On the Origin of Abstraction: Real and Imaginary Parts of Decidability-Making

Gilbert Giacomoni

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Gilbert Giacomoni
Institut de Recherche en Gestion - Université Paris 12 (UPEC)
61 Avenue de Général de Gaulle 94010 Créteil
gilbert.giacomoni@u-pec.fr

Centre de Gestion Scientifique – Chaire TMCI (FIMMM) - Mines ParisTech
60 Boulevard Saint-Michel 75006 Paris
gilbert.giacomoni@mines-paristech.fr

Abstract: Firms seeking an original standpoint, in strategy or design, need to break with imitation and uniformity. They specifically attempt to understand the cognitive processes by which decision-makers manage to work, individually or collectively, through undecidable situations generated by equivalent possible choices and design innovatively. The behavioral tradition has largely anchored on Simon's early conception of bounded rationality, it is important to engage more explicitly cognitive approaches particularly ones that might link to the issue of identifying novel competitive positions. The purpose of the study is to better understand the regeneration and meta-restructuring processes of knowledge systems triggered by decision makers in order to redefine their decidable space by abstraction. The theoretical breakthroughs liable to account a dual form of reasoning, deductive to prove (then make) equivalence and abstractive to represent (then unmake) it, in subtle mechanisms of decisional symmetry, indiscernibility (antisymmetry) and asymmetry, are presented. A development of a core analytical/conceptual apparatus is proposed as an extension of the most widespread models of rationality based on a real dimension (for preference-making), by adding a visible imaginary one (for abstraction-making) and open up vistas capacity in the fields of information systems, knowledge and decision. This extension takes complex numbers as generalizable objects.

Key words: decision-making, equality, indiscernibility, undecidability, imaginary, abstraction, knowledge, information, symmetry-breaking, identity, relation,

> "from similarity between elements it is possible to derive [by abstraction] another concept to which no name has yet been given. Instead of "the triangles are similar" we say that "the two triangles are of identical shape" or "the shape of one is identical with that of the other" (Frege, 1883).

An original standpoint, in strategy or design, requires breaking imitation and uniformity, that is, symmetry[1]. Some firms[2] seeking to distinguish themselves specifically attempt to understand the cognitive processes by which decision-makers (consumers, managers, etc.) manage to work through undecidable situations generated by equivalent possible choices and design innovatively. A theoretical understanding of the processes by which equivalence is conceived in turn helps explain how its symmetry gets broken. In this respect, it appears relevant to examine the theoretical models liable to account for this dual form of reasoning, deductive (to make an equivalence that is a loss of uniqueness[3] and discernibility) and abstractive (to unmake the equivalence recreating a common unique representation). Human thinking and human organizations produce knowledge to design and share representations. To prove the equality of objects from a situation where they are not known as such, a production of knowledge is required that makes a new common unique identity being constructible by abstraction. But from objects already known as equal[4], the knowledge produced at the origin to prove their equality is not spontaneously accessible and visible. Only the existing common identity is then obviously imposed on the mind, no other new one being easily constructible. The design of an identity requires a knowledge-based capacity. This is a current problem for design or creation activities. But it is also a problem to design an equality that would not have to be "bounded" in some known contexts only (a perfect universal equality being in fact unachievable and thus at the end undecidable). The behavioral tradition has largely anchored on Simon's early conception of bounded rationality (Simon, 1956, 1969, 1976), it is important to engage more explicitly cognitive approaches particularly ones that might link to the issue of identifying novel competitive positions. The purpose of the study is to better understand the regeneration and meta-restructuring processes of knowledge systems. It presents a development of a core analytical/conceptual apparatus that may potentially open up vistas in the fields of information systems, knowledge and decision. At a methodological level, the research was conducted inductively to trace the theoretical foundations of the most widespread models of rationality, particularly set theory (Cantor, 1883), and pinpoint their limitations (Nagel E. & al; 1989). By tracing those foundations, various fields were explored for approaches[5] seeking to complement set theory. The research was conducted deductively to show that those approaches all aim to explain the discernibility properties between equal things, which is inconceivable in set theory and leads to undecidable situations. The conclusion of this deductive phase is to propose a non-contradictory axiomatic system that is more comprehensive[6] than set theory and can describe the abstraction processes triggered by decision makers in order to redefine a decidable space by restructuring knowledge spaces. This axiomatic system opens up the possibility to generalize the mathematical formalism of partly real and partly imaginary numbers – notably utilized to describe variational phenomena (spread, etc.) - to the representation of objects.

