Flatness Characterization: Two Approaches

Felix Antritter, Jean Lévine

HAL Id: hal-00575197

https://minesparis-psl.hal.science/hal-00575197

Submitted on 9 Mar 2011

Table of Contents

**Advances in the Theory of Control, Signals and
Systems with Physical Modeling**

VI

Flatness Characterization: Two Approaches

Felix Antritter[1] and Jean Lévine[2]

[1] Automatisierungs- und Regelungstechnik, Universität der Bundeswehr München,
Werner-Heisenberg-Weg 37, DE-85579 Neubiberg, Germany
`felix.antritter@unibw.de`
[2] Mines ParisTech, CAS- Centre Automatique et Systèmes, Mathématiques et
Systèmes, 35, rue Saint-Honoré, 77305, Fontainebleau, France
`jean.levine@mines-paristech.fr`

Abstract. We survey two approaches to flatness necessary and sufficient
conditions and compare them on examples.

1 Introduction

In this survey we consider underdetermined implicit systems of the form

$$F(x, \dot{x}) = 0 \tag{1}$$

with $x \in X$, X being an inifnitely differentiable manifold of dimension
n, whose tangent bundle is denoted by TX, and $F : TX \to \mathbb{R}^{n-m}$
regular in the sense that rk $\frac{\partial F}{\partial \dot{x}} = n - m$ in a suitable open dense subset
of TX. Differential flatness, or more shortly, flatness was introduced in
1992 [20,11]. In the setting of implicit control systems it may be roughly
described as follows: there exists a smooth mapping $x = \varphi(y, \dot{y}, \ldots, y^{(r)})$
with $y = (y_1, \ldots, y_m)^T$ of dimension m, $r = (r_1, \ldots, r_m)^T \in \mathbb{N}^m$, such
that

$$F(\varphi(y, \dot{y}, \ldots, y^{(r)}), \dot{\varphi}(y, \dot{y}, \ldots, y^{(r+1)})) \equiv 0 \tag{2}$$

with φ invertible in the sense that there exists a locally defined smooth
mapping ψ and a multi-index s such that $y = \psi(x, \dot{x}, \ldots, x^{(s)})$.
The vector y is called a *flat output*.
This concept has inspired an important literature. See [10,21,19,26,27,31]
for surveys on flatness and its applications. Various formalisms have been
introduced: finite dimensional differential geometric approaches [4,14,30],
[32,28], differential algebra and related approaches [12,3,15], infinite di-
mensional differential geometry of jets and prolongations [13,33,19,6,7,23],
[22,24], which is adopted here. The interested reader may refer to [1,13,16],
[19,23,34] for more details.
The first part of the paper recalls the mathematical setting. In Section 3
the approch introduced in [19,2] for the characterization of differentially
flat systems is recalled. Then, in Section 4, we introduce a novel charac-
terization using the so-called Generalized Euler-Lagrange Operator. We
conclude the paper with examples.

2 Implicit control systems on manifolds of jets of infinite order

Given an infinitely differentiable manifold X of dimension n, we denote its tangent space at $x \in X$ by $T_x X$, and its tangent bundle by TX. Let F be a meromorphic function from TX to $\mathbb{R}^{n-m}$. We consider an underdetermined implicit system of the form (1) regular in the sense that rk $\frac{\partial F}{\partial \dot{x}} = n - m$ in a suitable open dense subset of TX.

Following [17,18], we consider the infinite dimensional manifold $\mathfrak{X}$ defined by $\mathfrak{X} \stackrel{\text{def}}{=} X \times \mathbb{R}^n_\infty \stackrel{\text{def}}{=} X \times \mathbb{R}^n \times \mathbb{R}^n \times \ldots$, made of an infinite (but countable) number of copies of $\mathbb{R}^n$, with the global infinite set of coordinates[3] $\overline{x} = \left(x, \dot{x}, \ldots, x^{(k)}, \ldots, \right)$, endowed with the product topology. Recall that, in this topology, a function φ from $\mathfrak{X}$ to $\mathbb{R}$ is *continuous* (resp. *differentiable*) if φ depends only on a finite (but otherwise arbitrary) number of variables and is continuous (resp. differentiable) with respect to these variables. C^∞ or analytic or meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ are then defined as in the usual finite dimensional case since they only depend on a finite number of variables. We endow $\mathfrak{X}$ with the so-called trivial Cartan field ([16,34]) $\tau_{\mathfrak{X}} = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}}$. We also denote by $L_{\tau_{\mathfrak{X}}} \gamma = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial \gamma}{\partial x_i^{(j)}} = \frac{d\gamma}{dt}$ the Lie derivative of a differentiable function γ along $\tau_{\mathfrak{X}}$ and $L_{\tau_{\mathfrak{X}}}^k \gamma$ its kth iterate. Since $\frac{d}{dt} x_i^{(j)} \stackrel{\text{def}}{=} \dot{x}_i^{(j)} = x_i^{(j+1)}$, the Cartan field acts on coordinates as a shift to the right. $\mathfrak{X}$ is thus called *manifold of jets of infinite order*.

