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A FULLY UNCONSTRAINED INTERIOR POINT ALGORITHM FOR MULTIVARIABLE STATE AND INPUT CONSTRAINED OPTIMAL CONTROL PROBLEMS.

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Abstract. *This paper exposes a methodology to solve constrained optimal control problems for non linear systems using interior penalty methods. A constructive choice for the penalty functions that are introduced to account for the constraints is established in the article. It is shown that this choice allows one to approach a solution of the non linear optimal control problem using a sequence of unconstrained problems, whose solutions are readily characterized by the simple calculus of variations. An illustrative example is given. The paper extends recent contributions, originally focused on single input single output systems.*

1 INTRODUCTION

This paper exposes a methodology allowing one to solve a constrained optimal control problem (COCP) for a general multi-input multi-output (MIMO) system with non linear dynamics. This methodology belongs to the class of *interior point methods* (IPMs) which consists in approaching the optimum by a path lying strictly inside the constraints. The reason for employing such a technique is that in the interior, optimality conditions are much easier to characterize and to explicit. For this purpose, penalty function approach commonly considered in finite dimensional optimization problem is employed.

Generally, in penalty methods, an augmented performance index is considered. This is the case for both finite optimization problems and optimal control problems. This augmented index is constructed as the sum of the original cost function and so-called *penalty functions* that have some diverging asymptotic behavior when the constraints are approached by any tentative solution. The optimum of this augmented performance index can then be readily characterized by simple stationarity conditions, yielding a (usually) biased estimate of the solution of the original problem. Then, gradually, the weight of the penalty functions is reduced to provide a converging sequence, hopefully diminishing the bias.

The penalty function methods are computationally appealing, as they yield *unconstrained* problems for which a vast range of highly effective algorithms are available. In finite dimensional optimization, outstanding algorithms have resulted from the careful analysis of the choice of penalty functions and the sequence of weights. In particular, the *interior points methods* [1] which are nowadays implemented in successful software packages such as KNITRO [2], OOQP [3] have their foundations in these approaches. We refer the interested reader to [4] for a historical perspective on this topic. In this article, we apply similar penalty methods to solve COCPs. COCPs represent a valuable formulation of objectives in numerous applications, especially because constraints are very natural in problems of engineering interest. Unfortunately, these constraints induce some serious difficulties [5, 6, 7]. In particular, it is a well known fact [7] that constraints bearing on state variables are difficult to characterize, as they generate both constrained and unconstrained arcs along the optimal trajectory. To determine optimality conditions, it is usually necessary to know (or to a-priori postulate) the sequence and the nature of the arcs constituting the desired optimal trajectory. Active or inactive parts of the trajectory split the optimality system in as many coupled subsets of algebraic and differential equations. Yet, not much is known on this sequence, and this often results in a high complexity. Therefore, it is often preferred to use a discretization based approach to this problem, and to treat it, e.g. through a collocation method [8], as a finite dimensional problem [9, 10, 11, 12, 13, 14, 15]. In this context, IPMs have been applied to optimal control problems by Wright [16], Vicente [17], Leibfritz and Sachs [18], Jockenhövel, Biegler and Wächter [19]. This is not the path that we explore, as we whish to use indirect methods (a.k.a. adjoint methods) to take advantage of their accuracy.

Although there is a well-established literature on the mathematical foundations of IPMs for finite-dimensional mathematical programming [3], this is not yet the case for optimal control problems. A main difficulty is to guarantee that the sequence of solutions is strictly interior. This point is critical since interiority is a requirement to avoid ill-posedness and computational failure of implemented algorithms. The problem of interiority in infinite dimensional optimization has been addressed in [20] for input-constrained optimal control, and in [21, 22] for single input single output (SISO) linear and nonlinear dynamics respectively. These contributions

provide penalty functions guaranteeing the interiority of the solutions. As shown in [21, 22], a constructive choice of the penalty functions guarantees that the state constraint is *strictly* satisfied. But, in these articles the choice of the control penalty relies on a strong assumption on the behavior of the control in the vicinity of the saturation. The purpose of the presented research work is to generalize the results obtained in the case of SISO systems [21, 22] to multi input multi output systems and to remove the assumption on the behavior of the control.
This paper is organized as follows: in Section 2, the COCP is presented together with a penalized optimal control problems (POCP) where the state constrained has been relaxed. In Section 3, sufficient conditions on the penalty functions are exhibited such that the optimal solution of the POCP is strictly interior to the constraints. In Section 4, a constructive choice of the penalty is given such that the aforementioned conditions hold and a completely unconstrained algorithm is given.The proposed algorithm is tested on an illustrative example in Section 5. Conclusions and perspectives are given in Section 6.

2 Notations, problem statement and penalty method.

2.1 Constrained optimal control problem and notations

In this article, we investigate the following state and input constrained COCP

$$\min_{u \in U^{\text{ad}}} \left[J(x^u, u) = \int_0^T \ell(x^u, u) dt \right] \tag{1}$$

where $\ell : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}$ is a Lipschtiz function of its arguments with Λ a Lipschitz constant, $x^u(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and the control of the following MIMO non linear dynamics

$$\dot{x} \;=\; f(x, u), \;\; x(0) = x_0 \tag{2}$$

Further, over the time interval $[0, T]$, $T > 0$ given, it is assumed that f is C^1 and that there exists a constant $0 < D < +\infty$ such that the following inequality (a.k.a. sub-linear growth condition) holds:

$$\| f(x, u) \| \leq D(1+ \| x \|), \; \forall x, \forall |u| \leq 1 \tag{3}$$

The control u is constrained to belong to the following set

$$\mathcal{U} = \{u \in L^\infty([0, T], \mathbb{R}^m) \text{ s.t. } \forall i = 1 \dots m \; \| u_i(t) \| \leq 1 \text{ a.e. } t \in [0, T]\} \tag{4}$$

which is the unit closed ball of Lebesgue essentially bounded measurable functions $[0, T] \mapsto \mathbb{R}^m$. The set U^{ad} in (1) is the following

$$U^{\text{ad}} \triangleq \{u \in \mathcal{U} \text{ s.t. } \; g(x^u(t)) \leq 0, \forall t \in [0, T]\} \tag{5}$$

where $g : \mathbb{R}^n \mapsto \mathbb{R}^q$ is assumed to be of class C^1. U^{ad} is the set of control whose corresponding solutions satisfy the state constraints. For the analysis developed in the rest of the paper, we make the following assumption:

