Towards a Computer Algebraic Algorithm for Flat Output Determination

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ABSTRACT

This contribution deals with nonlinear control systems. More precisely, we are interested in the formal computation of a so-called *flat output*, a particular generalized output whose property is, roughly speaking, that all the integral curves of the system may be expressed as smooth functions of the components of this flat output and their successive time derivatives up to a finite order (to be determined). Recently, a characterization of such flat output has been obtained in [14, 15], in the framework of manifolds of jets of infinite order (see e.g. [18, 9]), that yields an abstract algorithm for its computation. In this paper it is discussed how these conditions can be checked using computer algebra. All steps of the algorithm are discussed for the simple (but rich enough) example of a non holonomic car.

Categories and Subject Descriptors

I.1.4 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Applications*; G.4 [**Mathematics of Computing**]: Mathematical Software

General Terms

Theory

1. INTRODUCTION

We consider a nonlinear control system

$$\dot{x} = f(x, u) \quad (1)$$

where $x = (x_1, \ldots, x_n)$ is the state vector, $u = (u_1, \ldots, u_m)$ the control vector, $m \leq n$, and f is a meromorphic function of its arguments.

We say that this system is *differentially flat*, or shortly *flat* ([17, 7]), if and only if there exists a vector $y = (y_1, \ldots, y_m)$ such that:

(i) y and its successive time derivatives $\dot{y}, \ddot{y}, \ldots$ are functionnally independent,

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(ii) y is a function of x, u and a finite number of time derivatives of the components of u,

(iii) x and u can be expressed as functions of the components of y and a finite number of their successive time derivatives:

$$x = \varphi(y, \dot{y}, \ldots, y^{(\alpha)}), \quad u = \psi(y, \dot{y}, \ldots, y^{(\alpha+1)})$$

for some multi-integer $\alpha = (\alpha_1, \ldots, \alpha_m)$, and with the notation $y^{(\alpha)} = (\frac{d^{\alpha_1} y_1}{dt^{\alpha_1}}, \ldots, \frac{d^{\alpha_m} y_m}{dt^{\alpha_m}})$.

A vector y having these properties is called a *flat output*.

This concept has inspired an important literature and a large number of practical and industrial applications (see e.g. [18] for a survey). Its main advantages rely on the simplicity to solve the *motion planning* and *stable tracking* problems.

Various formalisms have been introduced to study this remarkable class of systems: finite dimensional differential geometric approaches ([2, 10, 28, 29]), differential algebra and related approaches ([8, 1, 12]), infinite dimensional differential geometry of jets and prolongations ([9, 30, 22, 20, 24]). Among these contributions, the characterization of differential flatness takes a large part ([1, 2, 4, 10, 12, 19, 21, 23, 24, 25, 28, 29, 14, 15]).

We follow here the results of [14, 15] in the formalism of manifolds of jets of infinite order ([9, 13, 22, 31]). For the stated flatness conditions implicit systems are considered which are obtained from (1) by eliminating the input vector u. We recall the notions of Lie-Bäcklund equivalence and Lie-Bäcklund isomorphism in this context and the flatness necessary and sufficient conditions in terms of polynomial matrices and differential forms. Note that this approach may be seen as an extension to nonlinear systems of [16] and provide flatness conditions that are invariant by endogeneous dynamic feedback extension.

The derived conditions use differential operators which combine differential geometric concepts like exterior derivative and wedge product as well as algebraic concepts as operations on skew polynomials with coefficients that are meromorphic functions of the coordinates. Existing computer algebra systems offer lots of functionalities for each of the mentioned fields but their combination is not considered. In this paper we show how to implement such operators in Maple 11, and include them in an algorithm to check the flatness necessary and sufficient conditions. Note that this algorithm doesn't necessarily finish in a finite number of steps.

The paper is organized as follows: Section 2 is devoted to the basic description of implicit control systems on manifolds of jets of infinite order. The notions of Lie-Bäcklund equivalence and Lie-Bäcklund isomorphism in this framework are recalled and the flatness property is defined precisely. In Section 3 the necessary and sufficient conditions for differential flatness derived in [14, 15] are stated and Section 4 discusses in more detail the introduced operators and conditions. General formulae for the operators are derived from its defining relations and it is explained how the operators and the algorithm developed in Section 3 can be implemented using a computer algebra system like *Maple 11*. Finally, in Section 5 the application of the implemented method is illustrated for the well-known nonholonomic car example. This example, though simple, may however call for all the resources of the algorithm.

A first draft of this algorithm can be obtained by sending an e-mail to the first author.

2. IMPLICIT CONTROL SYSTEMS ON MANIFOLDS OF JETS OF INFINITE ORDER

Given an infinitely differentiable manifold X of dimension n, we denote its tangent space at $x \in X$ by T_xX, and its tangent bundle by $\mathrm{T}X$. Let F be a meromorphic function from $\mathrm{T}X$ to $\mathbb{R}^{n-m}$. We consider an underdetermined implicit system of the form

$$F(x, \dot{x}) = 0 \tag{2}$$

regular in the sense that rk $\frac{\partial F}{\partial \dot{x}} = n - m$ in a suitable dense open subset of $\mathrm{T}X$.

According to the implicit function theorem, any explicit system (1) with $x \in X$, $(x, f(x, u)) \in \mathrm{T}_xX$ for every u in an open subset U of $\mathbb{R}^m$, and rk $\frac{\partial f}{\partial u} = m$ in a suitable open subset of $X \times U$, can be locally transformed into (2), and conversely.

A vector field f that depends, for every $x \in X$, on m independent variables $u \in \mathbb{R}^m$ in a meromorphic way with rk $\frac{\partial f}{\partial u} = m$ in a suitable open subset of $X \times \mathbb{R}^m$, satisfying $F(x, f(x, u)) = 0$ for every $u \in U$, is called *compatible* with (2).

Note that this elimination step, though easy for some classes of systems, e.g., affine with respect to u, may be non trivial in general. Remark also that the implicit representation (2), as opposed to (1), is invariant by endogeneous dynamic extension (see [9] for a precise definition).

In [9] (see also [22] where a similar approach has been developed independently), infinite systems of coordinates $(x, \overline{u}) = (x, u, \dot{u}, \ldots)$ have been introduced to deal with prolonged vector fields

$$\overline{f}(x, \overline{u}) = \sum_{i=1}^{n} f_i(x, u)\frac{\partial}{\partial x_i} + \sum_{j=1}^{m}\sum_{k\geq 0} u_j^{(k+1)}\frac{\partial}{\partial u_j^{(k)}}$$

the original system being in explicit form (1).

