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Flatness Characterization: Two Approaches

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Abstract. We survey two approaches to flatness necessary and sufficient conditions and compare them on examples.

1 Introduction

In this survey we consider underdetermined implicit systems of the form

$$F(x, \dot{x}) = 0 \tag{1}$$

with $x \in X$, X being an inifnitely differentiable manifold of dimension n, whose tangent bundle is denoted by $\mathrm{T}X$, and $F : \mathrm{T}X \to \mathbb{R}^{n-m}$ regular in the sense that $\mathrm{rk}\, \frac{\partial F}{\partial \dot{x}} = n - m$ in a suitable open dense subset of $\mathrm{T}X$. Differential flatness, or more shortly, flatness was introduced in 1992 [20,11]. In the setting of implicit control systems it may be roughly described as follows: there exists a smooth mapping $x = \varphi(y, \dot{y}, \ldots, y^{(r)})$ with $y = (y_1, \ldots, y_m)^T$ of dimension m, $r = (r_1, \ldots, r_m)^T \in \mathbb{N}^m$, such that

$$F(\varphi(y, \dot{y}, \ldots, y^{(r)}), \dot{\varphi}(y, \dot{y}, \ldots, y^{(r+1)})) \equiv 0 \tag{2}$$

with φ invertible in the sense that there exists a locally defined smooth mapping ψ and a multi-index s such that $y = \psi(x, \dot{x}, \ldots, x^{(s)})$.

The vector y is called a *flat output*.

This concept has inspired an important literature. See [10,21,19,26,27,31] for surveys on flatness and its applications. Various formalisms have been introduced: finite dimensional differential geometric approaches [4,14,30], [32,28], differential algebra and related approaches [12,3,15], infinite dimensional differential geometry of jets and prolongations [13,33,19,6,7,23], [22,24], which is adopted here. The interested reader may refer to [1,13,16], [19,23,34] for more details.

The first part of the paper recalls the mathematical setting. In Section 3 the approch introduced in [19,2] for the characterization of differentially flat systems is recalled. Then, in Section 4, we introduce a novel characterization using the so-called Generalized Euler-Lagrange Operator. We conclude the paper with examples.

2 Implicit control systems on manifolds of jets of infinite order

Given an infinitely differentiable manifold X of dimension n, we denote its tangent space at $x \in X$ by $\mathrm{T}_x X$, and its tangent bundle by $\mathrm{T}X$. Let F be a meromorphic function from $\mathrm{T}X$ to $\mathbb{R}^{n-m}$. We consider an underdetermined implicit system of the form (1) regular in the sense that $\mathrm{rk}\,\frac{\partial F}{\partial \dot{x}} = n - m$ in a suitable open dense subset of $\mathrm{T}X$.

Following [17,18], we consider the infinite dimensional manifold $\mathfrak{X}$ defined by $\mathfrak{X} \stackrel{\mathrm{def}}{=} X \times \mathbb{R}^n_\infty \stackrel{\mathrm{def}}{=} X \times \mathbb{R}^n \times \mathbb{R}^n \times \ldots$, made of an infinite (but countable) number of copies of $\mathbb{R}^n$, with the global infinite set of coordinates[3] $\overline{x} = \left(x, \dot{x}, \ldots, x^{(k)}, \ldots,\right)$, endowed with the product topology. Recall that, in this topology, a function φ from $\mathfrak{X}$ to $\mathbb{R}$ is *continuous* (resp. *differentiable*) if φ depends only on a finite (but otherwise arbitrary) number of variables and is continuous (resp. differentiable) with respect to these variables. C^∞ or analytic or meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ are then defined as in the usual finite dimensional case since they only depend on a finite number of variables. We endow $\mathfrak{X}$ with the so-called trivial Cartan field ([16,34]) $\tau_{\mathfrak{X}} = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}}$. We also denote by $L_{\tau_{\mathfrak{X}}} \gamma = \sum_{i=1}^n \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial \gamma}{\partial x_i^{(j)}} = \frac{d\gamma}{dt}$ the Lie derivative of a differentiable function γ along $\tau_{\mathfrak{X}}$ and $L^k_{\tau_{\mathfrak{X}}} \gamma$ its kth iterate. Since $\frac{d}{dt} x_i^{(j)} \stackrel{\mathrm{def}}{=} \dot{x}_i^{(j)} = x_i^{(j+1)}$, the Cartan field acts on coordinates as a shift to the right. $\mathfrak{X}$ is thus called *manifold of jets of infinite order.*