A regular implicit control system is defined as a triple $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *with* $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\tau_{\mathfrak{X}}$ *its associated trivial Cartan field, and* F *meromorphic from* TX *to* $\mathbb{R}^{n-m}$) *satisfying* rk $\frac{\partial F}{\partial \dot{x}} = n - m$ *in a suitable open subset of* TX.

We next consider the cotangent space $T^*_{\overline{x}} \mathfrak{X}$ with $dx_i^{(j)}$, $i = 1, \ldots, n$, $j \geq 0$ as basis, dual to the $\frac{\partial}{\partial x_i^{(j)}}$´s. 1-forms on $\mathfrak{X}$ are then defined in the usual way. The set of 1-forms is noted $\Lambda^1(\mathfrak{X})$. We also denote by $\Lambda^p(\mathfrak{X})$ the module of all the p-forms on $\mathfrak{X}$.

2.1 Flatness

We recall the following definitions and result [17,18,19]:

Given two regular implicit control systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$, with $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\dim X = n$ and rk $\frac{\partial F}{\partial \dot{x}} = n - m$, and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$, with $\mathfrak{Y} = Y \times \mathbb{R}^p_\infty$, $\dim Y = p$, $\tau_{\mathfrak{Y}}$ its trivial Cartan field, and rk $\frac{\partial G}{\partial \dot{y}} = p - q$, we set $\mathfrak{X}_0 = \{ \overline{x} \in \mathfrak{X} | L_{\tau_{\mathfrak{X}}}^k F(\overline{x}) = 0, \forall k \geq 0 \}$ and $\mathfrak{Y}_0 = \{ \overline{y} \in \mathfrak{Y} | L_{\tau_{\mathfrak{Y}}}^k G(\overline{y}) = 0, \forall k \geq 0 \}$. They are endowed with the topologies and differentiable structures induced by $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.

Definition 1 *The control systems* $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *and* $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ *are said* locally Lie-Bäcklund equivalent *(or shortly L-B equivalent) in a neighbourhood* $\mathfrak{X}_0 \times \mathfrak{Y}_0$ *of the pair* $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathfrak{Y}_0$ *if and only if*

---

[3] From now on, $\overline{x}\ \overline{y}$, … stand for the sequences of jets of infinite order of x, y,…

(i) there exists a one-to-one meromorphic mapping $\Phi = (\varphi, \dot{\varphi}, \ldots)$ from $\mathcal{Y}_0$ to $\mathcal{X}_0$ satisfying $\Phi(\overline{y}_0) = \overline{x}_0$ and such that $\Phi_* \tau_{\mathfrak{Y}} = \tau_{\mathfrak{X}}$;

(ii) there exists Ψ one-to-one and meromorphic from $\mathcal{X}_0$ to $\mathcal{Y}_0$, with $\Psi = (\psi, \dot{\psi}, \ldots)$, such that $\Psi(\overline{x}_0) = \overline{y}_0$ and $\Psi_* \tau_{\mathfrak{X}} = \tau_{\mathfrak{Y}}$.

The mappings Φ and Ψ are called mutually inverse Lie-Bäcklund isomorphisms *at $(\overline{x}_0, \overline{y}_0)$.*

Definition 2 *The implicit system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is* locally flat *in a neighborhood of $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}_\infty^m$ if and only if it is locally L-B equivalent around $(\overline{x}_0, \overline{y}_0)$ to the trivial implicit system $(\mathbb{R}_\infty^m, \tau_{\mathbb{R}_\infty^m}, 0)$. In this case, the mutually inverse L-B isomorphisms Φ and Ψ are called* inverse trivializations.

Theorem 1 *The system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is locally flat at $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}_\infty^m$ if and only if there exists a local meromorphic invertible mapping Φ from $\mathbb{R}_\infty^m$ to $\mathfrak{X}_0$, with meromorphic inverse, satisfying $\Phi(\overline{y}_0) = \overline{x}_0$, and such that*[4]

$$\Phi^* dF = 0. \tag{3}$$

3 Necessary and Sufficient Conditions: Generalized Moving Frame Structure Equations

3.1 Algebraic characterization of the differential of a trivialization

Consider the following matrix, polynomial with respect to the differential operator $\frac{d}{dt}$ (we use indifferently $\frac{d}{dt}$ for $L_{\tau_{\mathfrak{X}}}$ or $L_{\tau_{\mathbb{R}_\infty^m}}$, the context being unambiguous):