Following [14, 15], we adopt an external description of the prolonged manifold containing the solutions of (2): we consider the infinite dimensional manifold $\mathfrak{X}$ defined by $\mathfrak{X} \stackrel{\text{def}}{=} X \times \mathbb{R}^n_\infty \stackrel{\text{def}}{=} X \times \mathbb{R}^n \times \mathbb{R}^n \times \ldots$, made of an infinite (but countable) number of copies of $\mathbb{R}^n$, endowed with the product topology, and we assume that we are given the global infinite set of coordinates of $\mathfrak{X}$:

$$\overline{x} = \left(x, \dot{x}, \ldots, x^{(k)}, \ldots,\right). \tag{3}$$

Recall that, in this topology, a function φ from $\mathfrak{X}$ to $\mathbb{R}$ is *continuous* (resp. *differentiable*) if φ depends only on a finite (but otherwise arbitrary) number of variables and is continuous (resp. differentiable) with respect to these variables. C^∞ or analytic or meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ are then defined as in the usual finite dimensional case since they only depend on a finite number of variables. We endow $\mathfrak{X}$ with the so-called trivial Cartan field ([13, 31])

$$\tau_{\mathfrak{X}} = \sum_{i=1}^{n}\sum_{j\geq 0} x_i^{(j+1)}\frac{\partial}{\partial x_i^{(j)}}. \tag{4}$$

We also denote by $L_{\tau_{\mathfrak{X}}}\varphi = \sum_{i=1}^{n}\sum_{j\geq 0} x_i^{(j+1)}\frac{\partial \varphi}{\partial x_i^{(j)}} = \frac{d\varphi}{dt}$ the Lie derivative of a differentiable function φ along $\tau_{\mathfrak{X}}$ and $L^k_{\tau_{\mathfrak{X}}}\varphi$ its kth iterate. Thus $x_i^{(k)} = \frac{d^k x_i}{dt^k} = L^k_{\tau_{\mathfrak{X}}} x_i$ for every $i = 1, \ldots, n$ and $k \geq 1$, with the convention $x_i^{(0)} = x_i$. Since $\frac{d}{dt}x_i^{(j)} \stackrel{\text{def}}{=} \dot{x}_i^{(j)} = x_i^{(j+1)}$, the Cartan field acts on coordinates as a shift to the right. $\mathfrak{X}$ is thus called *manifold of jets of infinite order*. From now on, $\overline{x}$ $\overline{y}$, ... stand for the sequences of jets of infinite order of x, y,...

A regular implicit control system is defined as a triple $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *with* $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\tau_{\mathfrak{X}}$ *its associated trivial Cartan field, and* F *meromorphic from* $\mathrm{T}X$ *to* $\mathbb{R}^{n-m}$) *satisfying* rk $\frac{\partial F}{\partial \dot{x}} = n - m$ *in a suitable open subset of* $\mathrm{T}X$.

2.1 Lie-Bäcklund equivalence for implicit systems

We recall from [14, 15] the following definition:

Let us consider two regular implicit control systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$, with $\mathfrak{X} = X \times \mathbb{R}^n_\infty$, $\dim X = n$ and rk $\frac{\partial F}{\partial \dot{x}} = n - m$, and $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$, with $\mathfrak{Y} = Y \times \mathbb{R}^p_\infty$, $\dim Y = p$, $\tau_{\mathfrak{Y}}$ its trivial Cartan field, and rk $\frac{\partial G}{\partial \dot{y}} = p - q$.

Set $\mathfrak{X}_0 = \{\overline{x} \in \mathfrak{X} | L^k_{\tau_{\mathfrak{X}}} F(\overline{x}) = 0,\ \forall k \geq 0\}$ and $\mathfrak{Y}_0 = \{\overline{y} \in \mathfrak{Y} | L^k_{\tau_{\mathfrak{Y}}} G(\overline{y}) = 0,\ \forall k \geq 0\}$. They are endowed with the topologies and differentiable structures induced by $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.

DEFINITION 1. *We say that the regular implicit control systems* $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *and* $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ *are* Lie-Bäcklund equivalent *(or shortly L-B equivalent) at the pair of points* $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathfrak{Y}_0$ *if and only if*

(i) there exist neighborhoods $\mathcal{X}_0$ *and* $\mathcal{Y}_0$ *of* $\overline{x}_0$ *in* $\mathfrak{X}_0$ *and* $\overline{y}_0$ *in* $\mathfrak{Y}_0$ *respectively and a one-to-one meromorphic mapping* $\Phi = (\varphi_0, \varphi_1, \ldots)$ *from* $\mathcal{Y}_0$ *to* $\mathcal{X}_0$ *satisfying* $\Phi(\overline{y}_0) = \overline{x}_0$ *and such that the trivial Cartan fields are* Φ*-related, namely* $\Phi_*\tau_{\mathfrak{Y}} = \tau_{\mathfrak{X}}$*;*

(ii) there exists Ψ *one-to-one and meromorphic from* $\mathcal{X}_0$ *to* $\mathcal{Y}_0$, *with* $\Psi = (\psi_0, \psi_1, \ldots)$, *such that* $\Psi(\overline{x}_0) = \overline{y}_0$ *and* $\Psi_*\tau_{\mathfrak{X}} = \tau_{\mathfrak{Y}}$.

The mappings Φ *and* Ψ *are called* mutually inverse Lie-Bäcklund isomorphisms *at* $(\overline{x}_0, \overline{y}_0)$.

The two systems $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ *and* $(\mathfrak{Y}, \tau_{\mathfrak{Y}}, G)$ *are called* locally L-B equivalent *if they are L-B equivalent at every pair* $(\overline{x}, \Psi(\overline{x})) = (\Phi(\overline{y}), \overline{y})$ *of an open dense subset* $\mathcal{Z}$ *of* $\mathfrak{X}_0 \times \mathfrak{Y}_0$, *with* Φ *and* Ψ *mutually inverse Lie-Bäcklund isomorphisms on* $\mathcal{Z}$.

As a result, local L-B equivalence preserves equilibrium points, namely points $\tilde{y}$ (resp. $\tilde{x}$) such that $G(\tilde{y}, 0) = 0$ (resp. $F(\tilde{x}, 0) = 0$), and coranks $(m = q)$.

2.2 Differential Forms

Let us introduce a basis of the tangent space $\mathrm{T}_{\overline{x}}\mathfrak{X}$ of $\mathfrak{X}$ at a point $\overline{x} \in \mathfrak{X}$ consisting of the set of vectors

$$\{\frac{\partial}{\partial x_i^{(j)}} | i = 1, \ldots, n, j \geq 0\}.$$

A basis of the cotangent space $T^*_{\overline{x}}\mathfrak{X}$ at $\overline{x}$ is given by $\{dx_i^{(j)} | i = 1, \ldots, n, j \geq 0\}$ with $< dx_i^{(j)}, \frac{\partial}{\partial x_k^{(l)}} >= \delta_{i,k}\delta_{j,l}$, $\delta_{i,k}$ being the Kronecker symbol.

The differential of F is thus given, in matrix notations, by

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial \dot{x}}d\dot{x}. \tag{5}$$

Note that the shift property of $\frac{d}{dt}$ on coordinates extends to differentials: $\frac{d}{dt}dx = d\dot{x} = d\frac{d}{dt}x$, i.e. $\frac{d}{dt}$ commutes with d.