Assumption 1 *The initial condition of the problem x_0 is such that the following holds:*

$$\max_i g_i(x_0) \triangleq -\alpha_0 < 0 \;\text{ and }\; \{u \in \mathcal{U} \textit{ s.t. } \sup_{t \in [0,T]} \max_i g_i(x^u(t)) < 0\} \neq \emptyset$$

2.2 Presentation of the penalized problems

Following the approach of interior methods in their application to optimal control [20], we introduce two penalty functions

$$\begin{array}{lll}\gamma_g : & (-\infty, 0) & \to [0, +\infty) \\ \gamma_u : & [-1, 1] & \to [0, +\infty)\end{array}$$

The penalty γ_g is a strictly increasing function on $(-\infty, 0)$ going to infinity as its argument goes to zero by negative values. In the rest of the paper, we extend the state penalty on $\mathbb{R}$ as follows:

$$\gamma_g(x) \quad = \quad 0, \quad \forall x \in [0, +\infty) \tag{6}$$

The penalty function $\gamma_u \in C^1$ is a positive, symmetric, strictly convex function on $(-1, 1)$ taking its minimum value in 0 such that $\lim_{\alpha \downarrow 0} \gamma_u(1 - \alpha) = +\infty$

Remark 1 *The penalty function γ_u satisfies the aforementioned conditions if and only if its derivative $\gamma_u' : (-1, 1) \mapsto \mathbb{R}$ is an increasing bijective and symmetric with respect to zero mapping such that $\gamma_u'(0) = 0$.*

These functions serve to define the following POCP:
Note $\epsilon > 0$, solve:

$$\min_{u \in \mathcal{U}} \left[K(u, \epsilon) = \int_0^T \ell(x^u, u) + \epsilon \left[\sum_{i=1}^{q} \gamma_g \circ g_i(x^u) + \sum_{i=1}^{m} \gamma_u(u_i) \right] dt \right] \tag{7}$$

under the dynamics (2). We assume this POCP satifies the following assumption:

Assumption 2 *The penalized problem (7) has at least one solution.*

3 Feasibility of the optimal solution of the POCP

The objective of this section is to exhibit sufficient conditions on the penalty functions such that any optimal solution of POCP (7) belongs to U^{ad} so is admissible for COCP (1).
In Section 3.2 a sufficient condition on the state penalty γ_g guaranteeing that any optimal solution of POCP (7) strictly satisfies the state constraints is exhibited. Then, in Section (3.3) a sufficient condition on the control penalty γ_u guaranteeing that any optimal solution of POCP (7) strictly satisfies the input constraints is exhibited.
Thus, choosing these penalty functions guarantees that the optimal solution of POCP 7 are simply characterized by the classical stationarity conditions from the calculus of variations [5]. To exhibit these conditions some preliminary result on the topological properties of the admissible control sets are needed. This is the object of Section 3.1.

3.1 Preliminary analysis

In the following, we note

$$\mathcal{U}_0 \triangleq \{u \text{ s.t. } \forall i = 1 \dots m \ \text{ ess} \sup_{t \in [0,T]} \parallel u_i(t) \parallel < 1\} \tag{8}$$

First, let us introduce the following useful subset of U^{ad}.

$$\Psi = \{u \in \mathcal{U} \text{ s.t. } \sup_{t\in[0,T]} \max_i g_i(x^u(t)) < 0\} \quad (9)$$

$$\Psi_0 = \{u \in \mathcal{U}_0 \text{ s.t. } \sup_{t\in[0,T]} \max_i g_i(x^u(t)) < 0\} \quad (10)$$

The objective of this section is to prove that Ψ_0 is dense in Ψ in the L^∞ sense.

Proposition 1 *There exists $C < +\infty$ such that for all $u, v \in \mathcal{U}$ the following holds*

$$\| x^u - x^v \|_{L^\infty} \leq C \| u - v \|_{L^1} \quad (11)$$

Moreover the sets Ψ and Ψ_0 satisfy

$$\text{clos}(\Psi_0) = \text{clos}(\Psi) \quad (12)$$

where clos(.) *denotes the closure of its argument in the L^∞ sense.*

Proof: First, from equation (3) and using Grönwall lemma [23], $\| x^u \|$ is bounded for all $u \in \mathcal{U}$, moreover $f(.,.)$ being C^1 implies that f is Lipschitz with respect to its arguments. Thus $\| \dot{x}^u(t) - \dot{x}^v(t) \| \leq \lambda(\| x^u(t) - x^v(t) \| + \| u(t) - v(t) \|)$, $\lambda < +\infty$. Using again Grönwall lemma, there exists $C < \infty$ such that $\| x^u - x^v \|_{L^\infty} \leq C \| u - v \|_{L^1}$. This proves equation (11).
Let us now prove equation (12). $\Psi_0 \subset \Psi$, thus clos$(\Psi_0) \subset$ clos(Ψ). Now, let us prove the inverse inclusion. Consider any $v \in \Psi \setminus \Psi_0$. Define $-\beta \triangleq \sup_{t\in[0,T]} \max_i g_i(x^v(t))) < 0$. One can build a sequence $(u_n)_{n\in\mathbb{R}}$ with $u_n = (1 - \epsilon_n)v$, where $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence converging to 0, with $\epsilon_n > 0$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges to v in the topology of both L^1 and L^∞. From equation (11), one has $\| x^{u_n} - x^v \|_{L^\infty} \leq C \| u_n - v \|_{L^1}$. Therefore, x^{u_n} uniformly converges to x^v. Using the continuity of g, the sequence $(g(x^{u_n}))_{n\in\mathbb{N}}$ uniformly converges to $g(x^v)$. Thus, there exists N such that, $\forall n > N$, $\| g(x^{u_n}) - g(x^v) \|_{L^\infty} < \frac{\beta}{2}$. Thus, the sequence $(u_n)_{n>N}$ belongs to Ψ_0. Therefore, v is an adherent point to Ψ_0 and $\Psi \subset$ clos(Ψ_0). Eventually, this yields clos$(\Psi_0) =$ clos(Ψ).
■

3.2 Feasibility of the optimal constrained state

In this section, we exhibit a sufficient condition on the state penalty γ_g ensuring that any optimal solution of POCP (7) is admissible for COCP (1).