A regular implicit control system is defined as a triple $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *with* $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\tau_{\mathfrak{X}}$ *its associated trivial Cartan field, and* F *meromorphic from* $\mathrm{T}X$ *to* $\mathbb{R}^{n-m}$) *satisfying* $\mathrm{rk}\,\frac{\partial F}{\partial \dot{x}} = n - m$ *in a suitable open subset of* $\mathrm{T}X$.

We next consider the cotangent space $T^*_{\overline{x}}\mathfrak{X}$ with $dx_i^{(j)}$, $i = 1, \ldots, n$, $j \geq 0$ as basis, dual to the $\frac{\partial}{\partial x_i^{(j)}}$´s. 1-forms on $\mathfrak{X}$ are then defined in the usual way. The set of 1-forms is noted $\Lambda^1(\mathfrak{X})$. We also denote by $\Lambda^p(\mathfrak{X})$ the module of all the p-forms on $\mathfrak{X}$.

2.1 Flatness

We recall the following definitions and result [17,18,19]:

Given two regular implicit control systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$, with $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\dim X = n$ and $\mathrm{rk}\,\frac{\partial F}{\partial \dot{x}} = n - m$, and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$, with $\mathfrak{Y} = Y \times \mathbb{R}^p_\infty$, $\dim Y = p$, $\tau_{\mathfrak{Y}}$ its trivial Cartan field, and $\mathrm{rk}\,\frac{\partial G}{\partial \dot{y}} = p - q$, we set $\mathfrak{X}_0 = \{\overline{x} \in \mathfrak{X} | L^k_{\tau_{\mathfrak{X}}} F(\overline{x}) = 0,\ \forall k \geq 0\}$ and $\mathfrak{Y}_0 = \{\overline{y} \in \mathfrak{Y} | L^k_{\tau_{\mathfrak{Y}}} G(\overline{y}) = 0,\ \forall k \geq 0\}$. They are endowed with the topologies and differentiable structures induced by $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.

Definition 1 *The control systems* $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *and* $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ *are said* locally Lie-Bäcklund equivalent *(or shortly L-B equivalent) in a neighbourhood* $\mathcal{X}_0 \times \mathcal{Y}_0$ *of the pair* $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathfrak{Y}_0$ *if and only if*

[3] From now on, $\overline{x}$ $\overline{y}$, . . . stand for the sequences of jets of infinite order of x, y,. . .

(i) there exists a one-to-one meromorphic mapping $\Phi = (\varphi, \dot{\varphi}, \ldots)$ from $\mathcal{Y}_0$ to $\mathcal{X}_0$ satisfying $\Phi(\overline{y}_0) = \overline{x}_0$ and such that $\Phi_* \tau_{\mathfrak{Y}} = \tau_{\mathfrak{X}}$;
(ii) there exists Ψ one-to-one and meromorphic from $\mathcal{X}_0$ to $\mathcal{Y}_0$, with $\Psi = (\psi, \dot{\psi}, \ldots)$, such that $\Psi(\overline{x}_0) = \overline{y}_0$ and $\Psi_* \tau_{\mathfrak{X}} = \tau_{\mathfrak{Y}}$.

The mappings Φ and Ψ are called mutually inverse Lie-Bäcklund isomorphisms *at $(\overline{x}_0, \overline{y}_0)$.*

Definition 2 *The implicit system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is* locally flat *in a neighborhood of $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}^m_\infty$ if and only if it is locally L-B equivalent around $(\overline{x}_0, \overline{y}_0)$ to the trivial implicit system $(\mathbb{R}^m_\infty, \tau_{\mathbb{R}^m_\infty}, 0)$. In this case, the mutually inverse L-B isomorphisms Φ and Ψ are called* inverse trivializations*.*