Since a smooth function depends on a finite number of variables, its differential contains only a finite number of non zero terms. Accordingly, we define a 1-form on $\mathfrak{X}$ as a *finite* linear combination of the $dx_i^{(j)}$'s, with coefficients meromorphic from $\mathfrak{X}$ to $\mathbb{R}$ or, equivalently as a local meromorphic section of $T^*\mathfrak{X}$. The set of 1-forms is noted $\Lambda^1(\mathfrak{X})$. We also denote by $\Lambda^p(\mathfrak{X})$ the module of all the p-forms on $\mathfrak{X}$, by $(\Lambda^p(\mathfrak{X}))^m$ the space of all the m-dimensional vector p-forms on $\mathfrak{X}$, by $(\Lambda(\mathfrak{X}))^m$ the space of all the m-dimensional vector forms of arbitrary degree on $\mathfrak{X}$, and by $\mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right) = \mathcal{L}\left((\Lambda^p(\mathfrak{X}))^m, \left(\Lambda^{p+q}(\mathfrak{X})\right)^m, p \geq 1\right)$, the space of all linear operators from $(\Lambda^p(\mathfrak{X}))^m$ to $\left(\Lambda^{p+q}(\mathfrak{X})\right)^m$ for all $p \geq 1$, where $\mathcal{L}(\mathcal{P}, \mathcal{Q})$ denotes the set of linear mappings from a given space $\mathcal{P}$ to a given space $\mathcal{Q}$.

Note that if Φ is a meromorphic mapping from $\mathfrak{Y}$ to $\mathfrak{X}$, the definition of the (backward) image by Φ of a 1-form is the same as in the finite dimensional context.

2.3 Flatness

First recall from [9] that a system in explicit form is flat if and only if it is L-B equivalent to a trivial system. The reader may easily check that this definition is just a concise restatement of the definiton given in the Introduction. In our implicit context, it reads:

Definition 2. *The implicit system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is* flat *at $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}^m_\infty$ if and only if it is L-B equivalent at $(\overline{x}_0, \overline{y}_0)$ to the trivial implicit system $(\mathbb{R}^m_\infty, \tau_{\mathbb{R}^m_\infty}, 0)$. In this case, the mutually inverse L-B isomorphisms Φ and Ψ are called* inverse trivializations.

The next result is proven in [15].

Theorem 1. *The system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ is flat at $(\overline{x}_0, \overline{y}_0) \in \mathfrak{X}_0 \times \mathbb{R}^m_\infty$ if and only if there exists a local meromorphic invertible mapping Φ from $\mathbb{R}^m_\infty$ to $\mathfrak{X}_0$, with meromorphic inverse, satisfying $\Phi(\overline{y}_0) = \overline{x}_0$, and such that*

$$\Phi^* dF = 0. \tag{6}$$

3. NECESSARY AND SUFFICIENT CONDITIONS FOR DIFFERENTIAL FLATNESS

We now analyze condition (6) in more details: it characterizes the linear tangent mapping of Φ whose image entirely lies in the kernel of dF. The set of such mappings may be obtained in a systematic way in the framework of polynomial matrices by considering the following matrices polynomial with respect to the differential operator $\frac{d}{dt}$ (we use indifferently $\frac{d}{dt}$ for $L_{\tau_{\mathfrak{X}}}$ or $L_{\tau_{\mathbb{R}^m_\infty}}$, the context being unambiguous):

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}}\frac{d}{dt}, \quad P(\varphi_0) = \sum_{j\geq 0} \frac{\partial \varphi_0}{\partial y^{(j)}}\frac{d^j}{dt^j} \tag{7}$$

with $P(F)$ (resp. $P(\varphi_0)$) of size $(n-m) \times n$ (resp. $n \times m$. Equation (6) reads:

$$\Phi^* dF = P(F)P(\varphi_0)dy = 0. \tag{8}$$

Clearly, the entries of the matrices in (7) are polynomials of the differential operator $\frac{d}{dt}$ with meromorphic coefficients from $\mathfrak{X}$ to $\mathbb{R}$.

We denote by $\mathfrak{K}$ the field of meromorphic functions from $\mathfrak{X}$ to $\mathbb{R}$ and by $\mathfrak{K}[\frac{d}{dt}]$ the principal ideal ring of polynomials of $\frac{d}{dt}$ with coefficients in $\mathfrak{K}$. Note that $\mathfrak{K}[\frac{d}{dt}]$ is non commutative, even if $n = 1$: for every $a \in \mathfrak{K}$, $a \neq 0$, we have $\left(\frac{d}{dt} \cdot x - x \cdot \frac{d}{dt}\right)(a) = \dot{x}a + x\dot{a} - x\dot{a} = \dot{x}a \neq 0$, or $\frac{d}{dt} \cdot x - x \cdot \frac{d}{dt} = \dot{x}$.

For $r, s \in \mathbb{N}$, let us denote by $\mathcal{M}_{r,s}[\frac{d}{dt}]$ the module of $r \times s$ matrices over $\mathfrak{K}[\frac{d}{dt}]$ (see e.g. [5]). Recall that, for any $r \in \mathbb{N}$, the inverse of a square invertible matrix of $\mathcal{M}_{r,r}[\frac{d}{dt}]$ is not in general in $\mathcal{M}_{r,r}[\frac{d}{dt}]$. Matrices whose inverse belong to $\mathcal{M}_{r,r}[\frac{d}{dt}]$ are called *unimodular matrices* . They form a multiplicative group denoted by $\mathcal{U}_r[\frac{d}{dt}]$. Every matrix in $\mathcal{M}_{r,s}[\frac{d}{dt}]$ admits a *Smith decomposition* (or diagonal reduction). Without loss of generality, we only state its definition for $P(F) \in \mathcal{M}_{n-m,n}[\frac{d}{dt}]$:

$$VP(F)U = (\Delta, 0_{n-m,m}) \tag{9}$$

with $0_{n-m,m}$ the $(n-m) \times m$ matrix whose entries are all zeros, $V \in \mathcal{U}_{n-m}[\frac{d}{dt}]$, $U \in \mathcal{U}_n[\frac{d}{dt}]$ and $\Delta \in \mathcal{M}_{n-m,n-m}[\frac{d}{dt}]$ a diagonal matrix whose entries $d_{i,i}$ divide $d_{j,j}$ for all $0 \leq i \leq j \leq n-m$. Moreover, the degrees of the $d_{i,i}$'s are uniquely defined (see [5]).

Definition 3. *A matrix $M \in \mathcal{M}_{r,s}[\frac{d}{dt}]$ is said* hyper-regular *if and only if its Smith decomposition leads to either $(I_r, 0_{r,s-r})$ if $r < s$, or to I_r if $r = s$, or to $\begin{pmatrix} I_s \\ 0_{r-s,s} \end{pmatrix}$ if $r > s$.*

Note that a square matrix $M \in \mathcal{M}_{r,r}[\frac{d}{dt}]$ is hyper-regular if and only if it is unimodular.

According to the equivalence between flatness and controllability of the tangent linear system (see [9]) and controllability and freeness of the module associated to the tangent linear system (see [6]), it is proven in [15] that $P(F)$ is hyper-regular around every integral curve of the system $(\mathfrak{X}, \tau_{\mathfrak{X}}, F)$ if and only if its corresponding tangent module is free.