Proposition 2 *For any $u \in \mathcal{U}$ such that there exists at least one $i \leq q$ such that $\sup_t g_i(x^u(t)) \geq 0$, if the state penalty satisfies*

$$\lim_{\alpha\downarrow 0} \gamma_g(-\alpha)\mu_{g_i}(\alpha) = +\infty \quad (13)$$

where

$$\mu_{g_i}(\alpha) \triangleq \text{meas}\left(\{t \textit{ s.t. } 0 \geq g_i(x^u(t)) \geq -\alpha\}\right) \quad (14)$$

with meas(.) is the Lebesgue measure of its argument, then

$$K(u, \epsilon) = +\infty$$

for all $\epsilon > 0$.

Proof: From equation (6) we have:

$$\mathcal{I}_i \triangleq \int_0^T \gamma_g(g_i(x(t)))dt = \int_{0>g_i(x(t))} \gamma_g(g_i(x(t)))dt$$

Moreover, since $\gamma_g \geq 0$, we have

$$\mathcal{I}_i \geq \int_{0>g_i(x(t))\geq-\alpha} \gamma_g(g_i(x(t)))dt \triangleq \mathcal{J}_i(\alpha)$$

The state penalty satisfies $\gamma_g \geq 0$ on $(-\infty, 0)$, thus $\mathcal{J}_i(\alpha)$ is a non decreasing positive right continuous function of $\alpha > 0$. Therefore $\mathcal{J}_i(\alpha)$ is minimum in $\alpha = 0^+$

$$\mathcal{J}(0^+) = \lim_{\alpha\downarrow 0} \int_{0>g_i(x(t))\geq-\alpha} \gamma_g(g_i(x(t)))dt \geq \lim_{\alpha\downarrow 0} \gamma_g(-\alpha)\mu_{g_i}(\alpha)$$

with $\mu_{g_i}(.)$ the Lebesgue measure defined in equation (14). If (13) holds, then $\mathcal{J}_i(0^+) = +\infty$ which in turn implies that $\mathcal{I}_i = +\infty$. From equation (3) and using Grönwall lemma, the function $x^u : [0, T] \mapsto \mathbb{R}^n$ is bounded for all $u \in \mathcal{U}$. Plus, ℓ being Lipshitz yields that $|\int_0^T \ell(x^u, u)dt| < +\infty$ for all $u \in \mathcal{U}$. Moreover, $\sum_{i\leq m} \int_0^T \gamma_u(u_i)dt \geq 0$. Thus for any $u \in \mathcal{U} \setminus \Psi$ the cost $K(u, \epsilon) = \int_0^T \ell(x^u, u) + \sum_{i\leq m} \gamma_u(u_i)dt + \sum_{i\leq q} \mathcal{I}_i = +\infty$. This concludes the proof. ■

Since the measure μ_{g_i} appears in equation (13), it is handy to give a lower bound on it. This will be used in Section 4, in the explicit construction of suitable penalty functions. A lower bound is given by the following result.

Proposition 3 *Using Assumption 1, there exists a constant* $\Gamma < +\infty$ *such that for all* $\alpha \in [0, \alpha_0]$*, for all* $u \in \mathcal{U} \setminus \Psi$*, the measure* $\mu_{g_i}(\alpha)$ *defined in equation (14) is lower-bounded under the form*

$$\mu_{g_i}(\alpha) \geq \frac{\alpha}{\Gamma} \tag{15}$$

Proof: The proof is given in Appendix A.1 together with the expression of Γ. ■

Using Assumption 1 together with Propositions 2 and 3, one finally obtains the following lemma

Lemma 1 *If the state penalty* γ_g *is such that*

$$\lim_{\alpha\downarrow 0} \alpha\gamma_g(-\alpha) = +\infty \tag{16}$$

then any optimal solution u^* *of POCP (7) is admissible for COCP (1) in the sense where*

$$u^* \in \Psi$$

Proof: Let us consider a control $u^\sharp \in \mathcal{U} \setminus \Psi$. Using Propositions 2 and 3 yields that $K(u^\sharp, \epsilon) = +\infty$ for all $\epsilon > 0$. Using Assumption 2 yields that any optimal control for POCP (7) u^* belongs to Ψ. ■

3.3 Interiority of the optimal constrained control

In this Section, we assume that the state penalty satisfies condition (16) from Lemma 1. Then, a sufficient condition on the control penalty to guarantee that any optimal solution of POCP (7) satisfies $\| u \|_{L^\infty} < 1$ is exhibited.

3.3.1 Construction of an interior control u_2

Let us consider any control $u_1 \in \Psi \setminus \Psi_0$ and note $\sup_{t\in[0,T]} \max_i g_i(x^{u_1}(t)) \triangleq -2\beta_0 \leq 0$. From equation (12), we have the following existence result:

$$\exists \alpha_N > 0 \text{ s.t. } \forall u \text{ s.t. } \parallel u_1 - u \parallel_{L^\infty} \leq 2\alpha_N \text{ one has } \sup_{t\in[0,T]} \max_i g_i(x^u(t)) \leq -\beta_0$$

For each coordinate u_1^i of the control u_1, we construct the modified control u_2 coordinate by coordinate as follows:

$$u_2^i(t) = \begin{cases} u_1^i(t) & \text{if } |u_1^i(t)| < 1-\alpha \\ 1-2\alpha & \text{otherwise} \end{cases} \tag{17}$$

with $\alpha \in (0, \alpha_N]$.

