (assuming a flat non-rotating Earth) is described by

$$\begin{aligned} \dot{q} &= \frac{1}{2} q * \omega \\ \dot{V} &= A + q * a * q^{-1}, \end{aligned}$$

where

- q is the unit quaternion representing the orientation of the body-fixed frame with respect to the Earth-fixed frame
- ω is the instantaneous angular velocity vector
- V is the velocity vector of the center of mass with respect to the Earth-fixed frame
- $A = (0\ 0\ g)^T$ is the (constant) gravity vector in North-East-Down (NED) coordinates
- a is the specific acceleration vector, i.e. all the non-gravitational forces divided by the body mass.

The first equation describes the kinematics of the body, the second is Newton's force law. It is customary to use quaternions instead of Euler angles since they provide a global parametrization of the body orientation, and are well-suited for calculations and computer simulations. The basic facts used in this paper are summarized in the following section; for more detail see any good textbook on aircraft modeling, for instance [17].

B. Quaternions

A quaternion p can be thought of as a scalar $p_0 \in \mathbb{R}$ together with a vector $\vec{p} \in \mathbb{R}^3$, i.e., $p = (p_0, \vec{p})^T$. The (non commutative) quaternion product $*$ then reads

$$p * q \triangleq \begin{pmatrix} p_0 q_0 - \vec{p} \cdot \vec{q} \\ p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q} \end{pmatrix}.$$

To any quaternion q with unit norm is associated a rotation matrix $R_q \in SO(3)$ by $q^{-1} * \vec{p} * q = R_q \cdot \vec{p}$ for all $\vec{p} \in \mathbb{R}^3$. Any scalar $p_0 \in \mathbb{R}$ can be seen as the quaternion $(p_0, \vec{0})^T$, and any vector $\vec{p} \in \mathbb{R}^3$ can be seen as the quaternion $(0, \vec{p})^T$. We often use the formula $p \times q \triangleq \vec{p} \times \vec{q} = \frac{1}{2}(p * q - q * p)$ The quaternions (with non zero norm) form a group with $(1, \vec{0})^T$ as the identity element, and $(p * q)^{-1} = q^{-1} * p^{-1}$.

If q depends on time, then $\dot{q}^{-1} = -q^{-1} * \dot{q} * q^{-1}$. Finally, consider the differential equation $\dot{q} = q * u + v * q$ with $u, v \in \mathbb{R}^3$; then $\|q(t)\| = \|q(0)\|$ for all t.

C. Measurements

We use four triaxial sensors, yielding twelve scalar measurements: 3 gyros measure $\omega_m = \omega + \omega_b$, where ω_b is a constant vector bias; 3 accelerometers measure $a_m = a_s a$, where $a_s > 0$ is a constant scaling factor; 3 magnetometers measure $y_B = q^{-1} * B * q$, where $B = (B_1\ \ 0\ \ B_3)^T$ is the Earth magnetic field in NED coordinates; the velocity vector V is provided by the navigation solutions y_V of a GPS engine (the GPS velocity is obtained from the carrier phase and/or Doppler shift data, and not by differentiating the GPS position, hence is of rather good quality). There is some freedom in the modeling of the sensors imperfections, see [16] for a discussion. All the measurements are of course also corrupted by noise.

It is reasonable to assume each scalar sensor is corrupted by an additive gaussian white noise with identical variance for each of the three scalar sensors constituting a triaxial sensor, and all the noises mutually independent (this is technologically motivated for the acceleros, gyros and magnetic sensors, though much more questionable for the GPS engine). Hence we can see each triaxial sensor as corrupted by a "coordinate-free vector noise" whose coordinates are gaussian in the body frame as well as the Earth frame (or any other smooth time-varying frame). Indeed, the mean and the auto-correlation time of such a noise is not affected by a (smoothly) tyme-varying rotation.