3.1 Algebraic characterization of the differential of a trivialization

From now on, we assume that $P(F)$ is hyper-regular in a neighborhood of $\overline{x}_0$. In other words, there exist V and U such that

$$VP(F)U = (I_{n-m}, 0_{n-m,m}). \tag{10}$$

U and V satisfying (10) are indeed non unique. We say that $U \in \mathsf{R-Smith}(P(F))$ and $V \in \mathsf{L-Smith}(P(F))$ if they are such that $VP(F)U = (I_m, 0)$.

Accordingly, if $M \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ is hyper-regular with $m \leq n$, we say that $V \in \mathsf{L-Smith}(M)$ and $W \in \mathsf{R-Smith}(M)$ if $V \in \mathcal{U}_n[\frac{d}{dt}]$ and $W \in \mathcal{U}_m[\frac{d}{dt}]$ satisfy $VMW = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$.

In place of (8), we first solve the matrix equation:

$$P(F)\Theta = 0 \tag{11}$$

where the entries of $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ are not supposed to be gradients of some function φ_0.

Lemma 1. *The set of hyper-regular matrices $\Theta \in \mathcal{M}_{n,m}[\frac{d}{dt}]$ satisfying (11) is nonempty and given by*

$$\Theta = U\begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix}W \tag{12}$$

with $U \in \mathsf{R}-\mathsf{Smith}\,(P(F))$ and $W \in \mathfrak{U}_m[\frac{d}{dt}]$ arbitrary.

LEMMA 2. *For every $Q \in \mathsf{L}-\mathsf{Smith}\left(\hat{U}\right)$, with $\hat{U}$ given by*

$$\hat{U} = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} \tag{13}$$

there exists $Z \in \mathfrak{U}_m[\frac{d}{dt}]$ such that

$$Q\Theta = \begin{pmatrix} I_m \\ 0_{n-m,m} \end{pmatrix} Z. \tag{14}$$

Moreover, for every $Q \in \mathsf{L}-\mathsf{Smith}\left(\hat{U}\right)$, the submatrix $\hat{Q} = (0_{n-m,m}, I_{n-m})\,Q$ is equivalent to $P(F)$ ($\exists L \in \mathfrak{U}_{n-m}[\frac{d}{dt}]$ such that $P(F) = L\hat{Q}$).

3.2 Integrability

Let us denote by $Q_{i,j} = \sum_{k\geq 0} Q^k_{i,j}\frac{d^k}{dt^k}$ the (i,j)-th polynomial entry of $Q \in \mathsf{L}-\mathsf{Smith}\left(\hat{U}\right)$ obtained from Lemma 2. We also denote by ω the m-dimensional vector 1-form defined by

$$\begin{aligned} \omega(\overline{x}) &= \begin{pmatrix} \omega_1(\overline{x}) \\ \vdots \\ \omega_m(\overline{x}) \end{pmatrix} = (I_m, 0_{m,n-m})\,Q(\overline{x})dx_{|\mathfrak{X}_0} \\ &= \begin{pmatrix} \sum_{j=1}^n \sum_{k\geq 0} Q^k_{1,j}(\overline{x})dx^{(k)}_j{}_{|\mathfrak{X}_0} \\ \vdots \\ \sum_{j=1}^n \sum_{k\geq 0} Q^k_{m,j}(\overline{x})dx^{(k)}_j{}_{|\mathfrak{X}_0} \end{pmatrix} \end{aligned} \tag{15}$$

the restriction to $\mathfrak{X}_0$ meaning that $\overline{x} \in \mathfrak{X}_0$ satisfies $L^k_{\tau_\mathfrak{X}}F = 0$ for all k and that the $dx^{(k)}_j$ are such that $dL^k_{\tau_\mathfrak{X}}F = 0$ in $\mathfrak{X}_0$ for all k. Since $\hat{Q}$ is hyper-regular, the forms $\omega_1, \ldots, \omega_m$ are independent by construction.

Let us also recall that, if $\tau_1, \ldots, \tau_m$ are given independent 1-forms in $\Lambda^1(\mathfrak{X}_0)$, the $\mathfrak{K}[\frac{d}{dt}]$-ideal $\mathfrak{T}$ generated by $\tau_1, \ldots, \tau_m$ is the set of all combinations with coefficients in $\mathfrak{K}[\frac{d}{dt}]$ of forms $\eta \wedge \tau_i$ with η an arbitrary form on $\mathfrak{X}_0$ of arbitrary degree and $i = 1, \ldots, m$.

DEFINITION 4. *We say that the $\mathfrak{K}[\frac{d}{dt}]$-ideal $\mathfrak{T}$ generated by $\tau_1, \ldots, \tau_m$ is* strongly closed *if and only of there exists a matrix $M \in \mathfrak{U}_m[\frac{d}{dt}]$ such that $d(M\tau) = 0$.*

This definition is indeed independent of the choice of generators.

THEOREM 2. *A necessary and sufficient condition for system (2) to be flat at the point $(\overline{x}_0, \overline{y}_0)$ is that there exist $U \in \mathsf{R}-\mathsf{Smith}\,(P(F))$ and $Q \in \mathsf{L}-\mathsf{Smith}\left(\hat{U}\right)$, with $\hat{U}$ given by (13), such that the $\mathfrak{K}[\frac{d}{dt}]$-ideal Ω generated by the 1-forms $\omega_1, \ldots, \omega_m$ defined by (15) is strongly closed in $\mathfrak{X}_0$.*

In order to develop the expression $d(M\tau)$ for M polynomial matrix, we define the operator $\mathfrak{d}$ by:

$$\mathfrak{d}\,(H)\,\kappa = d(H\kappa) - Hd\kappa \tag{16}$$

for all m-dimensional vector p-form κ in $(\Lambda^p(\mathfrak{X}))^m$ and all $p \geq 1$. Note that (16) uniquely defines $\mathfrak{d}\,(H)$ as an element of $\mathcal{L}_1\left((\Lambda(\mathfrak{X}))^m\right)$.

We can prolong $\mathfrak{d}$ for all $\mu \in \mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right)$ and for all $\kappa \in (\Lambda^p(\mathfrak{X}))^m$ and all $p \geq 1$ by the formula:

$$\mathfrak{d}\,(\mu)\,\kappa = d(\mu\,\kappa) - (-1)^q\mu\,d\kappa. \tag{17}$$

THEOREM 3. *The $\mathfrak{K}[\frac{d}{dt}]$-ideal Ω generated by the 1-forms $\omega_1, \ldots, \omega_m$ defined by (15) is strongly closed in $\mathfrak{X}_0$ (or, equivalently, the system $(\mathfrak{X}, \tau_\mathfrak{X}, F)$ is flat) if and only if there exists $\mu \in \mathcal{L}_1\left((\Lambda(\mathfrak{X}))^m\right)$, and a matrix $M \in \mathfrak{U}_m[\frac{d}{dt}]$ such that*

$$d\omega = \mu\;\omega, \qquad \mathfrak{d}\,(\mu) = \mu^2, \qquad \mathfrak{d}\,(M) = -M\mu. \tag{18}$$

where we have noted $\mu^2 = \mu\mu$.

In addition, if (18) holds true, a flat output y is obtained by integration of $dy = M\omega$.

Note that conditions (18) may be seen as a generalization in the framework of manifolds of jets of infinite order of Cartan's well-known moving frame structure equations (see e.g. [3]).

