



# Flatness Characterization: Two Approaches

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# Flatness Characterization: Two Approaches

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**Abstract.** We survey two approaches to flatness necessary and sufficient conditions and compare them on examples.

## 1 Introduction

In this survey we consider underdetermined implicit systems of the form

$$F(x, \dot{x}) = 0 \tag{1}$$

with  $x \in X$ ,  $X$  being an infinitely differentiable manifold of dimension  $n$ , whose tangent bundle is denoted by  $TX$ , and  $F : TX \rightarrow \mathbb{R}^{n-m}$  regular in the sense that  $\text{rk} \frac{\partial F}{\partial \dot{x}} = n - m$  in a suitable open dense subset of  $TX$ . Differential flatness, or more shortly, flatness was introduced in 1992 [20,11]. In the setting of implicit control systems it may be roughly described as follows: there exists a smooth mapping  $x = \varphi(y, \dot{y}, \dots, y^{(r)})$  with  $y = (y_1, \dots, y_m)^T$  of dimension  $m$ ,  $r = (r_1, \dots, r_m)^T \in \mathbb{N}^m$ , such that

$$F(\varphi(y, \dot{y}, \dots, y^{(r)}), \dot{\varphi}(y, \dot{y}, \dots, y^{(r+1)})) \equiv 0 \tag{2}$$

with  $\varphi$  invertible in the sense that there exists a locally defined smooth mapping  $\psi$  and a multi-index  $s$  such that  $y = \psi(x, \dot{x}, \dots, x^{(s)})$ .

The vector  $y$  is called a *flat output*.

This concept has inspired an important literature. See [10,21,19,26,27,31] for surveys on flatness and its applications. Various formalisms have been introduced: finite dimensional differential geometric approaches [4,14,30], [32,28], differential algebra and related approaches [12,3,15], infinite dimensional differential geometry of jets and prolongations [13,33,19,6,7,23], [22,24], which is adopted here. The interested reader may refer to [1,13,16], [19,23,34] for more details.

The first part of the paper recalls the mathematical setting. In Section 3 the approach introduced in [19,2] for the characterization of differentially flat systems is recalled. Then, in Section 4, we introduce a novel characterization using the so-called Generalized Euler-Lagrange Operator. We conclude the paper with examples.

## 2 Implicit control systems on manifolds of jets of infinite order

Given an infinitely differentiable manifold  $X$  of dimension  $n$ , we denote its tangent space at  $x \in X$  by  $T_x X$ , and its tangent bundle by  $TX$ . Let  $F$  be a meromorphic function from  $TX$  to  $\mathbb{R}^{n-m}$ . We consider an underdetermined implicit system of the form (1) regular in the sense that  $\text{rk} \frac{\partial F}{\partial \bar{x}} = n - m$  in a suitable open dense subset of  $TX$ .

Following [17,18], we consider the infinite dimensional manifold  $\mathfrak{X}$  defined by  $\mathfrak{X} \stackrel{\text{def}}{=} X \times \mathbb{R}_\infty^n \stackrel{\text{def}}{=} X \times \mathbb{R}^n \times \mathbb{R}^n \times \dots$ , made of an infinite (but countable) number of copies of  $\mathbb{R}^n$ , with the global infinite set of coordinates<sup>3</sup>  $\bar{x} = (x, \dot{x}, \dots, x^{(k)}, \dots)$ , endowed with the product topology.

Recall that, in this topology, a function  $\varphi$  from  $\mathfrak{X}$  to  $\mathbb{R}$  is *continuous* (resp. *differentiable*) if  $\varphi$  depends only on a finite (but otherwise arbitrary) number of variables and is continuous (resp. differentiable) with respect to these variables.  $C^\infty$  or analytic or meromorphic functions from  $\mathfrak{X}$  to  $\mathbb{R}$  are then defined as in the usual finite dimensional case since they only depend on a finite number of variables. We endow  $\mathfrak{X}$  with the so-called trivial Cartan field ([16,34])  $\tau_{\mathfrak{X}} = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}}$ . We

also denote by  $L_{\tau_{\mathfrak{X}}} \gamma = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial \gamma}{\partial x_i^{(j)}} = \frac{d\gamma}{dt}$  the Lie derivative of a differentiable function  $\gamma$  along  $\tau_{\mathfrak{X}}$  and  $L_{\tau_{\mathfrak{X}}}^k \gamma$  its  $k$ th iterate. Since  $\frac{d}{dt} x_i^{(j)} \stackrel{\text{def}}{=} \dot{x}_i^{(j)} = x_i^{(j+1)}$ , the Cartan field acts on coordinates as a shift to the right.  $\mathfrak{X}$  is thus called *manifold of jets of infinite order*.