D. The considered system

To design our observers we therefore consider the system

$$\dot{q} = \frac{1}{2} q * (\omega_m - \omega_b) \tag{13}$$

$$\dot{V} = A + \frac{1}{a_s} q * a_m * q^{-1} \tag{14}$$

$$\dot{\omega}_b = 0 \tag{15}$$

$$\dot{a}_s = 0, \tag{16}$$

where ω_m and a_m are seen as known inputs, together with the output

$$\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} V \\ q^{-1} * B * q \end{pmatrix}. \tag{17}$$

IV. Multiplicative Extended Kalman Filter

We start with the design of a Multiplicative EKF in the spirit of [2], [3], [4], see also [18], [19]. The idea is to respect the geometry of the quaternion space, by using for the quaternion estimation a multiplicative correction term $\hat{q} * K_q E$ which preserves the unit norm, and by computing the error equation with the error $q^{-1} * \hat{q}$ (or equivalently $\hat{q}^{-1} * q = (q^{-1} * \hat{q})^{-1}$). Notice the standard linear correction term does not preserve the norm, hence some projection would be needed, whereas the standard linear error $\hat{q} - q$ does not really make sense for quaternions.

A. Problem setting

We consider the noise enters the system as

$$\dot{q} = \frac{1}{2} q * (\omega_m - \omega_b) + q * M_q w_q \tag{18}$$

$$\dot{V} = A + \frac{1}{a_s} q * a_m * q^{-1} + q * M_V w_V * q^{-1} \tag{19}$$

$$\dot{\omega}_b = M_\omega w_\omega \tag{20}$$

$$\dot{a}_s = M_a w_a, \tag{21}$$

and the output as

$$\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} V + N_V v_V \\ q^{-1} * B * q + N_B v_B \end{pmatrix}, \tag{22}$$

with $M_q, M_V, M_\omega, N_V, N_B$ diagonal matrices. The driving and observation noises are thus consistent with a scalar additive noise on each individual sensor.

B. MEKF equations

The MEKF then takes the form

$$\dot{\hat{q}} = \frac{1}{2} \hat{q} * (\omega_m - \hat{\omega}_b) + \hat{q} * K_q E \tag{23}$$

$$\dot{\hat{V}} = A + \frac{1}{\hat{a}_s} \hat{q} * a_m * \hat{q}^{-1} + K_V E \tag{24}$$

$$\dot{\hat{\omega}}_b = K_\omega E \tag{25}$$

$$\dot{\hat{a}}_s = K_a E. \tag{26}$$

where the output error is given by

$$E = \begin{pmatrix} \hat{y}_V - y_V \\ \hat{y}_B - y_B \end{pmatrix} = \begin{pmatrix} \hat{V} - V - N_V v_V \\ \hat{q}^{-1} * B * \hat{q} - y_B - N_B v_B \end{pmatrix}.$$

But for (23), the MEKF has the form of a standard EKF.

We consider the state error $\mu = q^{-1} * \hat{q}$, $\nu = \hat{V} - V$, $\beta = \hat{\omega}_b - \omega_b$ and $\alpha = \hat{a}_s - a_s$. A tedious but simple computation yields the error system

$$\begin{aligned}
\dot{\mu} &= -\frac{1}{2} \beta * \mu + \mu \times \hat{J}_\omega - M_q w_q * \mu + \mu * K_q E \\
\dot{\nu} &= \hat{I}_a - \frac{1}{\hat{a}_s - \alpha} \hat{q} \mu * a_m * \mu^{-1} * \hat{q}^{-1} \\
&\quad - \mu * \hat{q} * M_V w_V * \hat{q}^{-1} * \mu^{-1} + K_V E \\
\dot{\beta} &= K_\omega E - M_\omega w_\omega \\
\dot{\alpha} &= \alpha K_a E - \alpha M_a w_a,
\end{aligned}$$

where the output error is rewritten as

$$E = \begin{pmatrix} \nu - N_V v_V \\ \hat{J}_B - \mu^{-1} * \hat{J}_B * \mu - N_B v_B \end{pmatrix},$$

and $\hat{J}_\omega = \omega_m - \hat{\omega}_b$, $\hat{I}_a = \frac{1}{\hat{a}_s} \hat{q} * a_m * \hat{q}^{-1}$ and $\hat{J}_B = \hat{q}^{-1} * B * \hat{q}$.