3.3 A Theoretical Algorithm

From the necessary and sufficient conditions (18), we derive the following abstract algorithm:

1. We first compute a Smith decomposition[1] of $P(F)$ and then $\hat{U}$ as decribed in Lemmas 1 and 2. If $P(F)$ is not hyperregular, the system is non flat. Otherwise compute the vector 1-form ω defined by (15).

2. We compute the operator μ such that $d\omega = \mu\omega$ by componentwise identification. It is easy to prove that such μ always exists.

3. Among the possible μ's, only those satisfying $\mathfrak{d}\,(\mu) = \mu^2$ are kept. If no μ satisfy this relation, the system is non flat.

4. We then compute M such that $\mathfrak{d}\,(M) = -M\mu$, still by componentwise identification.

5. Finally, only those matrices M which are unimodular are kept. If there are no such M, the system is non flat. In the opposite case, a flat output is obtained by integration of $dy = M\omega$, which is possible since $d(M\omega) = 0$.

4. COMPUTER ALGEBRA IMPLEMENTATION

Before discussing the computer algebra implementation, useful general structures and formulae are derived. Note that computational cost aspects are not considered here. We only aim at showing that Algorithm 3.3 can be implemented using a standard computer algebra system.

4.1 The structure of elements of $\mathcal{M}_{r,s}[\frac{d}{dt}]$

The elements of matrices $A \in \mathcal{M}_{r,s}[\frac{d}{dt}]$ have the structure

$$[A]_{ij} = \sum_{k\geq 0} a_{ijk}\frac{d^k}{dt^k}, \quad i = 1,2,\ldots r;\; j = 1,2,\ldots,s \tag{19}$$

where the a_{ijk}'s are smooth functions on $\mathfrak{X}$. Thus, for $\omega \in (\Lambda^p(\mathfrak{X}))^s$, $A\omega \in (\Lambda^p(\mathfrak{X}))^r$ is the vector whose ith component is

$$[A\omega]_i = \sum_{j=1}^s \sum_{k\geq 0} a_{ijk}L^k_{\tau_\mathfrak{X}}\omega_j, \quad i = 1,2,\ldots,r \tag{20}$$

[1]Clearly, in this context, Smith-Decomposition is far from being optimal with respect to computational cost, but is only used to prove the existence of the resulting polynomial matrices. Improvements will be investigated in future work

The multiplication of two matrices $A \in \mathcal{M}_{r_1,r_2}[\frac{d}{dt}]$ and $B \in \mathcal{M}_{r_2,r_3}[\frac{d}{dt}]$ given by $[A]_{i,j} = \sum_{k\geq 0} a_{i,j,k}\frac{d^k}{dt^k}$, and $[B]_{i,j} = \sum_{k\geq 0} b_{i,j,k}\frac{d^k}{dt^k}$, is

$$
\begin{aligned}
&[AB]_{i,j} = \sum_{l=1}^{r_2}\sum_{k_1,k_2\geq 0} a_{i,l,k_1}\frac{d^{k_1}}{dt^{k_1}}\left(b_{l,j,k_2}\frac{d^{k_2}}{dt^{k_2}}\right)\\
&= \sum_{l=1}^{r_2}\sum_{k_1,k_2\geq 0}\sum_{k_3=0}^{k_1}\begin{pmatrix} k_1 \\ k_3\end{pmatrix} a_{i,l,k_1}\left(L_{\tau_{\mathfrak{X}}}^{k_1-k_3} b_{l,j,k_2}\right)\frac{d^{k_2+k_3}}{dt^{k_2+k_3}}
\end{aligned}
\tag{21}
$$

with $\begin{pmatrix} k_1 \\ k_3\end{pmatrix} = \frac{k_1!}{k_3!(k_1-k_3)!}$.

By explicitely indicating the dependence of the coordinates on the independent variable t it becomes possible to use the *mult* command (with DDt being used in our implementation to symbolize $\frac{d}{dt}$) of the *DETools* package of *Maple 11*. The function *mskew* (Input: $A \in \mathcal{M}_{r_1,r_2}[\frac{d}{dt}]$, $B \in \mathcal{M}_{r_2,r_3}[\frac{d}{dt}]$; Output: $C = AB \in \mathcal{M}_{r_1,r_3}[\frac{d}{dt}]$) provides a multiplication which corresponds to (21). Note furthermore that specifying the associated Cartan field to deal with functions of t only is not necessary (a discussion of this is done Section 4.4).

4.2 Smith-Decomposition of elements of $\mathcal{M}_{r,s}[\frac{d}{dt}]$

With the matrix multiplication over $\mathfrak{K}[\frac{d}{dt}]$ using *mskew*, a Smith decomposition of matrices $A \in \mathcal{M}_{r,s}[\frac{d}{dt}]$ can be implemented by adapting, e.g., the algorithm given in [11] for polynomial matrices with constant coefficients to the non-commutative case by constructing suitable unimodular matrices for left and right actions (see [5] for more details). The resulting *Maple* procedure has been called *Smith_sa* (Input: $A \in \mathcal{M}_{r,s}[\frac{d}{dt}]$; Output: $U \in \mathsf{R-Smith}(A)$ and $V \in \mathsf{L-Smith}(A)$). The *DETools* package of *Maple 11* provides all necessary operations.

4.3 The structure of elements of $\mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right)$

The elements of an operator $\mu \in \mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right)$ have the structure

$$
[\mu]_{ij} = \sum_{k\geq 0}\mu_{ijk}\wedge\frac{d^k}{dt^k},\quad i,j = 1,2,\ldots,m \tag{22}
$$

where μ_{ijk} is an arbitrary q-form, which means that, for every $\omega \in (\Lambda^p(\mathfrak{X}))^m$, $\mu\omega \in \left(\Lambda^{p+q}(\mathfrak{X})\right)^m$ is given by

$$
[\mu\omega]_i = \sum_{j=1}^{m}\sum_{k\geq 0}\mu_{ijk}\wedge L_{\tau_{\mathfrak{X}}}^k\omega_j,\quad i = 1,2,\ldots,m \tag{23}
$$

To implement operators $\mu \in \mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right)$ in *Maple 11* we define the operator itself by specifying it of (*Maple-*) type *Matrix*, the components being differential forms on a previously specified *Frame*. A *Frame* fixes the coordinates of a jet space in the *DifferentialGeometry* package. The coefficients of the differential forms are then specified as polynomials in $\frac{d}{dt}$. Thus the evaluation of μ on a p-form ω, according to (23), uses both $\wedge$ and $\frac{d}{dt}$. Therefore, a special function, called *Dtwedge* (Input: $\omega \in (\Lambda^p(\mathfrak{X}))^m$, $\omega \in (\Lambda^p(\mathfrak{X}))^m$; Output: $\mu\omega \in \left(\Lambda^{p+q}(\mathfrak{X})\right)^m$) is introduced. The core algorithm of this function, for fixed i and j, results to (assuming that μ has some finite degree l w.r.t $\frac{d}{dt}$)

- set k=0
- set $\gamma = 0$

while $k \leq l$ **do**
- extract the coefficient μ_{ijk} of μ_{ij} (using the commmand *DGmap* of the *DifferentialGeometry* package together with the standard *Maple* command *coeff*)
- compute $\frac{d^k}{dt^k}\omega$
- set $\gamma = \gamma + \mu_{ijk}\wedge\frac{d^k}{dt^k}\omega$
- set k=k+1

end while
- set $\mu\omega = \gamma$

Note that, in this case, the time derivative of ω in step 2 has to be implemented using the Lie-Derivative with respect to a cartan field on the previously defined jet space of a suitable prolongation order (a more detailed discussion of the choice of the truncation order is given in Section 4.4). It becomes clear that, though the evaluation of (23) is a combination of standard operations, its implementation is not obvious in standard computer algebra systems.