3.3.2 Condition guaranteeing the strict interiority of the optimal trajectory

The following result gives an upper estimate on the difference $K(u_2, \epsilon) - K(u_1, \epsilon)$. This estimate is the sum of three terms, representing respectively

(i) the integral variation of the original cost (1)

(ii) the integral variation of the state penalties $\epsilon \sum_{i\leq q} \gamma_g \circ g_i$

(iii) the integral variation of the input penalty $\epsilon \sum_{i\leq m} \gamma_u$

Proposition 4 *For any control $u_1 \in \Psi \setminus \Psi_0$, considering u_2 from equation (17), for any $\epsilon > 0$ one has*

$$K(u_2, \epsilon) - K(u_1, \epsilon) \leq \alpha \left[U_\ell + U_g(\epsilon) - L(\epsilon, \alpha_N)\right] \mu_{u_1}(\alpha) \tag{18}$$

with

$$\begin{aligned} U_\ell &\triangleq 2\Lambda\left[TC + 1\right] \\ U_g(\epsilon) &\triangleq 2\epsilon T K_g C \sum_{i=1}^{q} \gamma_g'(-\beta_0) \\ L(\epsilon, \alpha) &\triangleq \epsilon \gamma_u'(1-2\alpha) \end{aligned}$$

where K and K_g are positive constant (defined in Proposition 1 and Appendix A.2) and, for any measurable function u_1

$$\mu_{u_1}(s) \triangleq \textit{meas}\left(\left\{t \quad \textit{s.t.} \quad \max_i \parallel u_1^i(t) \parallel \geq 1-s\right\}\right) \tag{19}$$

where meas(.) is the Lebesgue measure of its argument.

Proof: See Appendix A.2. ■

Finally, using (18), the following result holds.

Lemma 2 *If any optimal control for POCP (7) belongs to Ψ, if the penalty function $\gamma_u \in C^1$ is a positive, symmetric, strictly convex function on $(-1, 1)$ taking its minimum value in 0 such that $\lim_{\alpha\downarrow 0} \gamma_u(1-\alpha) = +\infty$, then any optimal control u^* for POCP (7) satisfies:*

$$u^* \in \Psi_0$$

Proof: Using Remark 1 one has $\lim_{\alpha\downarrow 0}\gamma_u'(1-\alpha)=+\infty$. Moreover from the continuity of γ_u', for all $\epsilon>0$ there exists $\alpha_1>0$ such that $\epsilon\gamma_u'(1-2\alpha_1)=U_\ell+U_g(\epsilon)<+\infty$, where $U_\ell, U_g(\epsilon)$ are defined in Proposition 4. Assuming that an optimal control u_1 for (7) belongs to $\Psi\setminus\Psi_0$, the control $u_2\in\Psi_0$ defined in equation (17) with $\alpha\in(0,\min\{\alpha_N,\alpha_1\})$ has a penalized cost lower than u_1 which contradicts its optimality and yields the result. ■

4 Main results and algorithm

In Sections 3.2 and 3.3, conditions have been given, under the form of Lemmas 1 and 2 respectively, such that any optimal solution of POCP (7) is admissible for COCP (1). In this section, a class of penalty functions γ_g and γ_u are given such that these conditions actually hold.

4.1 Penalty design

Our main result, stated below, is a constructive result yielding a relatively direct application under the form of an algorithm detailed below.

Theorem 1 (Main Result) *Under Assumption 1, there exists penalty functions $\gamma_g(.)$ and $\gamma_u(.)$ such that any optimal solution u^* of POCP (7) belongs to Ψ_0. A particular choice of penalty is:*

$$\gamma_g\circ g(x) = -[g(x)]^{-n_g} \tag{20}$$
$$\gamma_u(u) = -\log(1-u^2) \tag{21}$$

with $n_g>1$

Proof: The existence is proven by showing that (20) and (21) are suitable penalties. The penalty (20) is such that equation (16) is satisfied, therefore any optimal solution of POCP (7) belongs to Ψ.
Now, let us prove that if any optimal solution u^* of (7) belongs to Ψ, then it belongs to Ψ_0. The control penalty (21) is such that $\lim_{\alpha_\downarrow 0} L(\epsilon,\alpha)\geq\lim_{\alpha\downarrow 0}\epsilon\gamma_u'(1-2\alpha)=+\infty$, $U_\ell<+\infty$ and $U_g(\epsilon)<+\infty$. Moreover, γ_u' is a continuous function of α. As a consequence, there always exists $\alpha\in(0,\alpha_N]$ such that Lemma 2 holds. Therefore $u^*\in\Psi_0$ which concludes the proof. ■

4.2 Change of variables

To employ the preceding result we now introduce a handy change of variables. Let ν be an element of $L^\infty([0,T],\mathbb{R}^m)$, the following change of variables

$$u \triangleq \phi(\nu)=\tanh(\nu) \tag{22}$$

is a bijective mapping from $L^\infty([0,T],\mathbb{R}^m)$ to $\mathcal{U}_0$. Using this change of variable the following POCP is defined:

POCP2:

$$\min_{\nu\in L^\infty([0,T],\mathbb{R}^m)}\left[P(\nu,\epsilon)=\int_0^T \ell(x,\phi(\nu))+\epsilon\left[\sum_{i\leq q}\gamma_g\circ g_i(x)+\sum_{i\leq m}\gamma_u\circ\phi(\nu_i)\right]dt\right] \tag{23}$$

where the penalty functions are given by equations (20) and (21).