We next linearize this error system around $(\overline{\mu}, \overline{\nu}, \overline{\beta}, \overline{\alpha}) = (1, 0, 0, 0)$, drop all the quadratic terms in noise and infinitesimal state error according to the approximation in section II-C, and eventually find

$$\begin{pmatrix} \delta\dot{\mu} \\ \delta\dot{\nu} \\ \delta\dot{\beta} \\ \delta\dot{\alpha} \end{pmatrix} = (A - KC) \begin{pmatrix} \delta\mu \\ \delta\nu \\ \delta\beta \\ \delta\alpha \end{pmatrix} - M \begin{pmatrix} w_q \\ w_V \\ w_\omega \\ w_a \end{pmatrix} + KN \begin{pmatrix} v_V \\ v_B \end{pmatrix},$$

which has the desired form (3) with

$$\begin{aligned}
A &= \begin{pmatrix} -\hat{J}_{\omega\times} & 0_{33} & -\frac{1}{2} I_3 & 0_{31} \\ 2\hat{J}_{a\times} R(\hat{q}) & 0_{33} & \hat{J}_{\omega\times} & -\hat{J}_a \\ 0_{33} & 0_{33} & 0_{33} & 0_{31} \\ 0_{31} & 0_{31} & 0_{31} & 0_{31} \end{pmatrix} \\
C &= \begin{pmatrix} 0_{33} & I_3 & 0_{33} & 0_{31} \\ -2\hat{J}_{B_\times} & 0_{33} & 0_{33} & 0_{31} \end{pmatrix} \\
M &= Diag(M_q, R(\hat{q}) M_V, M_\omega, M_a) \\
N &= Diag(N_V, N_B) \\
K &= -(K_q,\ K_V,\ K_\omega,\ K_a)^T.
\end{aligned}$$

We have used the matrices $I_\times$ and $R(q)$ defined by $I_\times u := I \times u$ and $R(q) u := q * u * q^{-1}$ for all for $u \in \mathbb{R}^3$.

C. Features of the MEKF

Sound geometric structure for the quaternion estimation equation: by construction equation (23) preserves the unit norm of the estimated quaternion.

Possible convergence issues in many situations: indeed, the matrices A and C used for computing the gain matrix K are constant only in level flight, i.e. $\hat{I}_\omega \simeq 0$ and $\hat{I}_a \simeq -A$, because of the trajectory-dependent terms $R(q)$ in A.

V. Left invariant extended Kalman filter

We now design a first Invariant Extended Kalman Filter, which can be seen as a generalization and an improvement of the MEKF. It is a direct application of the method presented in section II-C. It is termed "Left IEKF" (LIEKF) because in the transformation group defined below the quaternion q is multiplied on the left by a constant quaternion p_0.

A. Problem setting in terms of transformation groups

We notice the state space is a group for the law given by

$$\begin{pmatrix} p_0 \\ V_0 \\ \omega_0 \\ a_0 \end{pmatrix} \diamond \begin{pmatrix} q \\ V \\ \omega_b \\ a_s \end{pmatrix} := \begin{pmatrix} p_0 * q \\ p_0 * (V + V_0) * p_0^{-1} \\ \omega_b + \omega_0 \\ a_s a_0 \end{pmatrix},$$

hence acts on itself (the physical meaning is clear: rotation and translation in Earth axes, translation in body axes, and scaling). It also yields the transformation groups

$$\psi_{(p_0,V_0,\omega_0,a_0)} \begin{pmatrix} \omega_m \\ a_m \\ A \\ B \end{pmatrix} = \begin{pmatrix} \omega_m + \omega_0 \\ a_0 a_m \\ p_0 * A * p_0^{-1} \\ p_0 * B * p_0^{-1} \end{pmatrix}$$

$$\rho_{(p_0,V_0,\omega_0,a_0)} \begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} p_0 * (y_V + V_0) * p_0^{-1} \\ y_B \end{pmatrix}.$$

The system (13)–(16) is clearly invariant, for instance

$$\dot{\overbrace{p_0 * q}} = p_0 * \dot{q} = \frac{1}{2} (p_0 * q) * \big((\omega_m + \omega_0) - (\omega_b + \omega_0) \big),$$

whereas the output (17) is equivariant since

$$\begin{pmatrix} p_0 * (V + V_0) * p_0^{-1} \\ (p_0 * q)^{-1} * (p_0 * B * p_0^{-1}) * (p_0 * q) \end{pmatrix} = \rho_{(q_0,V_0,\omega_0,a_0)} \begin{pmatrix} V \\ q^{-1} * B * q \end{pmatrix}.$$

The complete set of invariants is given by $\psi_{\hat{x}^{-1}}(u)$, with $\hat{x}^{-1} = (\hat{q}^{-1}, -\hat{V}, -\hat{\omega}_b, \frac{1}{\hat{a}_s})$, hence reads

$$\begin{pmatrix} \hat{I}_\omega \\ \hat{I}_a \\ \hat{I}_A \\ \hat{I}_B \end{pmatrix} = \begin{pmatrix} \omega_m - \hat{\omega}_b \\ \frac{1}{\hat{a}_s} a_m \\ \hat{q}^{-1} * A * \hat{q} \\ \hat{q}^{-1} * B * \hat{q} \end{pmatrix}.$$