A *regular implicit control system* is defined as a triple  $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$  with  $\mathfrak{X} = X \times \mathbb{R}_\infty^n$ ,  $\tau_{\mathfrak{X}}$  its associated trivial Cartan field, and  $F$  meromorphic from  $TX$  to  $\mathbb{R}^{n-m}$  satisfying  $\text{rk} \frac{\partial F}{\partial \bar{x}} = n - m$  in a suitable open subset of  $TX$ .

We next consider the cotangent space  $T_{\bar{x}}^* \mathfrak{X}$  with  $dx_i^{(j)}$ ,  $i = 1, \dots, n$ ,  $j \geq 0$  as basis, dual to the  $\frac{\partial}{\partial x_i^{(j)}}$ 's. 1-forms on  $\mathfrak{X}$  are then defined in the usual way. The set of 1-forms is noted  $\Lambda^1(\mathfrak{X})$ . We also denote by  $\Lambda^p(\mathfrak{X})$  the module of all the  $p$ -forms on  $\mathfrak{X}$ .

### 2.1 Flatness

We recall the following definitions and result [17,18,19]:

Given two regular implicit control systems  $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ , with  $\mathfrak{X} = X \times \mathbb{R}_\infty^n$ ,  $\dim X = n$  and  $\text{rk} \frac{\partial F}{\partial \bar{x}} = n - m$ , and  $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ , with  $\mathfrak{Y} = Y \times \mathbb{R}_\infty^p$ ,  $\dim Y = p$ ,  $\tau_{\mathfrak{Y}}$  its trivial Cartan field, and  $\text{rk} \frac{\partial G}{\partial \bar{y}} = p - q$ , we set  $\mathfrak{X}_0 = \{\bar{x} \in \mathfrak{X} | L_{\tau_{\mathfrak{X}}}^k F(\bar{x}) = 0, \forall k \geq 0\}$  and  $\mathfrak{Y}_0 = \{\bar{y} \in \mathfrak{Y} | L_{\tau_{\mathfrak{Y}}}^k G(\bar{y}) = 0, \forall k \geq 0\}$ . They are endowed with the topologies and differentiable structures induced by  $\mathfrak{X}$  and  $\mathfrak{Y}$  respectively.

**Definition 1** *The control systems  $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$  and  $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$  are said locally Lie-Bäcklund equivalent (or shortly L-B equivalent) in a neighbourhood  $\mathcal{X}_0 \times \mathcal{Y}_0$  of the pair  $(\bar{x}_0, \bar{y}_0) \in \mathfrak{X}_0 \times \mathfrak{Y}_0$  if and only if*

<sup>3</sup> From now on,  $\bar{x} \bar{y}, \dots$  stand for the sequences of jets of infinite order of  $x, y, \dots$

- (i) there exists a one-to-one meromorphic mapping  $\Phi = (\varphi, \dot{\varphi}, \dots)$  from  $\mathcal{Y}_0$  to  $\mathcal{X}_0$  satisfying  $\Phi(\bar{y}_0) = \bar{x}_0$  and such that  $\Phi_*\tau_{\mathcal{Y}} = \tau_{\mathcal{X}}$ ;  
(ii) there exists  $\Psi$  one-to-one and meromorphic from  $\mathcal{X}_0$  to  $\mathcal{Y}_0$ , with  $\Psi = (\psi, \dot{\psi}, \dots)$ , such that  $\Psi(\bar{x}_0) = \bar{y}_0$  and  $\Psi_*\tau_{\mathcal{X}} = \tau_{\mathcal{Y}}$ .
- The mappings  $\Phi$  and  $\Psi$  are called mutually inverse Lie-Bäcklund isomorphisms at  $(\bar{x}_0, \bar{y}_0)$ .

**Definition 2** The implicit system  $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$  is locally flat in a neighborhood of  $(\bar{x}_0, \bar{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}_{\infty}^m$  if and only if it is locally L-B equivalent around  $(\bar{x}_0, \bar{y}_0)$  to the trivial implicit system  $(\mathbb{R}_{\infty}^m, \tau_{\mathbb{R}_{\infty}^m}, 0)$ . In this case, the mutually inverse L-B isomorphisms  $\Phi$  and  $\Psi$  are called inverse trivializations.