Corollary 1 *Under Assumption 1 and from Theorem 1, POCPs (7) and (23) are equivalent in the sense that*

$$\arg\min_{u\in\mathcal{U}} K(u,\epsilon) = \phi\left(\arg\min_{\nu\in L^\infty([0,T],\mathbb{R}^m)} P(\nu,\epsilon)\right)$$

Proof: Let us consider $u^* \in \mathcal{U}$ a minimizer of $K(.,\epsilon)$. From Theorem 1, $u^* \in \Psi_0 \subset \mathcal{U}_0$. Thus there exists $\nu^\sharp = \phi^{-1}(u^*)$. Moreover,

$$\begin{aligned} K(u^*,\epsilon) \leq &\quad K(u,\epsilon) \quad \forall u \in \mathcal{U}_0 \\ K(\phi(\nu^\sharp),\epsilon) \leq &\quad K(\phi(\nu),\epsilon) \quad \forall \nu \in L^\infty([0,T],\mathbb{R}^m) \\ P(\nu^\sharp,\epsilon) \leq &\quad P(\nu,\epsilon) \quad \forall \nu \in L^\infty([0,T],\mathbb{R}^m) \end{aligned}$$

Thus, $\nu^\sharp$ is a minimizer of $P(.,\epsilon)$ and

$$\arg\min_{u\in\mathcal{U}} K(u,\epsilon) \subset \phi\left(\arg\min_{\nu\in L^\infty([0,T],\mathbb{R}^m)} P(\nu,\epsilon)\right)$$

Let us consider $\nu^* \in L^\infty([0,T],\mathbb{R}^m)$ a minimizer of $P(.,\epsilon)$ and $u^\sharp = \phi^{-1}(\nu^*)$.

$$\begin{aligned} P(\nu^*,\epsilon) \leq &\quad P(\nu,\epsilon) \quad \forall \nu \in L^\infty([0,T],\mathbb{R}^m) \\ P(\phi^{-1}(u^\sharp),\epsilon) \leq &\quad P(\phi^{-1}(u),\epsilon) \quad \forall u \in \mathcal{U}_0 \\ K(u^\sharp,\epsilon) \leq &\quad K(u,\epsilon) \quad \forall u \in \mathcal{U}_0 \end{aligned}$$

Since $u^\sharp$ is a minimizer of $K(.,\epsilon)$ over $\mathcal{U}_0$, from Theorem 1 $u^\sharp$ is a minimizer also a minimizer of $K(.,\epsilon)$ over $\mathcal{U}$. Thus

$$\begin{aligned} \arg\min_{\nu\in L^\infty([0,T],\mathbb{R}^m)} P(\nu,\epsilon) \quad &\subset \quad \phi^{-1}\left(\arg\min_{u\in\mathcal{U}} K(u,\epsilon)\right) \\ \arg\min_{u\in\mathcal{U}} K(u,\epsilon) \quad &\supset \quad \phi\left(\arg\min_{\nu\in L^\infty([0,T],\mathbb{R}^m)} P(\nu,\epsilon)\right) \end{aligned}$$

Finally,

$$\arg\min_{u\in\mathcal{U}} K(u,\epsilon) = \phi\left(\arg\min_{\nu\in L^\infty([0,T],\mathbb{R}^m)} P(\nu,\epsilon)\right)$$

■

4.3 Algorithm

The purpose of the main result of this paper, i.e. Theorem 1 (and Corollary 1 which stems from it), is to allow one to solve a simple OCP (Problem (23)) instead of POCP (7) because they are equivalent. Each problem (23) penalized by ϵ from a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ can be solved using the calculus of variations. The sequence (ϵ_n) is used to gradually determine the solution, as the solution obtained with ϵ_n serves as an initial guess for the problem defined by ϵ_{n+1}. Define the Hamiltonian of the penalized problem (23) as follows

$$H_\epsilon(x,\nu,p) \triangleq \ell(x,\phi(\nu)) + \epsilon\left[\sum_{i\leq q}\gamma_g \circ g_i(x) + \sum_{i\leq m}\gamma_u \circ \phi(\nu_i)\right] + p^T f(x,\phi(\nu)) \tag{24}$$

where $p \in \mathbb{R}^n$ is the adjoint state of Pontryagin solution of $\frac{dp}{dt} = -\frac{\partial H_\epsilon}{\partial x}$ and where the penalty functions are chosen according to Theorem 1. Now, using the positive decreasing sequence $(\epsilon_n)_{n\in\mathbb{N}}$, one can approach the solution of (1).

- Step 1: Initialize the continuous functions $x(t)$ and $p(t)$ such that the initial values satisfy $g_i(x(t)) < 0$ for all $t \in [0, T]$, and set $\epsilon = \epsilon_0$. Note that $x(t)$ and $p(t)$ need not to satisfy any differential equation at this stage, even if it is better if they do.

- Step 2: Solve for each time $\frac{\partial H_\epsilon}{\partial \nu} = 0$, and note ν_ϵ^* the solution.

- Step 3: Solve the $2n$ differential equations $\frac{dx}{dt} = f(x, \phi(\nu_\epsilon^*))$ and $\frac{dp}{dt} = -\frac{\partial H_\epsilon}{\partial x}(x, \nu_\epsilon^*, p)$ forming a two point boundary values problem using bvp4c (see [24]), with the following boundary constraints $x(0) = x_0$ and $p(T) = 0$.

- Step 4: Decrease ϵ, initialize $x(t)$ and $p(t)$ with the solutions found at Step 3 and restart at Step 2.

5 Numerical Example

To illustrate the proposed methodology, we consider the following nonlinear dynamics

$$\ddot{x}(t) \;=\; x(t)^3 + x(t) - \dot{x}(t) + 10x(t)^2 u(t)$$

The optimal control problem is the following:

$$\min_u \left[J(u) = \int_0^1 -\frac{x(t)^2}{2} dt \right]$$

The boundary conditions are the following

$$x(0) \;=\; 0.3\ ,\ \dot{x}(0) = 0$$

the problem is solved under the following constraints:

$$\begin{aligned} |u| &\leq 1 \\ g_1(x, \dot{x}) &= x^3 - \dot{x}/2 - 0.7 - 0.3\cos\left(\frac{1}{1.05 - t}\right) \\ g_2(x, \dot{x}) &= 0.5 - \dot{x} \end{aligned}$$

these constraints are nonlinear and have strongly oscillatory behavior. The corresponding Hamiltonian is the following

$$\begin{aligned} H_\epsilon(x, \nu, p) \;=\; & -\frac{x(t)^2}{2} + \epsilon \left[\sum_{i=1}^{2} (\gamma_g \circ g_i(x(t), \dot{x}(t))) + \gamma_u \circ \phi(\nu(t)) \right] \cdots \\ & + p_1(t)\dot{x}(t) + p_2(t) \left[x(t)^3 + x(t) - \dot{x}(t) + 10x(t)^2 \phi(\nu(t)) \right] \end{aligned}$$