Moreover the driving noise as defined in (18)–(21) for the MEKF is also invariant. We finally define an invariant observation noise by

$$\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} V + q * N_V \nu_V * q^{-1} \\ q^{-1} * B * q + N_B \nu_B \end{pmatrix}. \tag{27}$$

Notice the noise on y_V is different from the noise in (22), which is not invariant: it is additive in body axes rather than in Earth axes, see section III-C for a tentative physical justification.

B. Left IEKF equations

Directly following section II-C, the LIEKF reads

$$\dot{\hat{q}} = \frac{1}{2} \hat{q} * (\omega_m - \hat{\omega}_b) + \hat{q} * (K_q E) \tag{28}$$

$$\dot{\hat{V}} = A + \frac{1}{\hat{a}_s} \hat{q} * a_m * \hat{q}^{-1} + \hat{q} * (K_V E) * \hat{q}^{-1} \tag{29}$$

$$\dot{\hat{\omega}}_b = K_\omega E \tag{30}$$

$$\dot{\hat{a}}_s = \hat{a}_s K_a E, \tag{31}$$

where the invariant output error is given by

$$E = \rho_{\hat{x}^{-1}} \begin{pmatrix} \hat{y}_V \\ \hat{y}_B \end{pmatrix} - \rho_{\hat{x}^{-1}} \begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} \hat{q}^{-1} * (\hat{V} - y_V) * \hat{q} \\ \hat{q}^{-1} * B * \hat{q} - y_B \end{pmatrix}.$$

Notice (28) and (30) are the same as (23) and (25) in the MEKF, while (29) and (31) are different from (24) and (26).

The invariant state error $x^{-1}\hat{x}$ reads

$$\begin{pmatrix} \mu \\ \nu \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} q^{-1} * \hat{q} \\ q^{-1} * (\hat{V} - V) * q \\ \hat{\omega}_b - \omega_b \\ \frac{\hat{a}_s}{a_s} \end{pmatrix},$$

hence we recover the quaternion error used in the MEKF.

The error system is

$$\begin{aligned} \dot{\mu} &= -\frac{1}{2} \beta * \mu + \mu \times \hat{J}_\omega - M_q w_q * \mu + \mu * K_q E \\ \dot{\nu} &= \mu * \hat{J}_a * \mu^{-1} - \alpha \hat{J}_a + \nu \times (\hat{J}_\omega + \beta) \\ &\quad - M_V w_V + 2\nu \times M_q w_q + \mu * K_V E * \mu^{-1} \\ \dot{\beta} &= K_\omega E - M_\omega w_\omega \\ \dot{\alpha} &= \alpha K_a E - \alpha M_a w_a, \end{aligned}$$

where the invariant output error is rewritten as

$$E = \begin{pmatrix} \mu^{-1} * (\nu - N_V \nu_V) * \mu \\ \hat{J}_B - \mu * \hat{J}_B * \mu^{-1} - N_B \nu_B \end{pmatrix}.$$

We then linearize this error system around the group identity element $(\overline{\mu}, \overline{\nu}, \overline{\beta}, \overline{\alpha}) = (1,0,0,1)$. We drop all the quadratic terms in noise and infinitesimal state error according to the approximation in section II-C, and eventually find

$$\begin{pmatrix} \delta\dot{\mu} \\ \delta\dot{\nu} \\ \delta\dot{\beta} \\ \delta\dot{\alpha} \end{pmatrix} = (A - KC) \begin{pmatrix} \delta\mu \\ \delta\nu \\ \delta\beta \\ \delta\alpha \end{pmatrix} - M \begin{pmatrix} w_q \\ w_V \\ w_\omega \\ w_a \end{pmatrix} + KN \begin{pmatrix} \nu_V \\ \nu_B \end{pmatrix},$$

which has the desired form (10) with

$$A = \begin{pmatrix} -\hat{J}_{\omega\times} & 0_{33} & -\frac{1}{2} I_3 & 0_{31} \\ -2\hat{J}_{a\times} & 0_{33} & -\hat{J}_{\omega\times} & -\hat{J}_a \\ 0_{33} & 0_{33} & 0_{33} & 0_{31} \\ 0_{31} & 0_{31} & 0_{31} & 0_{31} \end{pmatrix}$$