Theorem 1 *The system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is locally flat at $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}^m_\infty$ if and only if there exists a local meromorphic invertible mapping Φ from $\mathbb{R}^m_\infty$ to $\mathfrak{X}_0$, with meromorphic inverse, satisfying $\Phi(\overline{y}_0) = \overline{x}_0$, and such that*[4]

$$\Phi^* dF = 0. \tag{3}$$

3 Necessary and Sufficient Conditions: Generalized Moving Frame Structure Equations

3.1 Algebraic characterization of the differential of a trivialization

Consider the following matrix, polynomial with respect to the differential operator $\frac{d}{dt}$ (we use indifferently $\frac{d}{dt}$ for $L_{\tau_{\mathfrak{X}}}$ or $L_{\tau_{\mathbb{R}^m_\infty}}$, the context being unambiguous):

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt}, \quad P(\varphi) = \sum_{j \geq 0} \frac{\partial \varphi}{\partial y^{(j)}} \frac{d^j}{dt^j} \tag{4}$$

with $P(F)$ (resp. $P(\varphi)$) of size $(n-m) \times n$ (resp. $n \times m$).
Equation (3)) reads:

$$\Phi^* dF = P(F) P(\varphi) dy = 0. \tag{5}$$

Clearly, the entries of the matrices in (4) are polynomials in the differential operator $\frac{d}{dt}$ with meromorphic coefficients from $\mathfrak{X}$ to $\mathbb{R}$. We denote by $\mathfrak{K}$ the field of meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ and by $\mathfrak{K}[\frac{d}{dt}]$ the (non-commutative) principal ideal ring of polynomials in $\frac{d}{dt}$ with coefficients in $\mathfrak{K}$. For $r, s \in \mathbb{N}$, let us denote by $\mathcal{M}_{r,s}[\frac{d}{dt}]$ the module of $r \times s$ matrices over $\mathfrak{K}[\frac{d}{dt}]$ (see e.g. [8]). Matrices whose inverse belong to $\mathcal{M}_{r,r}[\frac{d}{dt}]$ are called *unimodular matrices* . They form a multiplicative group denoted by $\mathcal{U}_r[\frac{d}{dt}]$.

[4] Note that if Φ is a meromorphic mapping from $\mathfrak{Y}$ to $\mathfrak{X}$, the (backward) image by Φ of a 1-form is defined in the same way as in the finite dimensional context.

Every matrix $M \in \mathcal{M}_{r,s}[\frac{d}{dt}]$ admits a *Smith decomposition* (or diagonal reduction)

$$VMU = (\Delta, 0_{r,s-r}) \text{ if } r \leq s, \text{ and } \begin{pmatrix} \Delta \\ 0_{r-s,s} \end{pmatrix} \text{ if } s \leq r \tag{6}$$

with $V \in \mathcal{U}_r[\frac{d}{dt}]$ and $U \in \mathcal{U}_s[\frac{d}{dt}]$ and Δ diagonal (see e.g. [8]). U and V are indeed non unique. We say that $U \in \mathsf{R}-\mathsf{Smith}\,(M)$ and $V \in \mathsf{L}-\mathsf{Smith}\,(M)$.
A matrix $M \in \mathcal{M}_{r,s}[\frac{d}{dt}]$ is said *hyper-regular* if and only if its Smith decomposition leads to $\Delta = I$. An interpretation of this property in terms of controllability in the sense of [9], may be found in [18].
From now on, we assume that $P(F)$ is hyper-regular in a neighborhood of $\overline{x}_0$. In place of (5), we first solve the matrix equation:

$$P(F)\Theta = 0 \tag{7}$$

where $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ is not supposed to be of the form $P(\varphi)$. It may be verified that matrices $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ satisfying (7) have the structure

$$\Theta = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} W \tag{8}$$

with $U \in \mathsf{R}-\mathsf{Smith}\,(P(F))$ and $W \in \mathcal{U}_m[\frac{d}{dt}]$ arbitrary. Clearly Θ is itself hyper-regular and admits the Smith decomposition

$$Q\Theta Z = QU \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} WZ = Q\hat{U}R = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} \tag{9}$$

with $Q \in \mathcal{U}_n[\frac{d}{dt}]$, $Z \in \mathcal{U}_m[\frac{d}{dt}]$, $R = WZ$ and $\hat{U} = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}$.