The optimal control ν_ϵ^* is solution of the following equation:

$$\epsilon \gamma_u' \circ \phi(\nu(t)) + 10 p_2(t) x(t)^2 = 0$$

Using Lemma 2 and Remark 1, one can take γ_u' as a bijective increasing mapping from $(-1, 1)$ to $\mathbb{R}$. Moreover, ϕ being a bijective mapping from $\mathbb{R}$ to $(-1, 1)$, the function $\gamma_u' \circ \phi$ is a bijective mapping from $\mathbb{R}$ to $\mathbb{R}$. Conveniently, to have an analytical solution for Step 2 of the algorithm described in Section 4.3 we do not directly define γ_u but $\gamma_u \circ \phi$ instead. Setting

$$\gamma_u' \circ \phi(x) = \sinh(x)$$

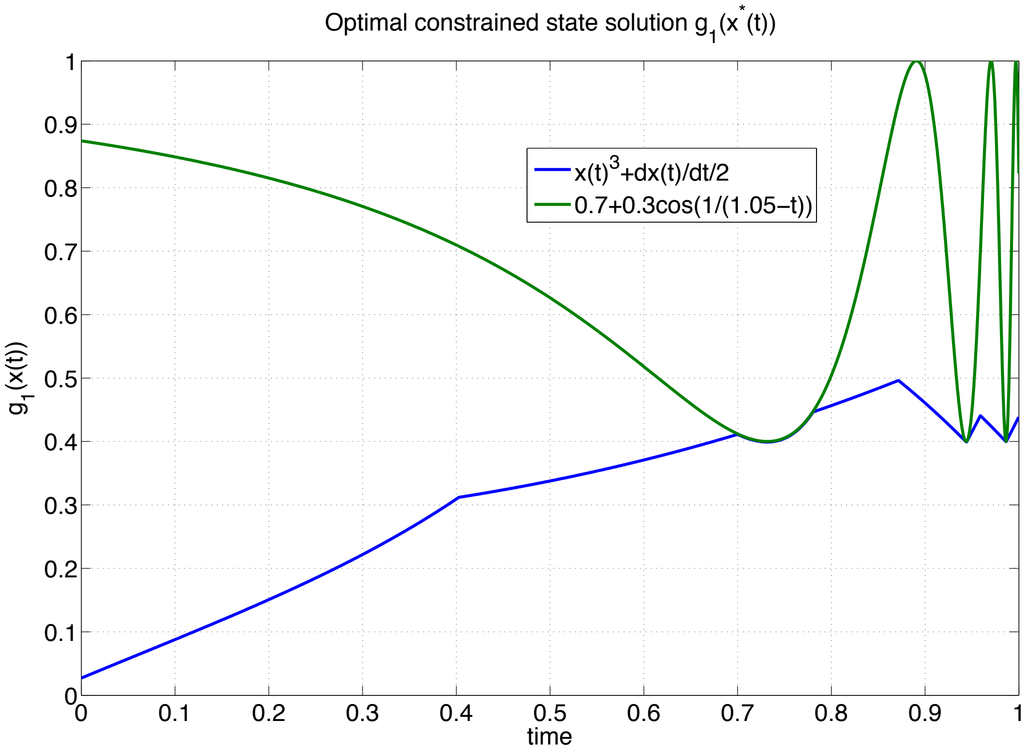

Figure 1: Optimal constrained state g_1 for POCP (7) with $\epsilon = 10^{-7}$

and

$$\gamma_g \circ g(x) = -\left[x(t)^3 - \dot{x}(t)/2 - 0.7 - 0.3\cos\left(\frac{1}{1.05 - t}\right)\right]^{-1.1}$$

satisfies the conditions from Lemmas 1 and 2 which yields that any optimal solution of this problem belongs to Ψ_0. The first state constraint $g_1(x, \dot{x})$ is displayed on figure 1, the second state constraint g_2 is displayed on figure 2, the optimal control obtained after 40 steps of $(\epsilon_n)$is displayed on figure 2 and the adjoint states are give on figure 3.

6 CONCLUSIONS

As a result of the proposed study, a practical method to solve constrained optimal control problems for non linear systems has been given. It solely requires the mathematical formulation of a suitably penalized OCP. A constructive choice has been given. This unconstrained problem can then be handled using a classic two-point boundary value problem solver. The presented iterative algorithm using an off-the-shelf routine is quite easy to implement and provides satisfactory results.

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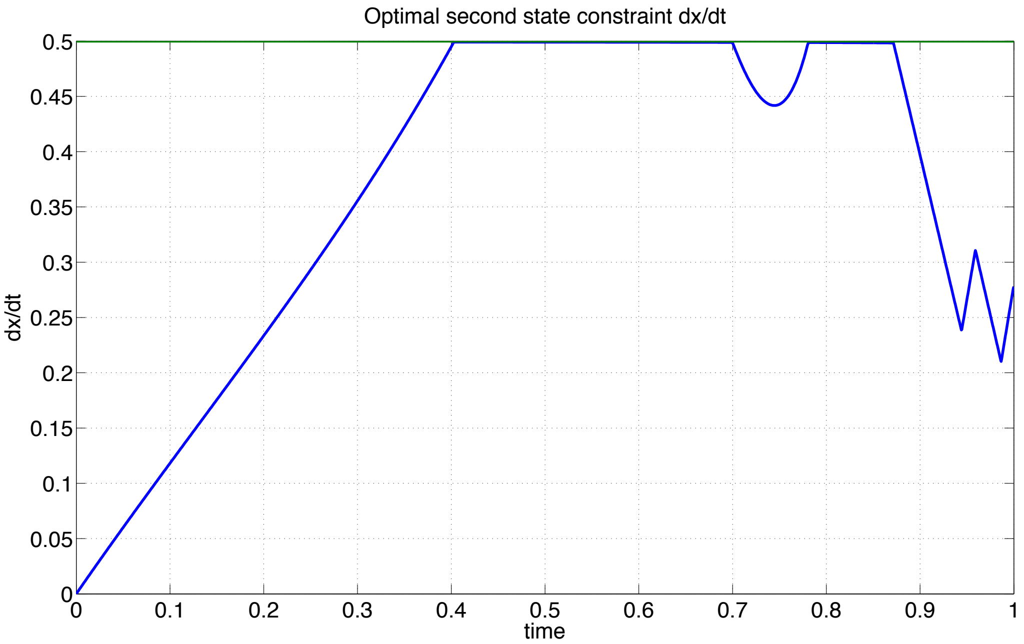