$$C = \begin{pmatrix} 0_{33} & I_3 & 0_{33} & 0_{31} \\ 2\hat{I}_{B_\times} & 0_{33} & 0_{33} & 0_{31} \end{pmatrix}$$

$$M = Diag(M_q, M_V, M_\omega, M_a)$$

$$N = Diag(N_V, N_B)$$

$$K = -(K_q,\ K_V,\ K_\omega,\ K_a)^T.$$

C. Features of the LIEKF

Symmetry-preserving structure: rotations, translations and scaling in the appropriated frames leave the error system unchanged, which is meaningful from an engineering point of view.

Sound geometric structure for the quaternion estimation equation: by construction equation (28) preserves the unit norm of the estimated quaternion.

Larger expected domain of convergence: the matrices A and C used for computing the gain matrix K are constant not only in level flight but also on every permanent trajectory defined by constant $\hat{J}_\omega, \hat{J}_a, \hat{J}_A, \hat{J}_B$. This is a much bigger set of trajectories, especially if K_q is kept small by choosing a large N_B (little confidence in the magnetic measurements) so that $\hat{J}_B$ does not really matter.

VI. Right invariant extended Kalman filter

We now design a second Invariant Extended Kalman Filter, with a different transformation group. It is termed "Right IEKF" (RIEKF) because the quaternion q is now multiplied on the right by a constant quaternion q_0.

A. Problem setting in terms of transformation groups

The state space is also a group for the law given by

$$\begin{pmatrix} q_0 \\ V_0 \\ \omega_0 \\ a_0 \end{pmatrix} \diamond \begin{pmatrix} q \\ V \\ \omega_b \\ a_s \end{pmatrix} := \begin{pmatrix} q * q_0 \\ V + V_0 \\ q_0^{-1} * \omega_b * q_0 + \omega_0 \\ a_s a_0 \end{pmatrix},$$

hence acts on itself (the physical meaning is now: translation in Earth axes, rotation and translation in body axes, and scaling). It also yields the transformation groups

$$\psi_{(q_0,V_0,\omega_0,a_0)} \begin{pmatrix} \omega_m \\ a_m \end{pmatrix} = \begin{pmatrix} q_0^{-1} * \omega_m * q_0 + \omega_0 \\ a_0 q_0^{-1} * a_m * q_0 \end{pmatrix}$$

$$\rho_{(q_0,V_0,\omega_0,a_0)} \begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} y_V + V_0 \\ q_0^{-1} * y_B * q_0 \end{pmatrix}.$$

The system (13)–(16) is invariant and the output (17) is equivariant. The complete set of invariants, given by $\psi_{\hat{x}^{-1}}(u)$ with $\hat{x}^{-1} = (\hat{q}^{-1}, -\hat{V}, -\hat{q} * \hat{\omega}_b * \hat{q}^{-1}, \frac{1}{\hat{a}_s})$, reads

$$\begin{pmatrix} \hat{I}_\omega \\ \hat{I}_a \end{pmatrix} = \begin{pmatrix} \hat{q} * (\omega_m - \hat{\omega}_b) * \hat{q}^{-1} \\ \frac{1}{\hat{a}_s} \hat{q} * a_m * \hat{q}^{-1} \end{pmatrix}.$$

To be invariant the driving noise must enter the system as

$$\dot{q} = \frac{1}{2} q * (\omega_m - \omega_b) + M_q w_q * q \tag{32}$$

$$\dot{V} = A + \frac{1}{a_s} q * a_m * q^{-1} + M_V w_V \tag{33}$$

$$\dot{\omega}_b = q^{-1} * M_\omega w_\omega * q \tag{34}$$

$$\dot{a}_s = a_s M_a w_a, \tag{35}$$

and the observation noise as

$$\begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} V + N_V v_V \\ q^{-1} * (B + N_B v_B) * q \end{pmatrix}. \tag{36}$$

The noise configuration used here is "dual" to the one used for the LIEKF, with Earth and body axes exchanged.