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[1] « Symmetry » comes from the Greek *sun* (together, with), meaning commensurability, proportion, harmony (Apollonius De Perge, 2009)
[2] One, in the domain of relief and virtual software computing is a partner since 10 years (in a longitudinal approach) and is looking in particular for new generations of images and virtual simulation sensors. Some others and partners of a research chair are focusing on innovation in the fields of transportation, automobile, software, (etc), and help address the questions behind this research.
[3] A=B establishes the existence of two copies of a same object named A or B.
[4] Such as 'These computers are the same…', 'It is the same person…', etc.,
[5] Fuzzy sets, quasi-sets, concept-knowledge, fuzzy logic, object theory, multidimensional coordinate spaces, abstraction principles, etc.
[6] More complete while staying consistent in Gödel's understanding (Gödel, 1940)

If to decide is to choose (note that an intentional non-choice can be considered as a choice[7]), one can start to examine situations in which choice is impossible, albeit desired. This applies to an undecidable situation as opposed to a decidable situation (considered as such in mathematical logic if there is an algorithm which decides step-by-step between yes or no answers). In mathematical logic, a decision problem involves determining whether a statement is universally valid (Cori & Lascar, 1993). Thus, undecidability appears as the devil of any decision-maker reaching for the heaven of equivalence between possible choices at hand, because, ultimately, there would only be one choice left to make. Contrary to appearances, equivalence appears to be incompatible with choice, as evidenced by any attempt to plan the act of choosing in a box containing equivalent objects. The instruction to choose "any" object does not translate into reality. Either the box is considered as a whole or the objects are differentiated. In the latter case, one way to go is to designate a specific object (for example, the first one if they were ordered beforehand), in other words, introduce probabilities (equiprobabilities, to be specific) which will necessarily result in the selection of one object over the other after the draw (because a random variable is an application between events and distinctive values). In all cases, equivalence will be interrupted so that choice can occur, because this choice is supposedly equivalent to any other. Indifference to equivalent choices may spring from a consumer as for any other decider (a doctor, a pilot, etc). An understanding of the process that leads them to conceive of this equivalence is key as it makes it easier, in turn, to understand the breaking of indifference as a seamlessly "symmetric" vision of possibilities, both individually discernible in space or in time, nameable (A, B, etc) and countable (1, 2, etc) and yet interchangeable and undistinguishable from one another (considered as strictly the same).

Undecidability is an essential feature of mathematics. Any property[8] that cannot be proved or disproved based on a given axiom theory is said to be undecidable, independent from this axiom, non trivial or absolute. Such a property is interesting as it helps define subsets of objects, for example, the subset of rubber tires which have the property "rain and hydroplaning resistant" in the universe of rubber tires. Conversely, "rubber" cannot be used to describe tires that lack this property; it is inseparable from the universe of tires and consistently applies to all of them (Hatchuel & Weil, 2007). Along the same lines, it is established that there is no algorithm that can decide from the source code of a program whether its output satisfies a non-trivial property[9] (that is not always true or false) such as "the program computes an accurate result according to specification" (Rice, 1953). Thus, the undecidability of a property in relation to a given axiomatic system[10] (Zemerlo, 1908) is essential to defining out of a set of objects the subset of those that do not have that property. As a matter of fact, the terms independence and undecidability are equally used. This holds for a set of objects as well as a set of strategies using properties for their formulation. In fact, equivalent things do not belong to the world of sets but that of collections[11] from which they stem. "A set can be described metaphorically as a "primary" box containing "secondary" boxes that never have equivalent contents, elements that in turn contain tertiary boxes themselves containing, etc." (Godement, 2001). Still, the validation of human formal constructions and the notion of mathematical truth are grounded on set theory and its meta-theoretical language. Thus, we fall outside this rationality when manipulating a myriad of

[7] Action of choosing something, someone over something or someone else; outcome of this action ; Power, possibility to choose ; set of things, etc. among which one can choose ; set of things chosen, selected for their qualities (French dictionary Larousse, 2012).
[8] An affirmation having a sense (meaning without any ambiguity a status of truth, right or wrong): it is the "excluded middle principle"
[9] Any property related to the function calculated by a Turing machine (see p 6) is undecidable.
[10] Zermelo-Frankel (ZFC) was selected here.
[11] The term collection is generally used to mean a set where order is ignored but multiplicity matters.

equivalent things that we do not fuse into a *unique whole*. " I use the term 'set' or 'system' to generally mean any multiplicity that may be conceived of as a unity, that is, any collection of determined elements that, through a law, may be combined into a whole. In that way, I believe I am defining something related to eidos or Plato's Theory of Forms (Cantor, 1883; Godin, 2002; Husserl, 1998)." In what world are we when we remain within the original collection? Levy (1987) notes: "(…) Identities are undefined and objects are uncompleted." We can only understand and compute different things in the world of sets whereas we can duplicate and compute equivalent things in the world of collections. This mapping suggests reconsidering decidability in relation to equivalence.