Figure 2: Optimal constrained state g_2 for POCP (7) with $\epsilon = 10^{-7}$

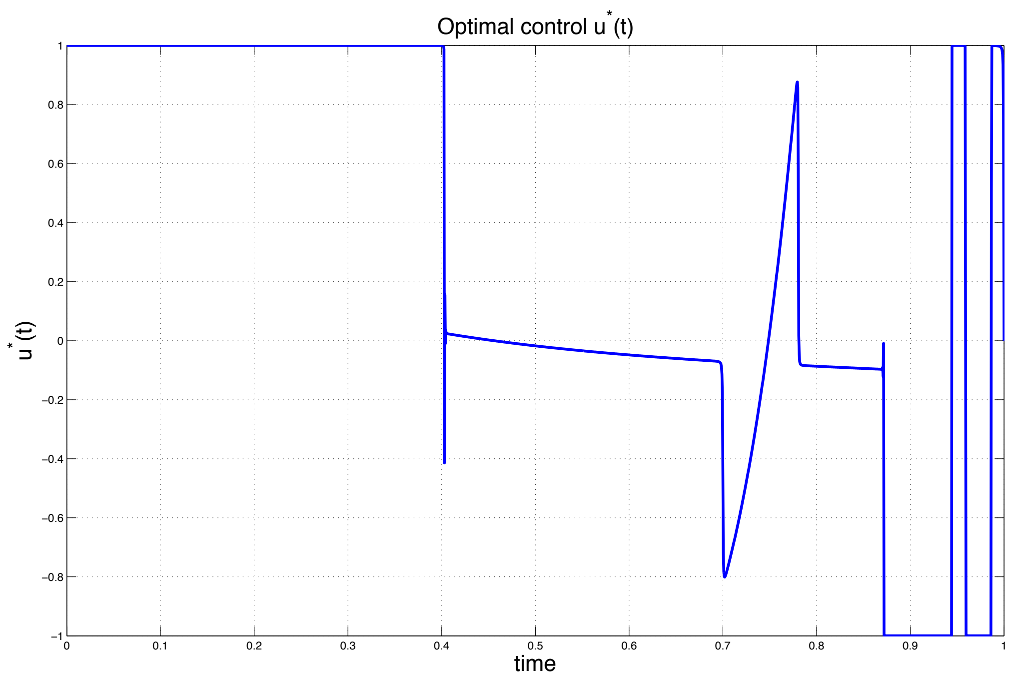

Figure 3: Optimal control for POCP (7) with $\epsilon = 10^{-7}$

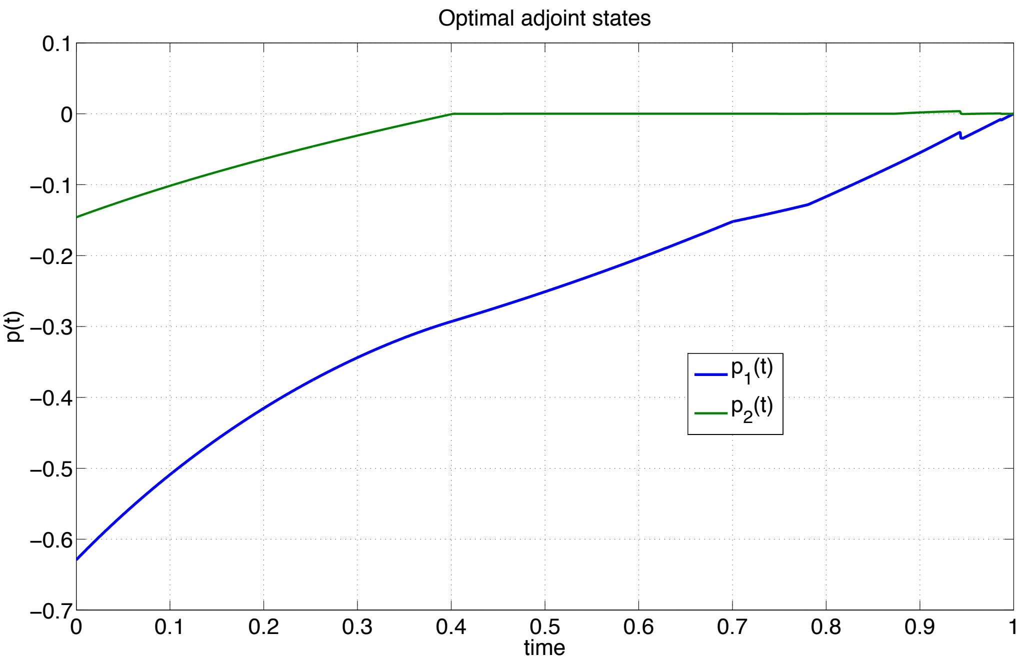

Figure 4: Adjoint state for POCP (7) with $\epsilon = 10^{-7}$

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A Variational calculus

A.1 Proof of Proposition 3

First, using Assumption (A1) together with Grönwall Lemma, one has for all $t \in [0,T]$ $\| x(t) \| \leq e^{DT}(1+ \| x_0 \|) - 1 \triangleq K_T$. Now, let us define:

$$K_x \triangleq \sup_{\|x\| \leq K_T, |u| \leq 1} \| f(x,u) \| \tag{25}$$

$$K_g \triangleq \max_i \sup_{\|x\| \leq K_T} \| \frac{\partial g_i}{\partial x}(x) \| \tag{26}$$