**Theorem 1** The system  $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$  is locally flat at  $(\bar{x}_0, \bar{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}_{\infty}^m$  if and only if there exists a local meromorphic invertible mapping  $\Phi$  from  $\mathbb{R}_{\infty}^m$  to  $\mathfrak{X}_0$ , with meromorphic inverse, satisfying  $\Phi(\bar{y}_0) = \bar{x}_0$ , and such that<sup>4</sup>

$$\Phi^* dF = 0. \quad (3)$$

### 3 Necessary and Sufficient Conditions: Generalized Moving Frame Structure Equations

#### 3.1 Algebraic characterization of the differential of a trivialization

Consider the following matrix, polynomial with respect to the differential operator  $\frac{d}{dt}$  (we use indifferently  $\frac{d}{dt}$  for  $L_{\tau_{\mathfrak{X}}}$  or  $L_{\tau_{\mathbb{R}_{\infty}^m}}$ , the context being unambiguous):

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt}, \quad P(\varphi) = \sum_{j \geq 0} \frac{\partial \varphi}{\partial y^{(j)}} \frac{d^j}{dt^j} \quad (4)$$

with  $P(F)$  (resp.  $P(\varphi)$ ) of size  $(n - m) \times n$  (resp.  $n \times m$ ). Equation (3) reads:

$$\Phi^* dF = P(F)P(\varphi)dy = 0. \quad (5)$$

Clearly, the entries of the matrices in (4) are polynomials in the differential operator  $\frac{d}{dt}$  with meromorphic coefficients from  $\mathfrak{X}$  to  $\mathbb{R}$ . We denote by  $\mathfrak{K}$  the field of meromorphic functions from  $\mathfrak{X}$  to  $\mathbb{R}$  and by  $\mathfrak{K}[\frac{d}{dt}]$  the (non-commutative) principal ideal ring of polynomials in  $\frac{d}{dt}$  with coefficients in  $\mathfrak{K}$ . For  $r, s \in \mathbb{N}$ , let us denote by  $\mathcal{M}_{r,s}[\frac{d}{dt}]$  the module of  $r \times s$  matrices over  $\mathfrak{K}[\frac{d}{dt}]$  (see e.g. [8]). Matrices whose inverse belong to  $\mathcal{M}_{r,r}[\frac{d}{dt}]$  are called *unimodular matrices*. They form a multiplicative group denoted by  $\mathcal{U}_r[\frac{d}{dt}]$ .

<sup>4</sup> Note that if  $\Phi$  is a meromorphic mapping from  $\mathcal{Y}$  to  $\mathfrak{X}$ , the (backward) image by  $\Phi$  of a 1-form is defined in the same way as in the finite dimensional context.

Every matrix  $M \in \mathcal{M}_{r,s}[\frac{d}{dt}]$  admits a *Smith decomposition* (or diagonal reduction)

$$VMU = (\Delta, 0_{r,s-r}) \text{ if } r \leq s, \text{ and } \begin{pmatrix} \Delta \\ 0_{r-s,s} \end{pmatrix} \text{ if } s \leq r \quad (6)$$

with  $V \in \mathcal{U}_r[\frac{d}{dt}]$  and  $U \in \mathcal{U}_s[\frac{d}{dt}]$  and  $\Delta$  diagonal (see e.g. [8]).  $U$  and  $V$  are indeed non unique. We say that  $U \in \mathbf{R} - \mathbf{Smith}(M)$  and  $V \in \mathbf{L} - \mathbf{Smith}(M)$ .

A matrix  $M \in \mathcal{M}_{r,s}[\frac{d}{dt}]$  is said *hyper-regular* if and only if its Smith decomposition leads to  $\Delta = I$ . An interpretation of this property in terms of controllability in the sense of [9], may be found in [18].