The first section of the study traces set theory, its origins, axiomatic system (ZFC) and principles in making subsets out of sets with non-trivial properties (that are, undecidable or independent from ZFC), in other words not common to all encapsulated sets. The equivalence relation, the choice function and the problem of the undecidability of questions that cannot be answered by any algorithm are also explained. The second section addresses decision theory and the models of rationality aimed at explaining individual or collective behaviors in choice situations. The decision-maker's vision is outlined as a "quotient" set resulting from a conception of equivalence relational schemes (Frege, 1879, 1884; Wright, 1983) based on an initial set of possible choices and knowledge standards. The study shows that the issue of choice representativeness requires steering clear of ZFC axiomatic theory to translate indiscernibility and multiplicity with a non-fusional (non-antisymmetric) equality relation. The third section explains through the example of 3D depth perception how the design process of a new decidable space functions. It demonstrates the necessity to revert to the original collections from which sets stem in order to repartition equivalence relations and knowledge frameworks in a subtle mechanism of decisional symmetry, antisymmetry and asymmetry. The fourth and last section shows the relevance of fuzzy logic in interpreting the design process of a new decidable space as well as its limitations. In contrast, the formalism of imaginary numbers can reveal the missing dimension likely to radically change the decision-maker's vision and engage their knowledge by abstraction to find the new relational key to reconsidering their new decidable space. Finally, an axiom system complementing ZFC and permitting a non-antisymmetric equivalence relation is proposed.

- Theoretical Basis -
The Choice Function And The Equivalence Relation In Set Theory:
The Unique And The Multiple

Set theory provides the observer with an overarching concept for describing his/her observations in the form of sets with the relational membership structures (algebra[12]), consistent with a language, rules and axioms[13] eliminating potential paradoxes (Russel, 1907) and helping to interpret the observations-based propositions in order to prove or disprove them (Tarski, 1972). This theory is a formal creation that can "rationalize" the real (Bell, Raiffa & Tversky, 1988). For example, relational databases operate on that basis. "(…) the dominant position of the relational model (…) is based on first-order predicate logic (that) took 2,000 years to develop, beginning with the ancient Greeks who discovered that the subject of logic could be intelligently discussed separately from the subject to which it might be applied, a major step in applying levels of abstraction" (Codd, 1970, 1990).

[12] Algebra, from the Arabic *al-jabr*, becomes algebra in Latin and means gathering (of pieces), reconstruction or connection.
[13] Zermelo-Fraenkel (ZF) axioms are widely admitted to axiomatize this set theory: the empty set exists, a gathering of sets is a set, etc.

A set is a collection of unrepeated objects. "A set is any collection into a whole of definite and distinct objects of our intuition and thought, with the objects referred to as elements of that set" (Belna, 2000). A set can exist but as a unique form and sets are constructed from an original collection whose identities can be multiple. Levy (1987) writes: "I call such multiplicities infinite or inconsistent multiplicities (unconceivable as a unity, a completed object) (…). Conversely, if all the elements of a multiplicity can be thought of as existing simultaneously, in the sense that it can be conceived as one single object, I call it a consistent multiplicity or set" (Levy, 1987)[14]. Accordingly, the conception of a unique object called set appears to be relative to the current knowledge that helps consider a multiplicity in a 'consistent' way.