B. Right IEKF equations

Following once again section II-C, the RIEKF reads

$$\dot{\hat{q}} = \frac{1}{2} \hat{q} * (\omega_m - \hat{\omega}_b) + K_q E * \hat{q} \tag{37}$$

$$\dot{\hat{V}} = \frac{1}{\hat{a}_s} \hat{q} * a_m * \hat{q}^{-1} + A + K_V E \tag{38}$$

$$\dot{\hat{\omega}}_b = \hat{q}^{-1} * K_\omega E * \hat{q} \tag{39}$$

$$\dot{\hat{a}}_s = \hat{a}_s K_a E, \tag{40}$$

where the invariant output error is given by

$$E = \rho_{\hat{x}^{-1}} \begin{pmatrix} \hat{y}_V \\ \hat{y}_B \end{pmatrix} - \rho_{\hat{x}^{-1}} \begin{pmatrix} y_V \\ y_B \end{pmatrix} = \begin{pmatrix} \hat{y}_V - y_V \\ B - \hat{q} * y_B * \hat{q}^{-1} \end{pmatrix}.$$

The invariant state error $x^{-1}\hat{x}$ reads

$$\begin{pmatrix} \mu \\ \nu \\ \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \hat{q} * q^{-1} \\ \hat{V} - V \\ q * (\hat{\omega}_b - \omega_b) * q^{-1} \\ \frac{\hat{a}_s}{a_s} \end{pmatrix}.$$

The error system is

$$\begin{aligned}
\dot{\mu} &= -\frac{1}{2} \mu * \beta - \mu * M_q w_q + K_q E \\
\dot{\nu} &= \hat{I}_a - \alpha \mu^{-1} * \hat{I}_a * \mu - M_V w_V + K_V E \\
\dot{\beta} &= (\mu^{-1} * \hat{I}_\omega * \mu) \times \beta \\
&\quad + \mu^{-1} * K_\omega E * \mu + M_q w_q \times \beta - M_\omega w_\omega \\
\dot{\alpha} &= -\alpha M_a w_a + \alpha K_a E.
\end{aligned}$$

where the invariant output error is rewritten as

$$E = \begin{pmatrix} \nu + N_V v_V \\ B - \mu * (B + N_B v_B) * \mu^{-1} \end{pmatrix}$$

We linearize this error system around the group identity element $(\overline{\mu}, \overline{\nu}, \overline{\beta}, \overline{\alpha}) = (1, 0, 0, 1)$. We drop all the quadratic terms in noise and infinitesimal state error according to the approximation in section II-C, and eventually find

$$\begin{pmatrix} \delta\dot{\mu} \\ \delta\dot{\nu} \\ \delta\dot{\beta} \\ \delta\dot{\alpha} \end{pmatrix} = (A - KC) \begin{pmatrix} \delta\mu \\ \delta\nu \\ \delta\beta \\ \delta\alpha \end{pmatrix} - M \begin{pmatrix} w_q \\ w_V \\ w_\omega \\ w_a \end{pmatrix} + KN \begin{pmatrix} v_V \\ v_B \end{pmatrix},$$

which has the desired form (10) with

$$A = \begin{pmatrix} 0_{33} & 0_{33} & -\frac{1}{2} I_3 & 0_{31} \\ -2\hat{I}_{a\times} & 0_{33} & 0_{33} & -\hat{I}_a \\ 0_{33} & 0_{33} & \hat{I}_{\omega\times} & 0_{31} \\ 0_{31} & 0_{31} & 0_{31} & 0_{31} \end{pmatrix}$$

$$C = \begin{pmatrix} 0_{33} & I_3 & 0_{33} & 0_{31} \\ 2B_\times & 0_{33} & 0_{33} & 0_{31} \end{pmatrix}$$

$$M = Diag(M_q, M_V, M_\omega, M_a)$$

$$N = Diag(N_V, N_B)$$

$$K = -(K_q,\ K_V,\ K_\omega,\ K_a)^T.$$

C. Features of the RIEKF

Symmetry-preserving structure: rotations, translations and scaling in the appropriated frames leave the error system unchanged, which is meaningful from an engineering point of view.

Sound geometric structure for the quaternion estimation equation: by construction equation (37) preserves the unit norm of the estimated quaternion.

Larger expected domain of convergence: the matrices A and C used for computing the gain matrix K are constant not only in level flight but also on every permanent trajectory defined by constant $\hat{I}_\omega, \hat{I}_a$. Since there are less invariant quantities than in the LIEKF, and in particular not $\hat{J}_B$, there are in consequence more permanent trajectories.