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt}, \quad P(\varphi) = \sum_{j \geq 0} \frac{\partial \varphi}{\partial y^{(j)}} \frac{d^j}{dt^j} \tag{4}$$

with $P(F)$ (resp. $P(\varphi)$) of size $(n-m) \times n$ (resp. $n \times m$).
Equation (3)) reads:

$$\Phi^* dF = P(F) P(\varphi) dy = 0. \tag{5}$$

Clearly, the entries of the matrices in (4) are polynomials in the differential operator $\frac{d}{dt}$ with meromorphic coefficients from $\mathfrak{X}$ to $\mathbb{R}$. We denote by $\mathfrak{K}$ the field of meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ and by $\mathfrak{K}[\frac{d}{dt}]$ the (non-commutative) principal ideal ring of polynomials in $\frac{d}{dt}$ with coefficients in $\mathfrak{K}$. For $r, s \in \mathbb{N}$, let us denote by $\mathcal{M}_{r,s}[\frac{d}{dt}]$ the module of $r \times s$ matrices over $\mathfrak{K}[\frac{d}{dt}]$ (see e.g. [8]). Matrices whose inverse belong to $\mathcal{M}_{r,r}[\frac{d}{dt}]$ are called *unimodular matrices* . They form a multiplicative group denoted by $\mathcal{U}_r[\frac{d}{dt}]$.

---

[4] Note that if Φ is a meromorphic mapping from $\mathfrak{Y}$ to $\mathfrak{X}$, the (backward) image by Φ of a 1-form is defined in the same way as in the finite dimensional context.

Every matrix $M \in \mathcal{M}_{r,s}[\frac{d}{dt}]$ admits a *Smith decomposition* (or diagonal reduction)

$$VMU = (\Delta, 0_{r,s-r}) \text{ if } r \leq s, \text{ and } \begin{pmatrix} \Delta \\ 0_{r-s,s} \end{pmatrix} \text{ if } s \leq r \qquad (6)$$

with $V \in \mathcal{U}_r[\frac{d}{dt}]$ and $U \in \mathcal{U}_s[\frac{d}{dt}]$ and Δ diagonal (see e.g. [8]). U and V are indeed non unique. We say that $U \in \mathsf{R} - \mathsf{Smith}\,(M)$ and $V \in \mathsf{L} - \mathsf{Smith}\,(M)$.

A matrix $M \in \mathcal{M}_{r,s}[\frac{d}{dt}]$ is said *hyper-regular* if and only if its Smith decomposition leads to $\Delta = I$. An interpretation of this property in terms of controllability in the sense of [9], may be found in [18].

From now on, we assume that $P(F)$ is hyper-regular in a neighborhood of $\overline{x}_0$. In place of (5), we first solve the matrix equation:

$$P(F)\Theta = 0 \qquad (7)$$

where $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ is not supposed to be of the form $P(\varphi)$. It may be verified that matrices $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ satisfying (7) have the structure

$$\Theta = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W \qquad (8)$$

with $U \in \mathsf{R} - \mathsf{Smith}\,(P(F))$ and $W \in \mathcal{U}_m[\frac{d}{dt}]$ arbitrary. Clearly Θ is itself hyper-regular and admits the Smith decomposition

$$Q\Theta Z = QU \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} WZ = Q\hat{U}R = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} \qquad (9)$$

with $Q \in \mathcal{U}_n[\frac{d}{dt}]$, $Z \in \mathcal{U}_m[\frac{d}{dt}]$, $R = WZ$ and $\hat{U} = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}$.

3.2 Integrability

We denote by ω the m-dimensional vector 1-form defined by

$$\omega(\overline{x}) = \begin{pmatrix} \omega_1(\overline{x}) \\ \vdots \\ \omega_m(\overline{x}) \end{pmatrix} = (I_m, 0_{m,n-m})\, Q(\overline{x})dx\big|_{\mathfrak{X}_0} \qquad (10)$$

with Q given by (9), the restriction to $\mathfrak{X}_0$ meaning that $\overline{x} \in \mathfrak{X}_0$ satisfies $L_{\tau_{\mathfrak{X}}}^k F = 0$ for all k and that the $dx_j^{(k)}$ are such that $dL_{\tau_{\mathfrak{X}}}^k F = 0$ in $\mathfrak{X}_0$ for all k. Since Q is hyper-regular, the forms $\omega_1, \dots, \omega_m$ are independent by construction.

Theorem 2 *A necessary and sufficient condition for system (1) to be locally flat around $(\overline{x}_0, \overline{y}_0)$ is that there exist $U \in \mathsf{R} - \mathsf{Smith}\,(P(F))$, $Q \in \mathsf{L} - \mathsf{Smith}\left(\hat{U}\right)$, with $\hat{U}$ given by (9) and a matrix $M \in \mathcal{U}_m[\frac{d}{dt}]$ such that $d(M\tau) = 0$.*

We denote by $(\Lambda^p(\mathfrak{X}))^m$ the space of m-dimensional vector p-forms on $\mathfrak{X}$, by $(\Lambda(\mathfrak{X}))^m$ the space of m-dimensional vector forms of arbitrary degree on $\mathfrak{X}$, and by $\mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right) = \bigcup_{p \geq 1} \mathcal{L}\left((\Lambda^p(\mathfrak{X}))^m, \left(\Lambda^{p+q}(\mathfrak{X})\right)^m\right)$ the space of linear operators from $(\Lambda^p(\mathfrak{X}))^m$ to $\left(\Lambda^{p+q}(\mathfrak{X})\right)^m$ for all $p \geq 1$, where $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ denotes the set of linear mappings from a given space $\mathcal{P}$ to a given space $\mathcal{Q}$.