The continuity of f and $\frac{\partial g_i}{\partial x}$ yields $K_x, K_g < +\infty$. Let us recall that $x(t)-x(s) = \int_s^t f(x(\tau), u(\tau))d\tau$. From Assumption 1 for all $\alpha \in [0, \alpha_0]$ there exists $s, t \in [0,T]$ such that $g_i(x^u(s)) = -\alpha$ and $g_i(x^u(t)) = 0$. This yields

$$g_i(x(t)) - g_i(x(s)) = \alpha \leq K_g \| x(t) - x(s) \| \leq K_g K_x (t-s)$$

This yields $t - s \geq \alpha(K_x K_g)^{-1}$. Now, let us define

$$\tau \triangleq \sup_{t \leq s}\{t \text{ s.t. } g_i(x(t)) = -\alpha\}$$

and the set

$$E(\alpha) \triangleq \{t \text{ s.t. } 0 \geq g_i(x(t)) \geq -\alpha\}$$

Then, we have $[\tau, s] \subset E(\alpha)$ wich yields

$$\mu_{g_i}(\alpha) = \text{meas}(E(\alpha)) \geq s - \tau \geq \alpha(K_x K_g)^{-1}$$

where meas(.) is the Lebesgue measure of its argument. Note $\Gamma \triangleq K_x K_g$. This concludes the proof.

A.2 Proof of Proposition 4

A.2.1 An upper bound on the possible increase K^+

To exhibit an upper bound on the possible increase, K^+ is split into two parts itself: the possible increase of the original cost $\int \ell(x, u, t)dt$ and the possible increase due to the penalties, separately.

Possible increase of the original cost There, an upper bound on the possible increase of $|\int_0^T \ell(x^{u_2}, u_2) - \ell(x^{u_1}, u_1)dt|$ is exhibited. Let us call K_ℓ this upper bound. Now, let us consider that the cost function $\int \ell(x, u, t)dt$ is Lipschitz with constant Λ, then from Proposition 1 equation (11) and equation (17), one has

$$\begin{aligned} K_\ell &\leq \Lambda \int_0^T \|x^{u_2} - x^{u_1}\|_{L^\infty} + \| u_2(t) - u_1(t) \| \, dt \\ &\leq \Lambda \left[TC + 1\right] \|u_2 - u_1\|_{L^1} \\ &\leq 2\Lambda\alpha \left[TC + 1\right] \mu_{u_1}(\alpha) \end{aligned} \quad (27)$$

We define this upper bound as follows:

$$\alpha U_\ell \mu_{u_1}(\alpha) \triangleq 2\alpha\Lambda \left[TC + 1\right] \mu_{u_1}(\alpha) \quad (28)$$

Possible increase due to the state penalty Note $K_{\gamma_g} \triangleq \epsilon \sum_{i=1}^q \int_0^T \gamma_g \circ g_i(x^{u_2}) - \gamma_g \circ g_i(x^{u_1})dt$. The integrand is positive when $g_i(x^{u_2}(t)) \geq g_i(x^{u_1}(t))$. But, from the construction of u_2 and equation (3.3.1), one has $\max_i g_i(x^{u_2}(t)) \leq -\beta_0$ for all $t \in [0, T]$. Using equation (26) and Proposition 1 equation (11) one obtains

$$\begin{aligned} K_{\gamma_g} &\leq \epsilon \int_0^T K_g \parallel x^{u_2} - x^{u_1} \parallel_{L^\infty} \sum_{i=1}^q \gamma_g'(-\beta_0)dt \\ K_{\gamma_g} &\leq \epsilon T K_g \sum_{i=1}^q \gamma_g'(-\beta_0) C \|u_2 - u_1\|_{L^1} \\ &\leq 2\epsilon T K_g \sum_{i=1}^q \gamma_g'(-\beta_0) C \alpha \mu_{u_1}(\alpha) \end{aligned} \quad (29)$$

We define this upper bound as follows:

$$\alpha U_g(\epsilon)\mu_{u_1}(\alpha) \triangleq 2\alpha\epsilon T K_g C \sum_{i=1}^q \gamma_g'(-\beta_0)\mu_{u_1}(\alpha) \quad (30)$$

Finally, using equations (28) and (30), we have:

$$K^+ \leq \alpha \left[U_\ell + U_g(\epsilon)\right] \mu_{u_1}(\alpha) \quad (31)$$

A.2.2 A lower bound on the possible decrease K^-

The aim of this part is to exhibit a lower bound on $|K^-|$. Here, we consider that the decrease can only be provided by the control penalty. Let us define $K_u \triangleq \epsilon \sum_{i=1}^m \int_0^T \gamma_u(u_2^i) - \gamma_u(u_1^i)dt$. Equation (3.3.1) yields that the integrand of the previous equation is never positive since $|u_2^i(t)| \leq |u_1^i(t)|$. Using convexity and symmetry properties of the penalty functions and equation (19) one has

$$\begin{aligned}
K^- &\leq \epsilon \sum_{i=1}^m \int_{|u_1^i| \geq 1-\alpha} \gamma_u(u_2^i) - \gamma_u(u_1^i)dt \\
K^- &\leq -\epsilon \sum_{i=1}^m \int_{|u_1^i| \geq 1-\alpha} \| u_2^i - u_1^i \|_{L^\infty} \gamma_u'(|u_2^i(t)|)dt \\
K^- &\leq -\epsilon\alpha\gamma_u'(1-2\alpha) \sum_{i=1}^m \int_{|u_1^i| \geq 1-\alpha} 1dt \\
K^- &\leq -\epsilon\alpha\gamma_u'(1-2\alpha)\mu_{u_1}(\alpha) \qquad (32)
\end{aligned}$$

We define this lower bound as follows:

$$K^- \leq -\alpha L(\epsilon, \alpha)\mu_{u_1}(\alpha) \triangleq -\alpha\epsilon\gamma_u'(1-2\alpha)\mu_{u_1}(\alpha) \qquad (33)$$

A.2.3 An upper bound on $K(u_2, \epsilon) - K(u_1, \epsilon)$

Gathering equations (31) and (33), one finally obtains

$$K(u_2, \epsilon) - K(u_1, \epsilon) \leq \alpha \left[U_\ell + U_g(\epsilon) - L(\epsilon)\right] \mu_{u_1}(\alpha) \qquad (34)$$

This concludes the proof of Proposition 4.