From now on, we assume that  $P(F)$  is hyper-regular in a neighborhood of  $\bar{x}_0$ . In place of (5), we first solve the matrix equation:

$$P(F)\Theta = 0 \quad (7)$$

where  $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$  is not supposed to be of the form  $P(\varphi)$ . It may be verified that matrices  $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$  satisfying (7) have the structure

$$\Theta = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W \quad (8)$$

with  $U \in \mathbf{R} - \mathbf{Smith}(P(F))$  and  $W \in \mathcal{U}_m[\frac{d}{dt}]$  arbitrary. Clearly  $\Theta$  is itself hyper-regular and admits the Smith decomposition

$$Q\Theta Z = QU \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} WZ = Q\hat{U}R = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} \quad (9)$$

with  $Q \in \mathcal{U}_n[\frac{d}{dt}]$ ,  $Z \in \mathcal{U}_m[\frac{d}{dt}]$ ,  $R = WZ$  and  $\hat{U} = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}$ .

### 3.2 Integrability

We denote by  $\omega$  the  $m$ -dimensional vector 1-form defined by

$$\omega(\bar{x}) = \begin{pmatrix} \omega_1(\bar{x}) \\ \vdots \\ \omega_m(\bar{x}) \end{pmatrix} = (I_m, 0_{m,n-m}) Q(\bar{x}) dx|_{\mathcal{X}_0} \quad (10)$$

with  $Q$  given by (9), the restriction to  $\mathcal{X}_0$  meaning that  $\bar{x} \in \mathcal{X}_0$  satisfies  $L_{\tau_{\bar{x}}}^k F = 0$  for all  $k$  and that the  $dx_j^{(k)}$  are such that  $dL_{\tau_{\bar{x}}}^k F = 0$  in  $\mathcal{X}_0$  for all  $k$ . Since  $Q$  is hyper-regular, the forms  $\omega_1, \dots, \omega_m$  are independent by construction.

**Theorem 2** *A necessary and sufficient condition for system (1) to be locally flat around  $(\bar{x}_0, \bar{y}_0)$  is that there exist  $U \in \mathbf{R} - \mathbf{Smith}(P(F))$ ,  $Q \in \mathbf{L} - \mathbf{Smith}(\hat{U})$ , with  $\hat{U}$  given by (9) and a matrix  $M \in \mathcal{U}_m[\frac{d}{dt}]$  such that  $d(M\tau) = 0$ .*

We denote by  $(\Lambda^p(\mathfrak{X}))^m$  the space of  $m$ -dimensional vector  $p$ -forms on  $\mathfrak{X}$ , by  $(\Lambda(\mathfrak{X}))^m$  the space of  $m$ -dimensional vector forms of arbitrary degree on  $\mathfrak{X}$ , and by  $\mathcal{L}_q((\Lambda(\mathfrak{X}))^m) = \bigcup_{p \geq 1} \mathcal{L}((\Lambda^p(\mathfrak{X}))^m, (\Lambda^{p+q}(\mathfrak{X}))^m)$  the space of linear operators from  $(\Lambda^p(\mathfrak{X}))^m$  to  $(\Lambda^{p+q}(\mathfrak{X}))^m$  for all  $p \geq 1$ , where  $\mathcal{L}(\mathcal{P}, \mathcal{Q})$  denotes the set of linear mappings from a given space  $\mathcal{P}$  to a given space  $\mathcal{Q}$ .

In order to develop the expression  $d(\mu\kappa)$  for  $\mu \in \mathcal{L}_q((\Lambda(\mathfrak{X}))^m)$  and for all  $\kappa \in (\Lambda^p(\mathfrak{X}))^m$  and all  $p \geq 1$ , we define the operator  $\mathfrak{d}$  by:

$$\mathfrak{d}(\mu)\kappa = d(\mu\kappa) - (-1)^q \mu d\kappa. \quad (11)$$

Note that (11) uniquely defines  $\mathfrak{d}(\mu)$  as an element of  $\mathcal{L}_{q+1}((\Lambda(\mathfrak{X}))^m)$ .

**Theorem 3** *The system  $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$  is locally flat iff there locally exists  $\mu \in \mathcal{L}_1((\Lambda(\mathfrak{X}))^m)$ , and a matrix  $M \in \mathcal{U}_m[\frac{d}{dt}]$  such that*

$$d\omega = \mu\omega, \quad \mathfrak{d}(\mu) = \mu^2, \quad \mathfrak{d}(M) = -M\mu. \quad (12)$$

with the notation  $\mu^2 = \mu\mu$  and where  $\omega$  is defined by (10). In addition, if (12) holds true, a flat output  $y$  is obtained by integration of  $dy = M\omega$ .