In order to develop the expression $d(\mu\kappa)$ for $\mu \in \mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right)$ and for all $\kappa \in (\Lambda^p(\mathfrak{X}))^m$ and all $p \geq 1$, we define the operator $\mathfrak{d}$ by:

$$\mathfrak{d}\left(\mu\right)\kappa = d(\mu\ \kappa) - (-1)^q \mu\ d\kappa. \tag{11}$$

Note that (11) uniquely defines $\mathfrak{d}\left(\mu\right)$ as an element of $\mathcal{L}_{q+1}\left((\Lambda(\mathfrak{X}))^m\right)$.

Theorem 3 *The system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is locally flat iff there locally exists $\mu \in \mathcal{L}_1\left((\Lambda(\mathfrak{X}))^m\right)$, and a matrix $M \in \mathfrak{U}_m[\frac{d}{dt}]$ such that*

$$d\omega = \mu\ \omega, \qquad \mathfrak{d}\left(\mu\right) = \mu^2, \qquad \mathfrak{d}\left(M\right) = -M\mu. \tag{12}$$

with the notation $\mu^2 = \mu\mu$ and where ω is defined by (10). In addition, if (12) holds true, a flat output y is obtained by integration of $dy = M\omega$.

Remark 1 *Note that the two first conditions of (12) are comparable to conditions (A) and (B) of [6,7]. However, the last condition of (12) is different from condition (C) of [6,7] and is easier to check.*
Note also that conditions (12) may be seen as a generalization in the framework of manifolds of jets of infinite order of Cartan's well-known moving frame structure equations (see e.g. [5]).

3.3 A Sequential Procedure

We start with $P(F)$ hyper-regular and compute the vector 1-form ω defined by (10).

1. We identify the operator μ such that $d\omega = \mu\omega$ componentwise. It is proven in [19] that such μ always exists.
2. Among the possible μ's, only those satisfying $\mathfrak{d}\left(\mu\right) = \mu^2$ are kept. It is shown in [19] that such μ always exists.
3. We then identify M such that $\mathfrak{d}\left(M\right) = -M\mu$ componentwise.
4. If, among such M's, there is a unimodular one, the system is flat and a flat output is obtained by integration of $dy = M\omega$. Otherwise the system is not flat.

More details and examples may be found in [18,19].

4 Necessary and Sufficient Conditions using the Generalized Euler-Lagrange Operator

Another way of analysing (3) consists in characterizing the change of coordinates corresponding to the mapping Φ in (3). More precisely (3) reads

$$\sum_{j=1}^{m} \sum_{k=0}^{r_j} \left(\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j^{(k)}} dy_j^{(k)} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt}\left(\frac{\partial \varphi}{\partial y_j^{(k)}} \right) dy_j^{(k)} + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(k)}} dy_j^{(k+1)} \right) = 0 \tag{13}$$

Since the one forms $dy_1, \ldots, dy_1^{(r_1)}, \ldots, dy_m, \ldots, dy_m^{(r_m)}$ are independent by assumption, (13) yields, for every $j = 1, \ldots, m$,

$$\begin{cases} \dfrac{\partial F}{\partial \dot{x}} \dfrac{\partial \varphi}{\partial y_j^{(r_j)}} = 0 \\ \dfrac{\partial F}{\partial x} \dfrac{\partial \varphi}{\partial y_j^{(k)}} + \dfrac{\partial F}{\partial \dot{x}} \dfrac{d}{dt}\left(\dfrac{\partial \varphi}{\partial y_j^{(k)}}\right) + \dfrac{\partial F}{\partial \dot{x}} \dfrac{\partial \varphi}{\partial y_j^{(k-1)}} = 0, \quad \forall k = 1, \ldots, r_j \\ \dfrac{\partial F}{\partial x} \dfrac{\partial \varphi}{\partial y_j} + \dfrac{\partial F}{\partial \dot{x}} \dfrac{d}{dt}\left(\dfrac{\partial \varphi}{\partial y_j}\right) = 0 \end{cases}$$

$$(14)$$

The Generalized Euler-Lagrange operator $\mathcal{E}_F$ associated to F is defined by

$$\mathcal{E}_F = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) \tag{15}$$

In the case $n - m = 1$, it is well-known that the curves that extremize the cost function $J = \int_0^T F(x, \dot{x})dt$ are those satisfying the Euler-Lagrange equation $\mathcal{E}_F = 0$, which justifies our terminology.