4.3.1 *Multiplication of two elements* $\mu \in \mathcal{L}_{q_1}\left((\Lambda(\mathfrak{X}))^m\right)$ *and* $\kappa \in \mathcal{L}_{q_2}\left((\Lambda(\mathfrak{X}))^m\right)$

We consider $\mu = \sum_{k\geq 0}\mu_k\wedge\frac{d^k}{dt^k}$ where the μ_k's are matrices whose entries are in $\Lambda^{q_1}(\mathfrak{X})$, and $\kappa = \sum_{k\geq 0}\kappa_k\wedge\frac{d^k}{dt^k}$ where the κ_k's are matrices whose entries are in $\Lambda^{q_2}(\mathfrak{X})$. The product $\mu\kappa$ is evaluated by considering the product $\mu\kappa\omega$ for every $\omega \in (\Lambda(\mathfrak{X}))^m$:

$$
\begin{aligned}
&\mu\kappa\omega = \mu(\kappa\omega) = \sum_{k_1\geq 0}\mu_{k_1}\wedge L_{\tau_{\mathfrak{X}}}^{k_1}\left(\sum_{k_2\geq 0}\kappa_{k_2}\wedge L_{\tau_{\mathfrak{X}}}^{k_2}\omega\right)\\
&= \sum_{k_1,k_2\geq 0}\sum_{k_3=0}^{k_1}\begin{pmatrix} k_1 \\ k_3\end{pmatrix}\mu_{k_1}\wedge\left(L_{\tau_{\mathfrak{X}}}^{k_1-k_3}\kappa_{k_2}\right)\wedge\left(L_{\tau_{\mathfrak{X}}}^{k_2+k_3}\omega\right)
\end{aligned}
$$

Therefore:

$$
\mu\kappa = \sum_{k_1,k_2\geq 0}\sum_{k_3=0}^{k_1}\begin{pmatrix} k_1 \\ k_3\end{pmatrix}\mu_{k_1}\wedge\left(L_{\tau_{\mathfrak{X}}}^{k_1-k_3}\kappa_{k_2}\right)\wedge\frac{d^{k_2+k_3}}{dt^{k_2+k_3}} \tag{24}
$$

In partcular, setting $\mu = \frac{d}{dt}$, we immediately deduce that:

$$
\frac{d}{dt}\kappa = \sum_{k}\left(L_{\tau_{\mathfrak{X}}}\kappa_k\wedge\frac{d^k}{dt^k} + \kappa_k\wedge\frac{d^{k+1}}{dt^{k+1}}\right) \tag{25}
$$

The multiplication of two such operators is done with the function *Dtwedgeop*. Its input is: $\mu \in \mathcal{L}_{q_1}\left((\Lambda(\mathfrak{X}))^m\right)$, $\kappa \in \mathcal{L}_{q_2}\left((\Lambda(\mathfrak{X}))^m\right)$ and output: $\mu\kappa \in \mathcal{L}_{q_1+q_2}\left((\Lambda(\mathfrak{X}))^m\right)$. Its core algorithm using *Maple* code results to (assuming again that μ has a finite degree l w.r.t. $\frac{d}{dt}$)

- set $j = 0$
- compute $\gamma = \mu\wedge\kappa$

while $k \leq l$ **do**
- compute $\kappa^{(j+1)} = \frac{d}{dt}\kappa^{(j)}$ according to (25)
- set $\gamma = \gamma + \mu_{j+1}\kappa^{(j+1)}$

end while
- set $\mu\kappa = \gamma$

4.3.2 *The operator* $\mathfrak{d}$

At this point we investigate the definition (17) of the operator $\mathfrak{d} \in \mathcal{L}_1\left((\Lambda(\mathfrak{X}))^m\right)$.

We first remark that, for $m = 1$, if μ is a 0th-order polynomial w.r.t. $\frac{d}{dt}$, i.e. $\mu = \mu_0\wedge$ with $\mu_0 \in \Lambda^q(\mathfrak{X})$, (17) boils down to the usual anti-derivation property of the exterior derivative, i.e. for every $\omega \in \Lambda^p(\mathfrak{X})$

$$
d(\mu\omega) = \underbrace{d\mu_0\wedge}_{\mathfrak{d}(\mu)}\omega + (-1)^q\mu_0\wedge d\omega \tag{26}
$$

Then, going back to the general case, with $\mu \in \mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right)$

and $\omega \in (\Lambda^p(\mathfrak{X}))^m$, we have, for $i = 1, 2, \ldots, m$,

$$\begin{aligned}
[d(\mu\omega)]_i &= d\left(\sum_{j=1}^m \sum_{k\geq 0} \mu_{ijk} \wedge L^k_{\tau_{\mathfrak{X}}} \omega_j\right) \\
&= \Big(\sum_{j=1}^m \sum_{k\geq 0} d\mu_{ijk} \wedge L^k_{\tau_{\mathfrak{X}}} \omega_j \\
&\qquad +(-1)^q \sum_{j=1}^m \sum_{k\geq 0} \mu_{ijk} \wedge d\left(L^k_{\tau_{\mathfrak{X}}} \omega_j\right)\Big) \\
&= \Big(\sum_{j=1}^m \sum_{k\geq 0} d\mu_{ijk} \wedge L^k_{\tau_{\mathfrak{X}}} \omega_j \\
&\qquad +(-1)^q \sum_{j=1}^m \sum_{k\geq 0} \mu_{ijk} \wedge L^k_{\tau_{\mathfrak{X}}} (d\omega_j)\Big)
\end{aligned} \tag{27}$$

We then calculate $\mathfrak{d}(\mu)\omega$ by combining (17) with (27):

$$\begin{aligned}
[\mathfrak{d}(\mu)\,\omega]_i &= [d(\mu\ \omega) - (-1)^q \mu\ d\omega]_i \\
&= \sum_{j=1}^m \sum_{k\geq 0} d\mu_{ijk} \wedge L^k_{\tau_{\mathfrak{X}}} \omega_j
\end{aligned} \tag{28}$$

As a consequence, the entries of $\mathfrak{d}(\mu)$ are simply given by

$$[\mathfrak{d}(\mu)]_{ij} = \sum_{k\geq 0} d\mu_{ijk} \wedge \frac{d^k}{dt^k}, \quad i, j = 1, 2, \ldots, m \tag{29}$$

Note that operators $\mu \in \mathcal{L}_q\left((\Lambda(\mathfrak{X}))^m\right)$ are specified for *Maple* as polynomials in *DDt*, the latter operator being a constant with respect to the Exterior Derivative on the Jet-Manifold $\mathfrak{X}$ with coordinates $(x, \dot{x}, \ldots)$. Thus, for the chosen implementation in *Maple*, $\mathfrak{d}$ boils down to applying the *ExteriorDerivative* command of the *DifferentialGeometry* package to the chosen representation of μ, which readily gives the procedure *gdmu* (Input: $\mu_0 \in \Lambda^q(\mathfrak{X})$, Output: $\mathfrak{d}(\mu_0) \in \Lambda^{q+1}(\mathfrak{X})$).