**Remark 1** *Note that the two first conditions of (12) are comparable to conditions (A) and (B) of [6,7]. However, the last condition of (12) is different from condition (C) of [6,7] and is easier to check.*

*Note also that conditions (12) may be seen as a generalization in the framework of manifolds of jets of infinite order of Cartan's well-known moving frame structure equations (see e.g. [5]).*

### 3.3 A Sequential Procedure

We start with  $P(F)$  hyper-regular and compute the vector 1-form  $\omega$  defined by (10).

1. We identify the operator  $\mu$  such that  $d\omega = \mu\omega$  componentwise. It is proven in [19] that such  $\mu$  always exists.
2. Among the possible  $\mu$ 's, only those satisfying  $\mathfrak{d}(\mu) = \mu^2$  are kept. It is shown in [19] that such  $\mu$  always exists.
3. We then identify  $M$  such that  $\mathfrak{d}(M) = -M\mu$  componentwise.
4. If, among such  $M$ 's, there is a unimodular one, the system is flat and a flat output is obtained by integration of  $dy = M\omega$ . Otherwise the system is not flat.

More details and examples may be found in [18,19].

## 4 Necessary and Sufficient Conditions using the Generalized Euler-Lagrange Operator

Another way of analysing (3) consists in characterizing the change of coordinates corresponding to the mapping  $\Phi$  in (3). More precisely (3) reads

$$\sum_{j=1}^m \sum_{k=0}^{r_j} \left( \frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j^{(k)}} dy_j^{(k)} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt} \left( \frac{\partial \varphi}{\partial y_j^{(k)}} \right) dy_j^{(k)} + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(k)}} dy_j^{(k+1)} \right) = 0 \quad (13)$$



Since the one forms  $dy_1, \dots, dy_1^{(r_1)}, \dots, dy_m, \dots, dy_m^{(r_m)}$  are independent by assumption, (13) yields, for every  $j = 1, \dots, m$ ,

$$\begin{cases} \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(r_j)}} = 0 \\ \frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j^{(k)}} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt} \left( \frac{\partial \varphi}{\partial y_j^{(k)}} \right) + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(k-1)}} = 0, \quad \forall k = 1, \dots, r_j \\ \frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt} \left( \frac{\partial \varphi}{\partial y_j} \right) = 0 \end{cases} \quad (14)$$

The Generalized Euler-Lagrange operator  $\mathcal{E}_F$  associated to  $F$  is defined by

$$\mathcal{E}_F = \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}} \right) \quad (15)$$

In the case  $n - m = 1$ , it is well-known that the curves that extremize the cost function  $J = \int_0^T F(x, \dot{x}) dt$  are those satisfying the Euler-Lagrange equation  $\mathcal{E}_F = 0$ , which justifies our terminology. Using (15) and elementary calculus, (14) yields:

**Theorem 4** *A necessary and sufficient condition for (1) to be differentially flat is that there exist  $(r_1, \dots, r_m)$  with  $\sum_{i=1}^m r_i + m \geq n$  and a solution  $\varphi$  of the following triangular system of PDEs in an open dense subset of  $\mathfrak{X}$*

$$\begin{cases} \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(r_j)}} = 0 \\ \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(l)}} = \sum_{k=0}^{r_j-l-1} (-1)^{k+1} \frac{d^k}{dt^k} \left( \mathcal{E}_F \frac{\partial \varphi}{\partial y_j^{(l+k+1)}} \right), \quad \forall l = 0, \dots, r_j - 1, \\ 0 = \sum_{k=0}^{r_j} (-1)^k \frac{d^k}{dt^k} \left( \mathcal{E}_F \frac{\partial \varphi}{\partial y_j^{(k)}} \right) \end{cases} \quad (16)$$

satisfying  $d\varphi_1 \wedge \dots \wedge d\varphi_n \neq 0$ .