Using (15) and elementary calculus, (14) yields:

Theorem 4 *A necessary and sufficient condition for (1) to be difffer-entially flat is that there exist $(r_1, \ldots, r_m)$ with $\sum_{i=1}^m r_i + m \geq n$ and a solution φ of the following triangular system of PDEs in an open dense subset of $\mathfrak{X}$*

$$\begin{cases} \dfrac{\partial F}{\partial \dot{x}} \dfrac{\partial \varphi}{\partial y_j^{(r_j)}} = 0 \\ \dfrac{\partial F}{\partial \dot{x}} \dfrac{\partial \varphi}{\partial y_j^{(l)}} = \displaystyle\sum_{k=0}^{r_j-l-1} (-1)^{k+1} \dfrac{d^k}{dt^k}\left(\mathcal{E}_F \dfrac{\partial \varphi}{\partial y_j^{(l+k+1)}}\right), \quad \forall l = 0, \ldots, r_j - 1 \\ 0 = \displaystyle\sum_{k=0}^{r_j} (-1)^k \dfrac{d^k}{dt^k}\left(\mathcal{E}_F \dfrac{\partial \varphi}{\partial y_j^{(k)}}\right) \end{cases}$$

$$(16)$$

satisfying $d\varphi_1 \wedge \ldots \wedge d\varphi_n \neq 0$.

Remark 2 *If there exists a coordinate transformation φ that satisfies the conditions of Theorem 4 with given $r_1, \ldots, r_m$, meaning that the system is flat, then $g_j = \sum_{i=1}^n \dfrac{\partial \varphi_i}{\partial y_j^{(r_j)}} \dfrac{\partial}{\partial \dot{x}_i}$, if non zero, defines a ruled direction [32,25,19].*

5 Examples

5.1 An Academic Example: Generalized Moving Frame Approach

We consider the 3-dimensional system with 2 inputs:

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = \sin\left(\frac{u_1}{u_2}\right) \tag{17}$$

or, in implicit form:

$$F(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) \triangleq \dot{x}_3 - \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right) = 0. \qquad (18)$$

It is readily seen that $P(F) = \left[-\cos(\frac{\dot{x}_1}{\dot{x}_2})\dot{x}_2^{-1}\frac{d}{dt} \left| \dot{x}_1 \cos(\frac{\dot{x}_1}{\dot{x}_2})\dot{x}_2^{-2}\frac{d}{dt} \right| \frac{d}{dt}\right]$ and that $VP(F)U = (1\ 0\ 0)$ with

$$V = 1, \quad U = \begin{pmatrix} \frac{\dot{x}_1}{a\dot{x}_2}\left|1 + \frac{\dot{x}_1}{a(\dot{x}_2)^2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\frac{d}{dt}\right| \frac{\dot{x}_1}{a\dot{x}_2}\frac{d}{dt} \\ \frac{1}{a} \qquad \frac{1}{a\dot{x}_2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\frac{d}{dt} \quad \left| -\frac{1}{a}\frac{d}{dt} \right. \\ 0 \qquad\qquad 0 \qquad\qquad 1 \end{pmatrix} \qquad (19)$$

where $a = -\frac{1}{\dot{x}_2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\left(\frac{\ddot{x}_1\dot{x}_2 - \dot{x}_1\ddot{x}_2}{(\dot{x}_2)^2}\right)$. Then, $Q\hat{U}R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is computed with

$$Q = \begin{pmatrix} 1 & \left| -\frac{\dot{x}_1}{\dot{x}_2} \right| & 0 \\ 0 & 0 & 1 \\ -\frac{1}{a\dot{x}_2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\frac{d}{dt} & \left| \frac{\dot{x}_1}{a(\dot{x}_2)^2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\frac{d}{dt} \right| & \frac{1}{a}\frac{d}{dt} \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (20)$$

So, $(\omega_1\ \omega_2)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} Qdx = \left(dx_1 - \frac{\dot{x}_1}{\dot{x}_2}dx_2\ \ dx_3\right)^T$ and $d\omega = \left(\frac{1}{\sqrt{1-(\dot{x}_3)^2}}dx_2 \wedge dx_3\ \ 0\right)^T$. According to section 3.3, step 1,

$$\mu = \begin{pmatrix} 0 & \left(-\frac{\dot{x}_3}{(1-(\dot{x}_3)^2)^{\frac{3}{2}}}dx_2 \wedge d\dot{x}_3 + \eta d\dot{x}_3\right) \wedge \frac{d}{dt} \\ 0 & 0 \end{pmatrix}. \qquad (21)$$

Step 2 yields $\eta = \frac{x_2\dot{x}_3}{(1-\dot{x}_3)^{\frac{3}{2}}} + \sigma(\dot{x}_3)$. For step 3 we set $M = \begin{pmatrix} 1 & m_{12}\frac{d}{dt} \\ 0 & 1 \end{pmatrix}$ which yields $m_{12} = -\left(\frac{x_2}{\sqrt{1-(\dot{x}_3)^2}} + \sigma_1(\dot{x}_3)\right)$ with σ_1 a primitive of σ. Thus, $d(M\omega) = 0$ and setting $(dy_1\ dy_2)^T = M\omega$, one obtains

$$y_1 = x_1 - \frac{\dot{x}_1}{\dot{x}_2}x_2 + \sigma_2(\dot{x}_3), \quad y_2 = x_3 \qquad (22)$$