4.4 Iterative increase of truncation order and degree

The Algorithm 3.3 includes a large number of degrees of freedom. There are many choices in the Smith decompositions yielding the vector 1-form ω, but any other choice of a basis of the ideal Ω is, at least algebraically, equivalent. Nevertheless, the set of operators μ such that $d\omega = \mu\omega$, which is always non empty, generally contains infinitely many elements. A lower bound of its degree w.r.t. $\frac{d}{dt}$ is easily computed but there is no a priori upper bound. The only restriction is that the equation $\mathfrak{d}(\mu) = \mu^2$ must be satisfied. If we note $\mu = \sum_{k\geq 0} \mu_k \wedge \frac{d^k}{dt^k}$, as a consequence of (29) and (24), the matrices μ_k must satisfy the infinite sequence of differential equations:

$$d\mu_k = \sum_{k_2=0}^{k} \sum_{k_1\geq k-k_2} \begin{pmatrix} k_1 \\ k-k_2 \end{pmatrix} \mu_{k_1} \wedge \left(L^{k_1+k_2-k}_{\tau_{\mathfrak{X}}} \mu_{k_2}\right) \tag{30}$$

for all $k \geq 0$. However, since the degree of μ w.r.t. $\frac{d}{dt}$ is finite, i.e. only finitely many μ_k's are non zero, on the one hand, and since μ depends only on a finite number of coordinates of $\mathfrak{X}$ on the other hand, the number of non trivial equations in (30) is indeed finite. Moreover, (30) establishes a link between the number of coordinates that are active via the expression of $d\mu_k$, and the polynomial degree of μ. Therefore, for a given truncation order [2] and a given degree which are compatible relatively to (30), solutions μ and M, if they exist, may be found using the above computer algebra program. If not, the truncation order and/or the degree can be increased. Unfortunately, there is no simple answer to the question "is this process ending?".

[2] by truncation, we refer to the elimination of the infinitely many coordinates of which μ doesn't depend

Once μ is determined, according to step 4, a suitable unimodular matrix $M \in \mathfrak{U}_p\left[\frac{d}{dt}\right]$ satisfying $\mathfrak{d}(M) = -M\mu$, has to be found. As a first guess an initial M can always be chosen as an upper triangular Matrix with ones on the main diagonal, the north-eastern entries of M being polynomials in $\frac{d}{dt}$ of suitable degree with coefficients depending on the truncated coordinates. If the provided free parameters in M are not sufficient to find a solution, more complicated unimodular matrices can be constructed by left and right multiplication with elementary left and right actions (see e.g. [5]).

If in step 4 no suitable matrix M can be found then it may be necessary to go back to step 3 and to increase the truncation order and/or the degree of μ to introduce additional degrees of freedom.

Using Maple 11 and the chosen implementation of the operators, construction methods of general operators $\mu \in$ and matrices $M \in$ for given truncation orders and degrees with respect to $\frac{d}{dt}$ could be implemented. This easily allows to iteratively increase the used truncation order and degrees which is essential for the application of this approach.

5. NON HOLONOMIC CAR

Consider the 3 dimensional system in the $x-y$ plane, representing a vehicle of length l, whose orientation is given by the angle θ, the coordinates (x, y) standing for the position of the middle of the rear axle, and controlled by the velocity modulus u and the angular position of the front wheels φ.

$$\begin{aligned}
\dot{x} &= u\cos\theta \\
\dot{y} &= u\sin\theta \\
\dot{\theta} &= \tfrac{u}{l}\tan\varphi
\end{aligned} \tag{31}$$

Since $n = 3$ and $m = 2$, $n - m = 1$ and (31) is equivalent to the single implicit equation obtained by eliminating the inputs u and φ:

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \dot{x}\sin\theta - \dot{y}\cos\theta = 0 \tag{32}$$

We immediately have:

$$\begin{aligned}
P(F) &= \begin{pmatrix} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}}\frac{d}{dt} & \frac{\partial F}{\partial y} + \frac{\partial F}{\partial \dot{y}}\frac{d}{dt} & \frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial \dot{\theta}}\frac{d}{dt} \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\frac{d}{dt} & -\cos\theta\frac{d}{dt} & \dot{x}\cos\theta + \dot{y}\sin\theta \end{pmatrix}.
\end{aligned} \tag{33}$$

In the following, all steps of Algorithm 3.3 are performed using the above presented Maple functions. Note that this well-known example has been chosen as it is at the same time challenging enough to illustrate most of the properties of the proposed approach and can at the same time be discussed in detail.

Step 1: Setting $E = \dot{x}\cos\theta + \dot{y}\sin\theta$, we apply the Smith decomposition algorithm (i.e. we apply the function *Smith_sa*) and we get $U \in \mathsf{R-Smith}\,(P(F))$ with

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{E} & \frac{\cos\theta}{E}\frac{d}{dt} & -\frac{\sin\theta}{E}\frac{d}{dt} \end{pmatrix}.$$

Thus

$$\hat{U} = U\begin{pmatrix} 0_{1,2} \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{\cos\theta}{E}\frac{d}{dt} & -\frac{\sin\theta}{E}\frac{d}{dt} \end{pmatrix}$$

with I_2 the identity matrix of $\mathbb{R}^2$. Again, computing $Q \in \mathsf{L-Smith}\left(\hat{U}\right)$ yields

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{\sin\theta}{E}\frac{d}{dt} & -\frac{\cos\theta}{E}\frac{d}{dt} & 1 \end{pmatrix}.$$

Multiplying Q by the vector $(dx, dy, d\theta)^T$, the last line reads $\frac{1}{E}(\sin\theta d\dot{x} - \cos\theta d\dot{y} + (\dot{x}\cos\theta + \dot{y}\sin\theta)d\theta) = \frac{1}{E}d(\dot{x}\sin\theta - \dot{y}\cos\theta)$ and, by (32), identically vanishes on $\mathfrak{X}_0$.

The remaining part of the system, namely

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} dx \\ dy \\ d\theta \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

is trivially strongly closed with $M = I_2$, which finally gives the flat output (which we denote here by y_f to avoid confusion with the coordinate y) $y_f = (y, x)^T$. We have thus recovered the flat output originally obtained in [27, 26], up to a permutation of the components of y.