VII. Numerical results

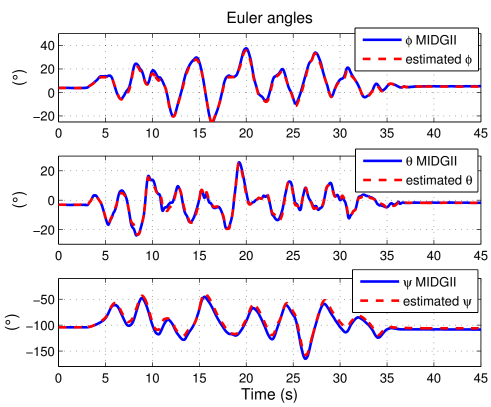

Fig. 1. Experiment: estimated Euler angles

We illustrate the behavior of the proposed filters on simulations and experimental data. The noises w_i, v_i (in the simulations) are independent normally distributed random 3-dimensional vectors with mean 0 and variance 1. The tuning of the EKF is made via the choice of covariance matrices $M_q = 0.5I_3$, $M_V = 0.01I_3$, $M_\omega = 0.001I_3$, $M_a = 0.1$, $N_V = 0.1I_3$, $N_B = 0.1I_3$. The (scaled) Earth magnetic field is taken as $B = (1\ 0\ 1)^T$ (roughly the value in France).

To enforce $\|\hat{q}\| = 1$ despite numerical round off, we systematically add the term $\lambda(1 - \|\hat{q}\|^2)\hat{q}$ in the estimated quaternion equation (otherwise the norm would slowly drift), which is a standard trick in numerical integration with quaternions. For instance for the right IEKF, we take

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} * (\omega_m - \hat{\omega}_b) + K_q E * \hat{q} + \lambda(1 - \|\hat{q}\|^2)\hat{q}.$$

Notice this correction term is invariant under both left and right multiplication by a constant quaternion. We have used $\lambda = 1$ (this value is not critical).

Since the RIEKF behaves better than the LIEKF, we do not show plots with the LIEKF for lack of space.

A. Experimental results

We first briefly compare the behavior of the RIEKF with the commercial INS-GPS device MIDG2 from Microbotics. The IEKF is fed with the raw measurements from the MIDG2 gyros, acceleros and magnetic sensors (update rate 50Hz), and the velocity provided by the navigation solutions of its GPS engine (update rate 4Hz). The IEKF estimations are compared with the MIDG2 estimations produced from the same raw data (and computed according to the user manual by some kind of Kalman filter).

The experiment consists in keeping the system at rest for a few minutes (for the biases to converge), and then moving it for about 35s. The IEKF and MIDG2 results are very similar, see Fig. 1 (only the Euler angles, converted from quaternions, are displayed).

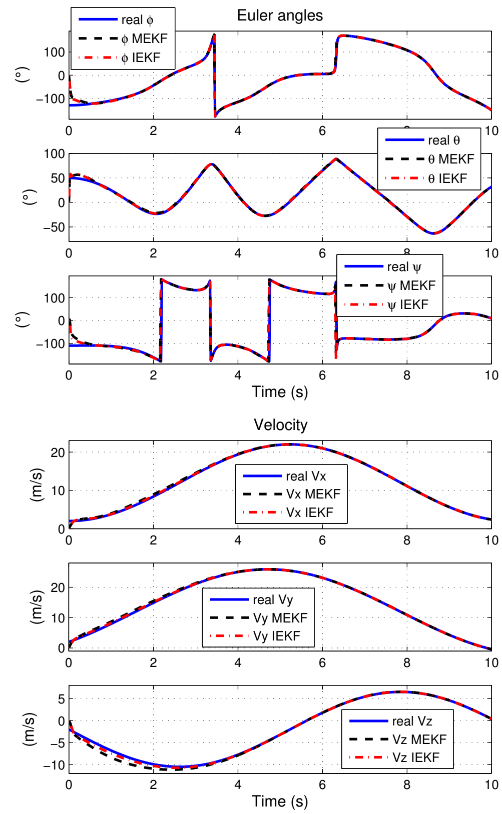

Fig. 2. Simulation: estimated Euler angles (top) and velocities (bottom)

B. Simulation results: comparison of MEKF and IEKF

The system follows a (nearly) permanent trajectory $\mathscr{T}_0$, quite representative of a small UAV flight. The MEKF and RIEKF are initialized with the same values. Both filters give correct estimations after the initial transient, see Fig. 2-3.