where $\sigma_2(\dot{x}_3)$ is an arbitrary meromorphic function (a primitive of σ_1). By inversion of (22) we get

$$x_1 = y_1 - \arcsin(\dot{y}_2)\frac{\sqrt{1-(\dot{y}_2)^2}}{\ddot{y}_2}(\dot{y}_1 - \sigma_1(\dot{y}_2)\ddot{y}_2) - \sigma_2(\dot{y}_2)$$

$$x_2 = -\frac{\sqrt{1-(\dot{y}_2)^2}}{\ddot{y}_2}(\dot{y}_1 - \sigma_1(\dot{y}_2)\ddot{y}_2) \qquad (23)$$

$$x_3 = y_2$$

5.2 Academic Example: Euler-Lagrange Operator

We consider once more the example (18). We have

$$\frac{\partial F}{\partial \dot{x}} = \left(-\dot{x}_2^{-1} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), \dot{x}_1 \dot{x}_2^{-2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), 1 \right), \quad \mathcal{E}_F = (\eta_1, \eta_2, 0) \quad (24)$$

with $\eta_1 = -\frac{\ddot{x}_2}{\dot{x}_2^2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) - \frac{\ddot{x}_1 \dot{x}_2 - \dot{x}_1 \ddot{x}_2}{\dot{x}_2^3} \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right)$ and
$\eta_2 = -\frac{\ddot{x}_1 \dot{x}_2 - 2\dot{x}_1 \ddot{x}_2}{\dot{x}_2^3} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) + \frac{\dot{x}_1 (\ddot{x}_1 \dot{x}_2 - \dot{x}_1 \ddot{x}_2)}{\dot{x}_2^4} \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right)$.
The first two equations of (16), with $r_1 = r_2 = 2$, read

$$-\frac{1}{\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \left(\frac{\partial \varphi_1}{\partial \dot{y}_j} - \frac{\dot{x}_1}{\dot{x}_2} \frac{\partial \varphi_2}{\partial \dot{y}_j} \right) + \frac{\partial \varphi_3}{\partial \dot{y}_j} = 0, \quad j = 1, 2 \quad (25)$$

If we assume that $\frac{\partial \varphi_3}{\partial \dot{y}_j} = \frac{\partial \varphi_3}{\partial \dot{y}_j} = 0, \quad j = 1, 2$ and introduce the variable

$$\psi = \frac{\dot{x}_1}{\dot{x}_2} \quad (26)$$

with $\frac{\partial}{\partial \ddot{y}} \psi = 0$ we obtain from (25)

$$\frac{\partial \varphi_1}{\partial \ddot{y}_j} - \psi \frac{\partial \varphi_2}{\partial \ddot{y}_j} = \frac{\partial}{\partial \ddot{y}_j} (\varphi_1 - \psi \varphi_2) = 0, \quad j = 1, 2$$

Setting $\kappa(y, \dot{y}) = \varphi_1 - \psi \varphi_2$, we get

$$\dot{\kappa} = \dot{\varphi}_1 - \psi \dot{\varphi}_2 - \dot{\psi} \varphi_2 = -\dot{\psi} \varphi_2 \quad (27)$$

Using the definition of κ and (27) we obtain:

$$\varphi_1 = \kappa - \frac{\dot{\kappa}\sqrt{1 - \dot{\varphi}_3}}{\ddot{\varphi}_3} \arcsin(\dot{\varphi}_3), \quad \varphi_2 = -\frac{\dot{\kappa}}{\ddot{\varphi}_3} \sqrt{1 - \dot{\varphi}_3}, \quad \varphi_3 = \varphi_3(y) \quad (28)$$

Choosing $\varphi_3 = y_2, \quad \kappa = y_1$, we arrive at the invertible transformation

$$x_1 = \varphi_1 = y_1 - \frac{\dot{y}_1}{\ddot{y}_2} \sqrt{1 - \dot{y}_2^2} \arcsin(\dot{y}_2), \quad x_2 = \varphi_2 = -\frac{\dot{y}_1}{\ddot{y}_2} \sqrt{1 - \dot{y}_2^2},$$

with $x_3 = \varphi_3 = y_2$, which gives the same formula as (23) with $\sigma_1 = \sigma_2 = 0$. Hence (y_1, y_2) is indeed a flat output, which implies that the remaining equations of (16) are satisfied.