3.2 Integrability

We denote by ω the m-dimensional vector 1-form defined by

$$\omega(\overline{x}) = \begin{pmatrix} \omega_1(\overline{x}) \\ \vdots \\ \omega_m(\overline{x}) \end{pmatrix} = (I_m, 0_{m,n-m})\, Q(\overline{x}) dx_{|\mathcal{X}_0} \tag{10}$$

with Q given by (9), the restriction to $\mathfrak{X}_0$ meaning that $\overline{x} \in \mathcal{X}_0$ satisfies $L^k_{\tau_{\mathfrak{X}}} F = 0$ for all k and that the $dx_j^{(k)}$ are such that $dL^k_{\tau_{\mathfrak{X}}} F = 0$ in $\mathcal{X}_0$ for all k. Since Q is hyper-regular, the forms $\omega_1, \ldots, \omega_m$ are independent by construction.

Theorem 2 *A necessary and sufficient condition for system (1) to be locally flat around* $(\overline{x}_0, \overline{y}_0)$ *is that there exist* $U \in \mathsf{R}-\mathsf{Smith}\,(P(F))$, $Q \in \mathsf{L}-\mathsf{Smith}\left(\hat{U}\right)$, *with* $\hat{U}$ *given by (9) and a matrix* $M \in \mathcal{U}_m[\frac{d}{dt}]$ *such that* $d(M\tau) = 0$.

We denote by $(\Lambda^p(\mathfrak{X}))^m$ the space of m-dimensional vector p-forms on $\mathfrak{X}$, by $(\Lambda(\mathfrak{X}))^m$ the space of m-dimensional vector forms of arbitrary degree on $\mathfrak{X}$, and by $\mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right) = \bigcup_{p\geq 1} \mathcal{L}\left((\Lambda^p(\mathfrak{X}))^m, \left(\Lambda^{p+q}(\mathfrak{X})\right)^m\right)$ the space of linear operators from $(\Lambda^p(\mathfrak{X}))^m$ to $\left(\Lambda^{p+q}(\mathfrak{X})\right)^m$ for all $p \geq 1$, where $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ denotes the set of linear mappings from a given space $\mathcal{P}$ to a given space $\mathcal{Q}$.
In order to develop the expression $d(\mu\kappa)$ for $\mu \in \mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right)$ and for all $\kappa \in (\Lambda^p(\mathfrak{X}))^m$ and all $p \geq 1$, we define the operator $\mathfrak{d}$ by:

$$\mathfrak{d}\,(\mu)\,\kappa = d(\mu\;\kappa) - (-1)^q \mu\; d\kappa. \tag{11}$$

Note that (11) uniquely defines $\mathfrak{d}\,(\mu)$ as an element of $\mathcal{L}_{q+1}\left((\Lambda(\mathfrak{X}))^m\right)$.

Theorem 3 *The system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is locally flat iff there locally exists $\mu \in \mathcal{L}_1\left((\Lambda(\mathfrak{X}))^m\right)$, and a matrix $M \in \mathfrak{U}_m[\frac{d}{dt}]$ such that*

$$d\omega = \mu\;\omega, \qquad \mathfrak{d}\,(\mu) = \mu^2, \qquad \mathfrak{d}\,(M) = -M\mu. \tag{12}$$

with the notation $\mu^2 = \mu\mu$ and where ω is defined by (10). In addition, if (12) holds true, a flat output y is obtained by integration of $dy = M\omega$.

Remark 1 *Note that the two first conditions of (12) are comparable to conditions (A) and (B) of [6,7]. However, the last condition of (12) is different from condition (C) of [6,7] and is easier to check.*
Note also that conditions (12) may be seen as a generalization in the framework of manifolds of jets of infinite order of Cartan's well-known moving frame structure equations (see e.g. [5]).

3.3 A Sequential Procedure

We start with $P(F)$ hyper-regular and compute the vector 1-form ω defined by (10).

1. We identify the operator μ such that $d\omega = \mu\omega$ componentwise. It is proven in [19] that such μ always exists.
2. Among the possible μ's, only those satisfying $\mathfrak{d}\,(\mu) = \mu^2$ are kept. It is shown in [19] that such μ always exists.
3. We then identify M such that $\mathfrak{d}\,(M) = -M\mu$ componentwise.
4. If, among such M's, there is a unimodular one, the system is flat and a flat output is obtained by integration of $dy = M\omega$. Otherwise the system is not flat.