**Remark 2** *If there exists a coordinate transformation  $\varphi$  that satisfies the conditions of Theorem 4 with given  $r_1, \dots, r_m$ , meaning that the system is flat, then  $g_j = \sum_{i=1}^n \frac{\partial \varphi_i}{\partial y_j^{(r_j)}} \frac{\partial}{\partial \dot{x}_i}$ , if non zero, defines a ruled direction [32,25,19].*

## 5 Examples

### 5.1 An Academic Example: Generalized Moving Frame Approach

We consider the 3-dimensional system with 2 inputs:

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = \sin \left( \frac{u_1}{u_2} \right) \quad (17)$$

or, in implicit form:

$$F(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) \triangleq \dot{x}_3 - \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right) = 0. \quad (18)$$

It is readily seen that  $P(F) = \left[ -\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\dot{x}_2^{-1}\frac{d}{dt} \middle| \dot{x}_1 \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\dot{x}_2^{-2}\frac{d}{dt} \middle| \frac{d}{dt} \right]$  and that  $VP(F)U = (1 \ 0 \ 0)$  with

$$V = 1, \quad U = \left( \begin{array}{c|c} \frac{\dot{x}_1}{a\dot{x}_2} & 1 + \frac{\dot{x}_1}{a(\dot{x}_2)^2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \frac{d}{dt} \\ \frac{1}{a} & \frac{1}{a\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \frac{d}{dt} \\ 0 & 0 \end{array} \middle| \begin{array}{c} \frac{\dot{x}_1}{a\dot{x}_2} \frac{d}{dt} \\ -\frac{1}{a} \frac{d}{dt} \\ 1 \end{array} \right) \quad (19)$$

where  $a = -\frac{1}{\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \left(\frac{\dot{x}_1\dot{x}_2 - \dot{x}_1\dot{x}_2}{(\dot{x}_2)^2}\right)$ . Then,  $Q\hat{U}R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$  is computed with

$$Q = \left( \begin{array}{c|c} 1 & -\frac{\dot{x}_1}{\dot{x}_2} \\ 0 & 0 \\ -\frac{1}{a\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \frac{d}{dt} & \frac{\dot{x}_1}{a(\dot{x}_2)^2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \frac{d}{dt} \end{array} \middle| \begin{array}{c} 0 \\ 1 \\ \frac{1}{a} \frac{d}{dt} \end{array} \right), \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (20)$$

So,  $(\omega_1 \ \omega_2)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} Q dx = \left( dx_1 - \frac{\dot{x}_1}{\dot{x}_2} dx_2 \ \ dx_3 \right)^T$  and  $d\omega = \left( \frac{1}{\sqrt{1-(\dot{x}_3)^2}} dx_2 \wedge dx_3 \ 0 \right)^T$ . According to section 3.3, step 1,

$$\mu = \begin{pmatrix} 0 & \left( -\frac{\dot{x}_3}{(1-(\dot{x}_3)^2)^{\frac{3}{2}}} dx_2 \wedge d\dot{x}_3 + \eta d\dot{x}_3 \right) \wedge \frac{d}{dt} \\ 0 & 0 \end{pmatrix}. \quad (21)$$

Step 2 yields  $\eta = \frac{x_2\dot{x}_3}{(1-\dot{x}_3)^{\frac{3}{2}}} + \sigma(\dot{x}_3)$ . For step 3 we set  $M = \begin{pmatrix} 1 & m_{12} \frac{d}{dt} \\ 0 & 1 \end{pmatrix}$  which yields  $m_{12} = -\left( \frac{x_2}{\sqrt{1-(\dot{x}_3)^2}} + \sigma_1(\dot{x}_3) \right)$  with  $\sigma_1$  a primitive of  $\sigma$ . Thus,  $d(M\omega) = 0$  and setting  $(dy_1 \ dy_2)^T = M\omega$ , one obtains

$$y_1 = x_1 - \frac{\dot{x}_1}{\dot{x}_2} x_2 + \sigma_2(\dot{x}_3), \quad y_2 = x_3 \quad (22)$$

where  $\sigma_2(\dot{x}_3)$  is an arbitrary meromorphic function (a primitive of  $\sigma_1$ ). By inversion of (22) we get

$$\begin{aligned} x_1 &= y_1 - \arcsin(\dot{y}_2) \frac{\sqrt{1-(\dot{y}_2)^2}}{\dot{y}_2} (\dot{y}_1 - \sigma_1(\dot{y}_2)\dot{y}_2) - \sigma_2(\dot{y}_2) \\ x_2 &= -\frac{\sqrt{1-(\dot{y}_2)^2}}{\dot{y}_2} (\dot{y}_1 - \sigma_1(\dot{y}_2)\dot{y}_2) \\ x_3 &= y_2 \end{aligned} \quad (23)$$