5.3 An Example Proposed by P. Rouchon

Consider the implicit control system

$$F(x, \dot{x}) = \dot{x}_1 \dot{x}_3 - (\dot{x}_2)^2 = 0. \quad (29)$$

We thus have $\frac{\partial F}{\partial x} = (0 \quad 0 \quad 0)$, $\frac{\partial F}{\partial \dot{x}} = (\dot{x}_3 \quad -2\dot{x}_2 \quad \dot{x}_1)$ and

$$\mathcal{E}_F = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = (-\ddot{x}_3 \quad 2\ddot{x}_2 \quad -\ddot{x}_1).$$

The lowest possible choice of (r_1, r_2) in Theorem 4 is $r_1 = r_2 = 1$. However, there is no solution of (16) for these values, and we choose $r_1 = r_2 = 2$. The two first equations of (16) read

$$\dot\varphi_3 \frac{\partial \varphi_1}{\partial \ddot y_j} - 2\dot\varphi_2 \frac{\partial \varphi_2}{\partial \ddot y_j} + \dot\varphi_1 \frac{\partial \varphi_3}{\partial \ddot y_j} = 0, \quad j = 1, 2 \tag{30}$$

We divide (30) by $\dot\varphi_3$ to obtain

$$\frac{\partial \varphi_1}{\partial \ddot y_j} - 2\psi \frac{\partial \varphi_2}{\partial \ddot y_j} + \psi^2 \frac{\partial \varphi_3}{\partial \ddot y_j} = 0, \quad j = 1, 2 \tag{31}$$

where, taking account of the system equation (29),

$$\psi = \frac{\dot\varphi_2}{\dot\varphi_3} = \sqrt{\frac{\dot\varphi_1}{\dot\varphi_3}}. \tag{32}$$

If we assume that ψ doesn't depend on $\ddot y_1$ and $\ddot y_2$, equation (31) reads $\frac{\partial}{\partial \ddot y_j} \left(\varphi_1 - 2\psi\varphi_2 + \psi^2 \varphi_3 \right) = 0$, for $j = 1, 2$. In other words, there exists a function κ satisfying $\frac{\partial \kappa}{\partial \ddot y_j} = 0$ for $j = 1, 2$, such that

$$\varphi_1 - 2\psi\varphi_2 + \psi^2 \varphi_3 = \kappa \tag{33}$$

Differentiating the latter relation with respect to t, and taking into account the relation $\dot\varphi_1 - 2\psi\dot\varphi_2 + \psi^2 \dot\varphi_3 = 0$ obtained from (29) and (32), we get

$$\varphi_2 - \psi\varphi_3 = -\frac{\dot\kappa}{2\dot\psi}. \tag{34}$$

We again differentiate the latter relation with respect to t to obtain

$$\varphi_3 = \frac{\ddot\kappa\dot\psi - \dot\kappa\ddot\psi}{2\dot\psi^3} \tag{35}$$

thanks to $\dot\varphi_2 - \psi\dot\varphi_3 = 0$ from (32). Thus, solving the system (33)–(35), we immediately obtain

$$\varphi_1 = \kappa - \psi\frac{\dot\kappa}{\dot\psi} + \psi^2 \left(\frac{\ddot\kappa\dot\psi - \dot\kappa\ddot\psi}{2\dot\psi^3} \right)$$
$$\varphi_2 = -\frac{\dot\kappa}{2\dot\psi} + \psi \left(\frac{\ddot\kappa\dot\psi - \dot\kappa\ddot\psi}{2\dot\psi^3} \right) \tag{36}$$
$$\varphi_3 = \frac{\ddot\kappa\dot\psi - \dot\kappa\ddot\psi}{2\dot\psi^3}$$

where κ and ψ are arbitrary functions of $y_1, y_2, \dot y_1, \dot y_2$.

Note that choosing $\kappa = y_1$ and $\psi = y_2$ yields, after inversion of (36) with (32):

$$y_1 = x_1 - 2x_2 \frac{\dot x_2}{\dot x_3} + x_3 \frac{\dot x_1}{\dot x_3}, \quad y_2 = \frac{\dot x_2}{\dot x_3},$$

which is similar to the solution obtained by F. Ollivier[5].

Similarly, the solution of K. Schlacher and M. Schöberl [29] may be

---

[5] personal communication

recovered by posing $\kappa = y_1 - y_2 \frac{\dot{y}_1}{\dot{y}_2}$ and $\psi = \frac{\dot{y}_1}{2\dot{y}_2}$ which, again after inversion of (36) with (32), yields:

$$y_1 = x_1 - x_3 \frac{\dot{x}_1}{\dot{x}_3}, \quad y_2 = x_2 - x_3 \frac{\dot{x}_2}{\dot{x}_3}.$$

6   Conclusion

In this survey we presented two dual approaches to flatness necessary and sufficient conditions, one based on the integration of 1-forms and the second based on the integration of a set of PDEs involving a generalized Euler-Lagrange operator. Their complexity is compared on examples.