More details and examples may be found in [18,19].

4 Necessary and Sufficient Conditions using the Generalized Euler-Lagrange Operator

Another way of analysing (3) consists in characterizing the change of coordinates corresponding to the mapping Φ in (3). More precisely (3) reads

$$\sum_{j=1}^{m}\sum_{k=0}^{r_j}\left(\frac{\partial F}{\partial x}\frac{\partial \varphi}{\partial y_j^{(k)}}dy_j^{(k)} + \frac{\partial F}{\partial \dot{x}}\frac{d}{dt}\left(\frac{\partial \varphi}{\partial y_j^{(k)}}\right)dy_j^{(k)} + \frac{\partial F}{\partial \dot{x}}\frac{\partial \varphi}{\partial y_j^{(k)}}dy_j^{(k+1)}\right) = 0 \tag{13}$$

Since the one forms $dy_1, \ldots, dy_1^{(r_1)}, \ldots, dy_m, \ldots, dy_m^{(r_m)}$ are independent by assumption, (13) yields, for every $j = 1, \ldots, m$,

$$\left\{\begin{array}{l} \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(r_j)}} = 0 \\ \frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j^{(k)}} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt} \left(\frac{\partial \varphi}{\partial y_j^{(k)}} \right) + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(k-1)}} = 0, \quad \forall k = 1, \ldots, r_j \\ \frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j} + \frac{\partial F}{\partial \dot{x}} \frac{d}{dt} \left(\frac{\partial \varphi}{\partial y_j} \right) = 0 \end{array}\right. \tag{14}$$

The Generalized Euler-Lagrange operator $\mathcal{E}_F$ associated to F is defined by

$$\mathcal{E}_F = \frac{\partial F}{\partial x} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \right) \tag{15}$$

In the case $n - m = 1$, it is well-known that the curves that extremize the cost function $J = \int_0^T F(x, \dot{x}) dt$ are those satisfying the Euler-Lagrange equation $\mathcal{E}_F = 0$, which justifies our terminology.
Using (15) and elementary calculus, (14) yields:

Theorem 4 *A necessary and sufficient condition for (1) to be diffferentially flat is that there exist* $(r_1, \ldots, r_m)$ *with* $\sum_{i=1}^m r_i + m \geq n$ *and a solution* φ *of the following triangular system of PDEs in an open dense subset of* $\mathfrak{X}$

$$\left\{\begin{array}{l} \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(r_j)}} = 0 \\ \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(l)}} = \sum_{k=0}^{r_j - l - 1} (-1)^{k+1} \frac{d^k}{dt^k} \left(\mathcal{E}_F \frac{\partial \varphi}{\partial y_j^{(l+k+1)}} \right), \quad \forall l = 0, \ldots, r_j - 1 \ , \\ 0 = \sum_{k=0}^{r_j} (-1)^k \frac{d^k}{dt^k} \left(\mathcal{E}_F \frac{\partial \varphi}{\partial y_j^{(k)}} \right) \end{array}\right. \tag{16}$$

satisfying $d\varphi_1 \wedge \ldots \wedge d\varphi_n \neq 0$.

Remark 2 *If there exists a coordinate transformation* φ *that satisfies the conditions of Theorem 4 with given* $r_1, \ldots, r_m$*, meaning that the system is flat, then* $g_j = \sum_{i=1}^n \frac{\partial \varphi_i}{\partial y_j^{(r_j)}} \frac{\partial}{\partial \dot{x}_i}$ *, if non zero, defines a ruled direction [32,25,19].*

5 Examples

5.1 An Academic Example: Generalized Moving Frame Approach

We consider the 3-dimensional system with 2 inputs:

$$\dot{x}_1 = u_1, \ \ \dot{x}_2 = u_2, \ \ \dot{x}_3 = \sin \left(\frac{u_1}{u_2} \right) \tag{17}$$

or, in implicit form:

$$F(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3) \triangleq \dot{x}_3 - \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right) = 0. \tag{18}$$