Step 1 b: Other decompositions of $P(F)$, given by (33), may indeed be obtained. They are all equivalent in the sense that one decomposition may be deduced from another one by multiplication by a unimodular matrix. However, the resulting vector 1-form ω, contrarily to what happens in the previous example, may not be integrable. Our aim is here to show how the generalized moving frame structure equations (18) may be used to obtain an integrable $M\omega$. Such an example is provided by restarting the right-Smith decomposition of $P(F)$ by right-mutiplying it by $\begin{pmatrix} \cos\theta & 0 & 0 \\ \sin\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and using the formula $\sin\theta\frac{d}{dt}(\cos\theta) - \cos\theta\frac{d}{dt}(\sin\theta) = -\dot{\theta}$. The Smith decomposition algorithm yields

$$U = \begin{pmatrix} \cos\theta & -\frac{1}{\dot{\theta}}\cos^2\theta\frac{d}{dt} & \frac{1}{\dot{\theta}}E\cos\theta \\ \sin\theta & 1 - \frac{1}{\dot{\theta}}\sin\theta\cos\theta\frac{d}{dt} & \frac{1}{\dot{\theta}}E\sin\theta \\ 0 & 0 & 1 \end{pmatrix},$$

i.e.

$$\hat{U} = \begin{pmatrix} -\frac{1}{\dot{\theta}}\cos^2\theta\frac{d}{dt} & \frac{1}{\dot{\theta}}E\cos\theta \\ 1 - \frac{1}{\dot{\theta}}\sin\theta\cos\theta\frac{d}{dt} & \frac{1}{\dot{\theta}}E\sin\theta \\ 0 & 1 \end{pmatrix}$$

The Smith decomposition of $\hat{U}$ then yields $Q \in \mathsf{L-Smith}\left(\hat{U}\right)$ with

$$Q = \begin{pmatrix} -\tan\theta & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\dot{\theta}}\sin\theta\cos\theta\frac{d}{dt} & \frac{1}{\dot{\theta}}\cos^2\theta\frac{d}{dt} & -\frac{1}{\dot{\theta}}E\cos\theta \end{pmatrix}.$$

Step 2: In this case ω is obtained as

$$\omega = (\omega_1, \omega_2)^T = \hat{Q}\,(dx, dy, d\theta)^T = (-\tan\theta dx + dy, d\theta)^T$$

Its exterior derivative is non zero:

$$d\omega = (d\omega_1, d\omega_2)^T = (-\frac{1}{\cos^2\theta}d\theta \wedge dx, 0)^T$$

showing that ω is not closed. The simplest possible operator μ is of truncated order 0 and degree 0:

$$\mu = \begin{pmatrix} 0 & \mu_{120} \\ 0 & 0 \end{pmatrix}\wedge$$

with $\mu_{120} = \mu_{1210}(x,y,\theta)dx + \mu_{1220}(x,y,\theta)dy + \mu_{1230}(x,y,\theta)d\theta$. Evaluation of $\mu\omega$ using the *Dtwedge* function yields

$$\mu\omega = \begin{pmatrix} \mu_{120} \wedge d\theta \\ 0 \end{pmatrix}$$

with $\mu_{120} \wedge d\theta = \mu_{1210}dx \wedge d\theta + \mu_{1220}dy \wedge d\theta$. The comparison of $d\omega$ and $\mu\omega$ yields $\mu_{1210} = \frac{1}{\cos^2\theta}$ and $\mu_{1220} = 0$, or

$$\mu = \begin{pmatrix} 0 & \frac{1}{\cos^2\theta}dx + \mu_{1230}(x,y,\theta)d\theta \\ 0 & 0 \end{pmatrix}\wedge$$

Step 3: By direct computation (using *Dtwedgeop*):

$$\mu^2 = \begin{pmatrix} 0 & \mu_{120} \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & \mu_{120} \\ 0 & 0 \end{pmatrix} = 0$$

On the other hand, using *gdmu*, we have

$$\mathfrak{d}(\mu) = \begin{pmatrix} 0 & d\left(\frac{1}{\cos^2\theta}dx + \mu_{1230}(x,y,\theta)d\theta\right) \\ 0 & 0 \end{pmatrix}\wedge$$

From $\mu^2 = \mathfrak{d}(\mu)$ we obtain the system of P.D.E.'s

$$\begin{aligned} \frac{\partial}{\partial x}\mu_{1230}(x,y,\theta) &= \frac{2\sin(\theta)}{\cos^3(\theta)} \\ \frac{\partial}{\partial y}\mu_{1230}(x,y,\theta) &= 0 \end{aligned}$$

whose solution, using *pdsolve*, is

$$\mu_{1230}(x,y,\theta) = \frac{2\sin(\theta)x}{\cos^3(\theta)} + C_1(\theta). \tag{34}$$

Thus

$$\mu = \begin{pmatrix} 0 & \frac{1}{\cos^2\theta}dx + \left(\frac{2\sin(\theta)x}{\cos^3(\theta)} + C_1(\theta)\right)d\theta \\ 0 & 0 \end{pmatrix}.$$

Step 4/5: The simplest unimodular matrix has truncation order 0 and degree 0:

$$M = \begin{pmatrix} 1 & m_{120}(x,y,\theta) \\ 0 & 1 \end{pmatrix}.$$

Computing $\mathfrak{d}(M) = -M\mu$, we get $\begin{pmatrix} 0 & dm_{120} \\ 0 & 0 \end{pmatrix} = \mu$, i.e.

$$\begin{aligned} \frac{\partial}{\partial x}m_{120}(x,y,\theta) &= -\frac{1}{\cos^2\theta} \\ \frac{\partial}{\partial y}m_{120}(x,y,\theta) &= 0 \\ \frac{\partial}{\partial \theta}m_{120}(x,y,\theta) &= -\left(\frac{2\sin(\theta)x}{\cos^3(\theta)} + C_1(\theta)\right) \end{aligned}$$

whose solution is (determined again with *pdsolve*)

$$m_{120} = -\frac{x}{\cos^2(\theta)} + C_2(\theta) \tag{35}$$

with $C_1(\theta)$ in (34) given by $C_1(\theta) = -\frac{d}{d\theta}C_2(\theta)$. It results that

$$M = \begin{pmatrix} 1 & -\frac{x}{\cos^2(\theta)} + C_2(\theta) \\ 0 & 1 \end{pmatrix}.$$

Thus, we get as differential of a flat output, which we denote here as above by y_f

$$dy_f = M\omega = \begin{pmatrix} -\tan(\theta)dx + dy + (-\frac{x}{\cos^2(\theta)} + C_2(\theta))d\theta \\ d\theta \end{pmatrix}$$

This one-form is closed and, using *pdsolve* we obtain the flat output

$$y_f = (y - x\tan(\theta) + C_3(\theta), \theta)^T.$$

6. CONCLUSIONS

In this contribution the necessary and sufficient conditions for differential flatness of nonlinear control systems have been discussed with the aim of arriving at a computer algebra implementation. To this end general formulae for the used operators have been deduced from its defining relations. It could be shown that all used opeartors can be implemented using, e.g., the computer algebra package *Maple 11*. However, as operations from differential geometry as well as from algebra were needed, the operators could not be directly implemented but special functions had to be created to implement the action of the results on differential forms or other operators. We want to emphasize that this paper is a first step towards the formal computation of flat output where computational costs are voluntarily ignored. Obtaining more efficient algorithms will be the subject of future work.

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