Acknowledgements

References

1. R. L. Anderson and N. H. Ibragimov. *Lie-Bäcklund Transformations in Applications*. SIAM, Philadelphia, 1979.
2. F. Antritter and J. Lévine. Towards a computer algebraic algorithm for flat output determination. In *Proc. ISSAC 08*, 2008.
3. E. Aranda-Bricaire, C. Moog, and J.-B. Pomet. A linear algebraic framework for dynamic feedback linearization. *IEEE Trans. Automat. Control*, 40(1):127–132, 1995.
4. B. Charlet, J. Lévine, and R. Marino. Sufficient conditions for dynamic state feedback linearization. *SIAM J. Control Optimiz.*, 29(1):38–57, 1991.
5. S. Chern, W. Chen, and K. Lam. *Lectures on Differential Geometry*, volume 1 of *Series on University Mathematics*. World Scientific, 2000.
6. V. Chetverikov. New flatness conditions for control systems. In *Proceedings of NOLCOS'01, St. Petersburg*, pages 168–173, 2001.
7. V. Chetverikov. Flatness conditions for control systems. *Preprint DIPS*, 2002. available via internet : http://www.diffiety.ac.ru.
8. P. Cohn. *Free Rings and Their Relations*. Academic Press, London, 1985.
9. M. Fliess. A remark on Willems' trajectory characterization of linear controllability. *Systems & Control Letters*, 19:43–45, 1992.
10. M. Fliess, J. Lévine, P. Martin, F. Ollivier, and P. Rouchon. Controlling nonlinear systems by flatness. In C. Byrnes, B. Datta, D. Gilliam, and C. Martin, editors, *Systems and Control in the Twenty-First Century*, pages 137–154. Birkhäuser, Boston, 1997.

11. M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *C.R. Acad. Sci. Paris*, I–315:619–624, 1992.

12. M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *Int. J. Control*, 61(6):1327–1361, 1995.

13. M. Fliess, J. Lévine, P. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Trans. Automat. Control*, 44(5):922–937, 1999.

14. J. Franch. *Flatness, Tangent Systems and Flat Outputs*. PhD thesis, Universitat Politècnica de Catalunya Jordi Girona, 1999.

15. B. Jakubczyk. Invariants of dynamic feedback and free systems. In *Proc. ECC'93, Groningen*, pages 1510–1513, 1993.

16. I. S. Krasil'shchik, V. V. Lychagin, and A. M. Vinogradov. *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*. Gordon and Breach, New York, 1986.

17. J. Lévine. On necessary and sufficient conditions for differential flatness. In *Proc. of IFAC NOLCOS 2004 Conference*, Stuttgart, 2004.

18. J. Lévine. On necessary and sufficient conditions for differential flatness. *arXiv:math.OC/0605405*, 2006. available via internet: http://www.arxiv.org.

19. J. Lévine. *Analysis and Control of Nonlinear Systems: A Flatness-based Approach*. Mathematical Engineering Series. Springer, 2009.

20. P. Martin. *Contribution à l'Étude des Systèmes Différentiellement Plats*. PhD thesis, École des Mines de Paris, 1992.

21. P. Martin, R. Murray, and P. Rouchon. Flat systems. In G. Bastin and M. Gevers, editors, *Plenary Lectures and Minicourses, Proc. ECC 97, Brussels*, pages 211–264, 1997.

22. P. Pereira da Silva and C. C. Filho. Relative flatness and flatness of implicit systems. *SIAM J. Control and Optimization*, 39(6):1929–1951, 2001.

23. J.-B. Pomet. A differential geometric setting for dynamic equivalence and dynamic linearization. In B. Jakubczyk, W. Respondek, and T. Rzeżuchowski, editors, *Geometry in Nonlinear Control and Differential Inclusions*, pages 319–339. Banach Center Publications, Warsaw, 1993.

24. M. Rathinam and R. Murray. Configuration flatness of Lagrangian systems underactuated by one control. *SIAM J. Control Optimiz.*, 36(1):164–179, 1998.

25. P. Rouchon. Necessary condition and genericity of dynamic feedback linearization. *J. Math. Systems Estim. & Control*, 4(2):257–260, 1994.

26. J. Rudolph. *Flatness Based Control of Distributed Parameter Systems*. Shaker Verlag, Aachen, 2003.

27. J. Rudolph, J. Winkler, and F. Woittenek. *Flatness Based Control of Distributed Parameter Systems: Examples and Computer Exercises from Various Technological Domains*. Shaker Verlag, Aachen, 2003.

28. K. Schlacher and M. Schöberl. Construction of flat outputs by reduction and elimination. In *Proc. 7th IFAC Symposium on Nonlin-*

ear Control Systems, Pretoria, South Africa, pages 666–671, August 2007.

29. K. Schlacher and M. Schöberl. Construction of flat outputs by reduction and elimination. In J. Lévine and P. Müllhaupt, editors, *Advances in the Theory of Control, Signals and Systems, with Physical Modeling*. Springer, 2010.

30. W. Shadwick. Absolute equivalence and dynamic feedback linearization. *Systems & Control Letters*, 15:35–39, 1990.

31. H. Sira-Ramirez and S. Agrawal. *Differentially Flat Systems*. Marcel Dekker, New York, 2004.

32. W. Sluis. A necessary condition for dynamic feedback linearization. *Systems & Control Letters*, 21:277–283, 1993.

33. M. van Nieuwstadt, M. Rathinam, and R. Murray. Differential flatness and absolute equivalence of nonlinear control systems. *SIAM J. Control Optim.*, 36(4):1225–1239, 1998.

34. V. Zharinov. *Geometrical Aspect of Partial Differential Equations*. World Scientific, Singapore, 1992.