It is readily seen that $P(F) = \left[-\cos(\frac{\dot{x}_1}{\dot{x}_2})\dot{x}_2^{-1}\frac{d}{dt} \middle| \dot{x}_1 \cos(\frac{\dot{x}_1}{\dot{x}_2})\dot{x}_2^{-2}\frac{d}{dt} \middle| \frac{d}{dt} \right]$ and that $VP(F)U = (1\ 0\ 0)$ with

$$V = 1, \quad U = \left(\begin{array}{c|c|c} \frac{\dot{x}_1}{a\dot{x}_2} & 1 + \frac{\dot{x}_1}{a(\dot{x}_2)^2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\frac{d}{dt} & \frac{\dot{x}_1}{a\dot{x}_2}\frac{d}{dt} \\ \frac{1}{a} & \frac{1}{a\dot{x}_2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\frac{d}{dt} & -\frac{1}{a}\frac{d}{dt} \\ 0 & 0 & 1 \end{array}\right) \tag{19}$$

where $a = -\frac{1}{\dot{x}_2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\left(\frac{\ddot{x}_1\dot{x}_2 - \dot{x}_1\ddot{x}_2}{(\dot{x}_2)^2}\right)$. Then, $Q\hat{U}R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is computed with

$$Q = \left(\begin{array}{c|c|c} 1 & -\frac{\dot{x}_1}{\dot{x}_2} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{a\dot{x}_2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\frac{d}{dt} & \frac{\dot{x}_1}{a(\dot{x}_2)^2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\frac{d}{dt} & \frac{1}{a}\frac{d}{dt} \end{array}\right), \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{20}$$

So, $(\omega_1\ \omega_2)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} Qdx = \left(dx_1 - \frac{\dot{x}_1}{\dot{x}_2}dx_2\ \ dx_3\right)^T$ and $d\omega = \left(\frac{1}{\sqrt{1-(\dot{x}_3)^2}}dx_2 \wedge dx_3\ \ 0\right)^T$. According to section 3.3, step 1,

$$\mu = \begin{pmatrix} 0 & \left(-\frac{\dot{x}_3}{(1-(\dot{x}_3)^2)^{\frac{3}{2}}}dx_2 \wedge d\dot{x}_3 + \eta d\dot{x}_3\right) \wedge \frac{d}{dt} \\ 0 & 0 \end{pmatrix}. \tag{21}$$

Step 2 yields $\eta = \frac{x_2\dot{x}_3}{(1-\dot{x}_3)^{\frac{3}{2}}} + \sigma(\dot{x}_3)$. For step 3 we set $M = \begin{pmatrix} 1 & m_{12}\frac{d}{dt} \\ 0 & 1 \end{pmatrix}$ which yields $m_{12} = -\left(\frac{x_2}{\sqrt{1-(\dot{x}_3)^2}} + \sigma_1(\dot{x}_3)\right)$ with σ_1 a primitive of σ. Thus, $d(M\omega) = 0$ and setting $(dy_1\ dy_2)^T = M\omega$, one obtains

$$y_1 = x_1 - \frac{\dot{x}_1}{\dot{x}_2}x_2 + \sigma_2(\dot{x}_3), \quad y_2 = x_3 \tag{22}$$

where $\sigma_2(\dot{x}_3)$ is an arbitrary meromorphic function (a primitive of σ_1). By inversion of (22) we get

$$\begin{aligned} x_1 &= y_1 - \arcsin(\dot{y}_2)\frac{\sqrt{1-(\dot{y}_2)^2}}{\ddot{y}_2}(\dot{y}_1 - \sigma_1(\dot{y}_2)\ddot{y}_2) - \sigma_2(\dot{y}_2) \\ x_2 &= -\frac{\sqrt{1-(\dot{y}_2)^2}}{\ddot{y}_2}(\dot{y}_1 - \sigma_1(\dot{y}_2)\ddot{y}_2) \\ x_3 &= y_2 \end{aligned} \tag{23}$$