## 5.2 Academic Example: Euler-Lagrange Operator

We consider once more the example (18). We have

$$\frac{\partial F}{\partial \dot{x}} = \left( -\dot{x}_2^{-1} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), \dot{x}_1 \dot{x}_2^{-2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), 1 \right), \quad \mathcal{E}_F = (\eta_1, \eta_2, 0) \quad (24)$$

with  $\eta_1 = -\frac{\ddot{x}_2}{\dot{x}_2^2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) - \frac{\dot{x}_1 \dot{x}_2 - \dot{x}_1 \ddot{x}_2}{\dot{x}_2^3} \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right)$  and  
 $\eta_2 = -\frac{\dot{x}_1 \dot{x}_2 - 2\dot{x}_1 \ddot{x}_2}{\dot{x}_2^3} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) + \frac{\dot{x}_1(\dot{x}_1 \dot{x}_2 - \dot{x}_1 \ddot{x}_2)}{\dot{x}_2^4} \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right)$ .

The first two equations of (16), with  $r_1 = r_2 = 2$ , read

$$-\frac{1}{\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \left( \frac{\partial \varphi_1}{\partial \dot{y}_j} - \frac{\dot{x}_1}{\dot{x}_2} \frac{\partial \varphi_2}{\partial \dot{y}_j} \right) + \frac{\partial \varphi_3}{\partial \dot{y}_j} = 0, \quad j = 1, 2 \quad (25)$$

If we assume that  $\frac{\partial \varphi_3}{\partial \dot{y}_j} = \frac{\partial \varphi_3}{\partial \dot{y}_j} = 0$ ,  $j = 1, 2$  and introduce the variable

$$\psi = \frac{\dot{x}_1}{\dot{x}_2} \quad (26)$$

with  $\frac{\partial}{\partial \dot{y}} \psi = 0$  we obtain from (25)

$$\frac{\partial \varphi_1}{\partial \dot{y}_j} - \psi \frac{\partial \varphi_2}{\partial \dot{y}_j} = \frac{\partial}{\partial \dot{y}_j} (\varphi_1 - \psi \varphi_2) = 0, \quad j = 1, 2$$

Setting  $\kappa(y, \dot{y}) = \varphi_1 - \psi \varphi_2$ , we get

$$\dot{\kappa} = \dot{\varphi}_1 - \psi \dot{\varphi}_2 - \dot{\psi} \varphi_2 = -\dot{\psi} \varphi_2 \quad (27)$$

Using the definition of  $\kappa$  and (27) we obtain:

$$\varphi_1 = \kappa - \frac{\dot{\kappa} \sqrt{1 - \dot{\varphi}_3}}{\dot{\varphi}_3} \arcsin(\dot{\varphi}_3), \quad \varphi_2 = -\frac{\dot{\kappa}}{\dot{\varphi}_3} \sqrt{1 - \dot{\varphi}_3}, \quad \varphi_3 = \varphi_3(y) \quad (28)$$

Choosing  $\varphi_3 = y_2$ ,  $\kappa = y_1$ , we arrive at the invertible transformation

$$x_1 = \varphi_1 = y_1 - \frac{\dot{y}_1}{\dot{y}_2} \sqrt{1 - \dot{y}_2^2} \arcsin(\dot{y}_2), \quad x_2 = \varphi_2 = -\frac{\dot{y}_1}{\dot{y}_2} \sqrt{1 - \dot{y}_2^2},$$

with  $x_3 = \varphi_3 = y_2$ , which gives the same formula as (23) with  $\sigma_1 = \sigma_2 = 0$ . Hence  $(y_1, y_2)$  is indeed a flat output, which implies that the remaining equations of (16) are satisfied.

## 5.3 An Example Proposed by P. Rouchon

Consider the implicit control system

$$F(x, \dot{x}) = \dot{x}_1 \dot{x}_3 - (\dot{x}_2)^2 = 0. \quad (29)$$

We thus have  $\frac{\partial F}{\partial x} = (0 \quad 0 \quad 0)$ ,  $\frac{\partial F}{\partial \dot{x}} = (\dot{x}_3 \quad -2\dot{x}_2 \quad \dot{x}_1)$  and

$$\mathcal{E}_F = \frac{\partial F}{\partial x} - \frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}} \right) = -\frac{d}{dt} \left( \frac{\partial F}{\partial \dot{x}} \right) = (-\ddot{x}_3 \quad 2\ddot{x}_2 \quad -\ddot{x}_1).$$