We now illustrate the invariance property of the IEKF: both IEKF are initialized with three different initial conditions having the same norms. The MEKF behavior does depend on the initial conditions, while the RIEKF behavior does not, see Fig. 4 (for lack of space only the norm $E_V = \|v\|$ of the velocity error is displayed).

Finally we show the RIEKF gain matrix K becomes as expected constant on the permanent trajectory $\mathscr{T}_0$, while the MEKF gain does not, see Fig. 5. This is remarkable since $\mathscr{T}_0$ is far from being an equilibrium point.

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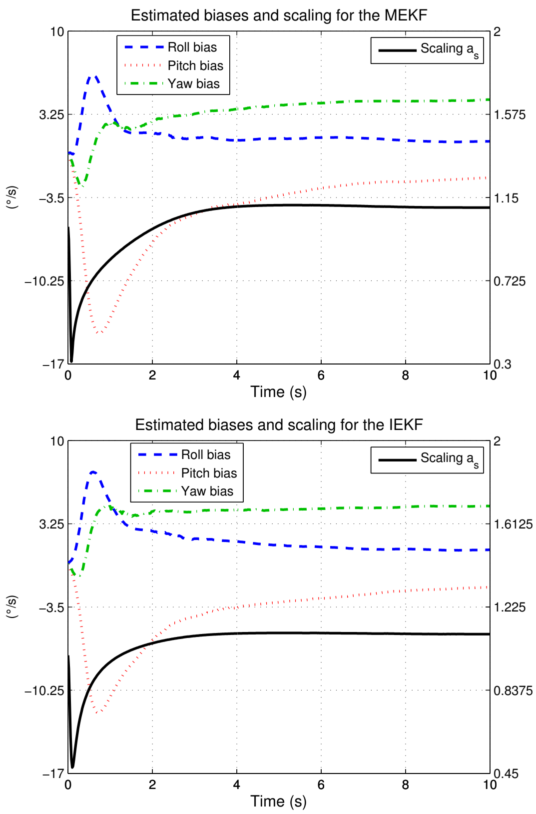

Fig. 3. Simulation: estimated biases (MEKF top, RIEKF bottom)

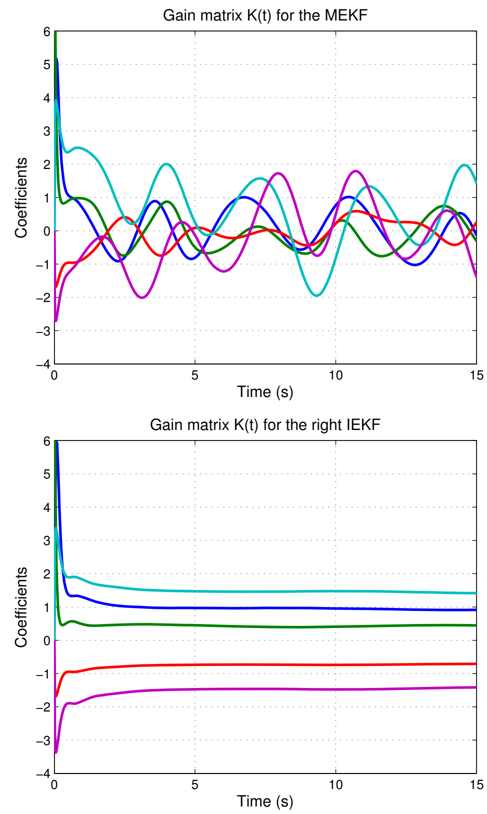

Fig. 5. Simulation: evolution of K (MEKF top, RIEKF bottom)

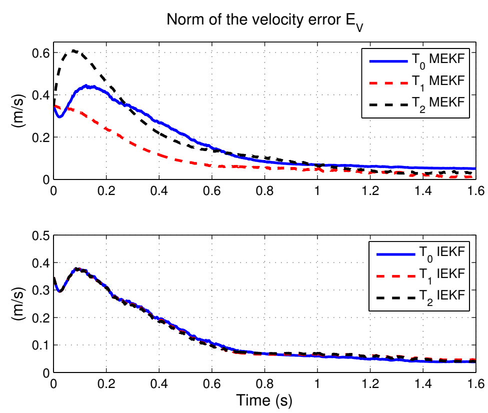

Fig. 4. Simulation: evolution of $E_V = \|v\|$ for three initial conditions

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