5.2 Academic Example: Euler-Lagrange Operator

We consider once more the example (18). We have

$$\frac{\partial F}{\partial \dot{x}} = \left(-\dot{x}_2^{-1} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), \dot{x}_1 \dot{x}_2^{-2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), 1 \right), \ \mathcal{E}_F = (\eta_1, \eta_2, 0) \quad (24)$$

with $\eta_1 = -\frac{\ddot{x}_2}{\dot{x}_2^2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) - \frac{\ddot{x}_1 \dot{x}_2 - \dot{x}_1 \ddot{x}_2}{\dot{x}_2^3} \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right)$ and
$\eta_2 = -\frac{\ddot{x}_1 \dot{x}_2 - 2\dot{x}_1 \ddot{x}_2}{\dot{x}_2^3} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) + \frac{\dot{x}_1(\ddot{x}_1 \dot{x}_2 - \dot{x}_1 \ddot{x}_2)}{\dot{x}_2^4} \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right)$.
The first two equations of (16), with $r_1 = r_2 = 2$, read

$$-\frac{1}{\dot{x}_2} \cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) \left(\frac{\partial \varphi_1}{\partial \dot{y}_j} - \frac{\dot{x}_1}{\dot{x}_2} \frac{\partial \varphi_2}{\partial \dot{y}_j} \right) + \frac{\partial \varphi_3}{\partial \dot{y}_j} = 0, \quad j = 1, 2 \quad (25)$$

If we assume that $\frac{\partial \varphi_3}{\partial \ddot{y}_j} = \frac{\partial \varphi_3}{\partial \dot{y}_j} = 0, \quad j = 1, 2$ and introduce the variable

$$\psi = \frac{\dot{x}_1}{\dot{x}_2} \quad (26)$$

with $\frac{\partial}{\partial \ddot{y}} \psi = 0$ we obtain from (25)

$$\frac{\partial \varphi_1}{\partial \ddot{y}_j} - \psi \frac{\partial \varphi_2}{\partial \ddot{y}_j} = \frac{\partial}{\partial \ddot{y}_j} (\varphi_1 - \psi \varphi_2) = 0, \quad j = 1, 2$$

Setting $\kappa(y, \dot{y}) = \varphi_1 - \psi \varphi_2$, we get

$$\dot{\kappa} = \dot{\varphi}_1 - \psi \dot{\varphi}_2 - \dot{\psi} \varphi_2 = -\dot{\psi} \varphi_2 \quad (27)$$

Using the definition of κ and (27) we obtain:

$$\varphi_1 = \kappa - \frac{\dot{\kappa}\sqrt{1 - \dot{\varphi}_3}}{\ddot{\varphi}_3} \arcsin(\dot{\varphi}_3), \quad \varphi_2 = -\frac{\dot{\kappa}}{\ddot{\varphi}_3} \sqrt{1 - \dot{\varphi}_3}, \quad \varphi_3 = \varphi_3(y) \quad (28)$$

Choosing $\varphi_3 = y_2, \ \ \kappa = y_1$, we arrive at the invertible transformation

$$x_1 = \varphi_1 = y_1 - \frac{\dot{y}_1}{\ddot{y}_2} \sqrt{1 - \dot{y}_2^2} \arcsin(\dot{y}_2), \ x_2 = \varphi_2 = -\frac{\dot{y}_1}{\ddot{y}_2} \sqrt{1 - \dot{y}_2^2},$$

with $x_3 = \varphi_3 = y_2$, which gives the same formula as (23) with $\sigma_1 = \sigma_2 = 0$. Hence (y_1, y_2) is indeed a flat output, which implies that the remaining equations of (16) are satisfied.

5.3 An Example Proposed by P. Rouchon

Consider the implicit control system

$$F(x, \dot{x}) = \dot{x}_1 \dot{x}_3 - (\dot{x}_2)^2 = 0. \quad (29)$$

We thus have $\frac{\partial F}{\partial x} = (0 \quad 0 \quad 0)$, $\frac{\partial F}{\partial \dot{x}} = (\dot{x}_3 \quad -2\dot{x}_2 \ \ \dot{x}_1)$ and

$$\mathcal{E}_F = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = (-\ddot{x}_3 \ \ 2\ddot{x}_2 \ \ -\ddot{x}_1).$$