The lowest possible choice of  $(r_1, r_2)$  in Theorem 4 is  $r_1 = r_2 = 1$ . However, there is no solution of (16) for these values, and we choose  $r_1 = r_2 = 2$ . The two first equations of (16) read

$$\dot{\varphi}_3 \frac{\partial \varphi_1}{\partial \ddot{y}_j} - 2\dot{\varphi}_2 \frac{\partial \varphi_2}{\partial \ddot{y}_j} + \dot{\varphi}_1 \frac{\partial \varphi_3}{\partial \ddot{y}_j} = 0, \quad j = 1, 2 \quad (30)$$

We divide (30) by  $\dot{\varphi}_3$  to obtain

$$\frac{\partial \varphi_1}{\partial \ddot{y}_j} - 2\psi \frac{\partial \varphi_2}{\partial \ddot{y}_j} + \psi^2 \frac{\partial \varphi_3}{\partial \ddot{y}_j} = 0, \quad j = 1, 2 \quad (31)$$

where, taking account of the system equation (29),

$$\psi = \frac{\dot{\varphi}_2}{\dot{\varphi}_3} = \sqrt{\frac{\dot{\varphi}_1}{\dot{\varphi}_3}}. \quad (32)$$

If we assume that  $\psi$  doesn't depend on  $\dot{y}_1$  and  $\dot{y}_2$ , equation (31) reads  $\frac{\partial}{\partial \ddot{y}_j} (\varphi_1 - 2\psi\varphi_2 + \psi^2\varphi_3) = 0$ , for  $j = 1, 2$ . In other words, there exists a function  $\kappa$  satisfying  $\frac{\partial \kappa}{\partial \ddot{y}_j} = 0$  for  $j = 1, 2$ , such that

$$\varphi_1 - 2\psi\varphi_2 + \psi^2\varphi_3 = \kappa \quad (33)$$

Differentiating the latter relation with respect to  $t$ , and taking into account the relation  $\dot{\varphi}_1 - 2\psi\dot{\varphi}_2 + \psi^2\dot{\varphi}_3 = 0$  obtained from (29) and (32), we get

$$\varphi_2 - \psi\varphi_3 = -\frac{\dot{\kappa}}{2\dot{\psi}}. \quad (34)$$

We again differentiate the latter relation with respect to  $t$  to obtain

$$\varphi_3 = \frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \quad (35)$$

thanks to  $\dot{\varphi}_2 - \psi\dot{\varphi}_3 = 0$  from (32). Thus, solving the system (33)–(35), we immediately obtain

$$\begin{aligned} \varphi_1 &= \kappa - \psi \frac{\dot{\kappa}}{\dot{\psi}} + \psi^2 \left( \frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \right) \\ \varphi_2 &= -\frac{\dot{\kappa}}{2\dot{\psi}} + \psi \left( \frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \right) \\ \varphi_3 &= \frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \end{aligned} \quad (36)$$

where  $\kappa$  and  $\psi$  are arbitrary functions of  $y_1, y_2, \dot{y}_1, \dot{y}_2$ .

Note that choosing  $\kappa = y_1$  and  $\psi = y_2$  yields, after inversion of (36) with (32):

$$y_1 = x_1 - 2x_2 \frac{\dot{x}_2}{\dot{x}_3} + x_3 \frac{\dot{x}_1}{\dot{x}_3}, \quad y_2 = \frac{\dot{x}_2}{\dot{x}_3},$$

which is similar to the solution obtained by F. Ollivier<sup>5</sup>.

Similarly, the solution of K. Schlacher and M. Schöberl [29] may be

<sup>5</sup> personal communication

recovered by posing  $\kappa = y_1 - y_2 \frac{\dot{y}_1}{y_2}$  and  $\psi = \frac{\dot{y}_1}{2y_2}$  which, again after inversion of (36) with (32), yields:

$$y_1 = x_1 - x_3 \frac{\dot{x}_1}{\dot{x}_3}, \quad y_2 = x_2 - x_3 \frac{\dot{x}_2}{\dot{x}_3}.$$

## 6 Conclusion

In this survey we presented two dual approaches to flatness necessary and sufficient conditions, one based on the integration of 1-forms and the second based on the integration of a set of PDEs involving a generalized Euler-Lagrange operator. Their complexity is compared on examples.

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