The lowest possible choice of (r_1, r_2) in Theorem 4 is $r_1 = r_2 = 1$. However, there is no solution of (16) for these values, and we choose $r_1 = r_2 = 2$. The two first equations of (16) read

$$\dot{\varphi}_3 \frac{\partial \varphi_1}{\partial \ddot{y}_j} - 2\dot{\varphi}_2 \frac{\partial \varphi_2}{\partial \ddot{y}_j} + \dot{\varphi}_1 \frac{\partial \varphi_3}{\partial \ddot{y}_j} = 0, \quad j = 1, 2 \tag{30}$$

We divide (30) by $\dot{\varphi}_3$ to obtain

$$\frac{\partial \varphi_1}{\partial \ddot{y}_j} - 2\psi \frac{\partial \varphi_2}{\partial \ddot{y}_j} + \psi^2 \frac{\partial \varphi_3}{\partial \ddot{y}_j} = 0, \quad j = 1, 2 \tag{31}$$

where, taking account of the system equation (29),

$$\psi = \frac{\dot{\varphi}_2}{\dot{\varphi}_3} = \sqrt{\frac{\dot{\varphi}_1}{\dot{\varphi}_3}}. \tag{32}$$

If we assume that ψ doesn't depend on $\ddot{y}_1$ and $\ddot{y}_2$, equation (31) reads $\frac{\partial}{\partial \ddot{y}_j}\left(\varphi_1 - 2\psi\varphi_2 + \psi^2\varphi_3\right) = 0$, for $j = 1, 2$. In other words, there exists a function κ satisfying $\frac{\partial \kappa}{\partial \ddot{y}_j} = 0$ for $j = 1, 2$, such that

$$\varphi_1 - 2\psi\varphi_2 + \psi^2\varphi_3 = \kappa \tag{33}$$

Differentiating the latter relation with respect to t, and taking into account the relation $\dot{\varphi}_1 - 2\psi\dot{\varphi}_2 + \psi^2\dot{\varphi}_3 = 0$ obtained from (29) and (32), we get

$$\varphi_2 - \psi\varphi_3 = -\frac{\dot{\kappa}}{2\dot{\psi}}. \tag{34}$$

We again differentiate the latter relation with respect to t to obtain

$$\varphi_3 = \frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \tag{35}$$

thanks to $\dot{\varphi}_2 - \psi\dot{\varphi}_3 = 0$ from (32). Thus, solving the system (33)–(35), we immediately obtain

$$\begin{aligned} \varphi_1 &= \kappa - \psi\frac{\dot{\kappa}}{\dot{\psi}} + \psi^2\left(\frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3}\right) \\ \varphi_2 &= -\frac{\dot{\kappa}}{2\dot{\psi}} + \psi\left(\frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3}\right) \\ \varphi_3 &= \frac{\ddot{\kappa}\dot{\psi} - \dot{\kappa}\ddot{\psi}}{2\dot{\psi}^3} \end{aligned} \tag{36}$$

where κ and ψ are arbitrary functions of $y_1, y_2, \dot{y}_1, \dot{y}_2$.
Note that choosing $\kappa = y_1$ and $\psi = y_2$ yields, after inversion of (36) with (32):

$$y_1 = x_1 - 2x_2\frac{\dot{x}_2}{\dot{x}_3} + x_3\frac{\dot{x}_1}{\dot{x}_3}, \quad y_2 = \frac{\dot{x}_2}{\dot{x}_3},$$

which is similar to the solution obtained by F. Ollivier[5].
Similarly, the solution of K. Schlacher and M. Schöberl [29] may be

[5] personal communication

recovered by posing $\kappa = y_1 - y_2\frac{\dot{y}_1}{\dot{y}_2}$ and $\psi = \frac{\dot{y}_1}{2\dot{y}_2}$ which, again after inversion of (36) with (32), yields:

$$y_1 = x_1 - x_3\frac{\dot{x}_1}{\dot{x}_3}, \quad y_2 = x_2 - x_3\frac{\dot{x}_2}{\dot{x}_3}.$$

6 Conclusion

In this survey we presented two dual approaches to flatness necessary and sufficient conditions, one based on the integration of 1-forms and the second based on the integration of a set of PDEs involving a generalized Euler-Lagrange operator. Their complexity is compared on examples.

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