Set theory operates with an appropriate equivalence relation – equality – based on the principle of indiscernibility (Leibnitz, 1714). "Statements in a=b form often have an invaluable content for the progress of knowledge and not always an a priori grounding. The discovery that every morning the same sun comes up and not a new sun was certainly one of the most critical breakthroughs of astronomy (…). I use this term [denotation] in the sense of identity and I mean 'a=b' in the sense of 'a' is the same as b' or 'a and b match' (…). Arguably, the statement a=b does not refer to the thing per se but the way we designate it." (Frege, 1994). The equality relation is[15] reflexive (a=a), symmetric (a=b if and only if b=a) as well as antisymmetric (a=b if and only if a and b merge, that is, if a is included in b and b is included in a). When referring to two finite sets that are definable by extension, that is, by their content (on which properties depend), content equality makes for set equality (there are no distinct sets with identical contents). When referring to infinite sets[16], those must be defined by *comprehension* (by *intention*)[17] (fig.A). To do so, one must agree to work inside a larger set (which may be called a superset) that is itself defined by *comprehension*, and so forth. And to make this inclusion process consistent, everything must be immersed in a universe that does not itself qualify as a set, in order to avert paradoxes. Also, there must be a characteristic property (expressed in the language) of the set that one wishes to define, one that is common to all its elements. But obviously, such a property must not be trivial, meaning not already common to all included sets of the universe in which we stand. It is described as *independent* (or *undecidable*) from the axiomatic theory from which the sets of the universe are constructed. In other words, one uses a property that is not the universal brand of sets made by the axiomatic machine. Thus, they are elements that verify it, namely those of the set that one wishes to define by comprehension, and others that do not as they only exist outside the set, within the superset.

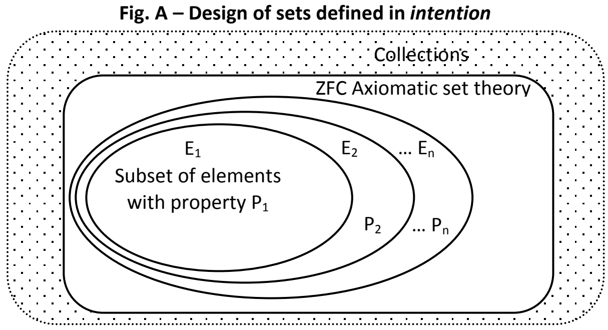


Fig. A – Design of sets defined in *intention*

The equality of two sets defined by comprehension is verified when they have the same properties and can thus be distinguished from one another. In another words, there cannot be differences between sets unless there are differences between their properties. Frege (1884) concurs: "(a=b). These cannot be differentiated unless the difference of the signs (a;b)

[14] Quoted comments of Georg Cantor
[15] It is also transitive, meaning that if a=b and b=c then a=c
[16] for example, numbers (and they make up the numerical world)
[17] As a parallel to *extension* (listing contents)

corresponds to a difference in how the designated object is given. (…) Therefore, the statement contains actual knowledge". Leibnitz (1998) notes: "For sets to have the same properties, they must be substitutable for one another. (…) are the same things of which one can be substituted with the other without compromising the truth." The criterion of object substitutability operates with the identity of the concepts that define them, which involves "replacing everywhere the objects by the concepts into which they fall (…) If one designates the *extension of a concept* as the collection of objects that it incorporates, it is possible to say that if line a is parallel to line b, then the extension of the concept *line parallel to line a* is identical to the extension of the concept *line parallel to line b*" (Frege, 1884). "(…) To say that a=b if and only if a and b behave similarly in any context (…) is in fact equivalent to the general case (Leibniz equality) but structurally more simple (Girard, 2009)."

In set theory, the equivalence relation translates to a reflexive, symmetric, antisymmetric and transitive relation of equality that must be verified in all contexts to become established. Equivalence thus holds in absolute terms. If not, two objects can be identical in some contexts (those known, for example) but distinct in other contexts (those yet unknown), meaning they are not always and everywhere the same. Equivalence thus holds in relative terms (Engel, 1989; Geach, 1980; Longeart-Roth, 1981; Wiggins, 2002; Monnoyer & al; 2006; Giacomoni & Sardas, 2010). The equivalence (named equality) of two sets can be established by verifying that one is included in the other and vice versa. In other words, they can mutually serve as context for one another (content, respectively).

The identity of objects as conceived by set theory is not natural to the decision-maker, if one only looks at our usual conception of numbers, which exist as both unique entities and individually as an indeterminate and arbitrary group of units[18]. The result of the operation 1+1+1 fits into the same space as 2+1 and 3. "If we designate each element of a set as A, then we attribute the same sign to different things, but if we give 1 distinctive indices, then 1 becomes unsuitable to arithmetic. It seems that we need to ascribe two contradictory properties to units: identities and discernibility" (Frege, 1884). For arithmetic, 'one' is (for collections)[19] not of the same nature as 'unity' (for sets)[20]. This is a reason why Frege defined cardinal numbers with equivalence relations [classes of] as an 'abstraction'[21] (Frege, 1893, 1903) with respect to some principles[22] and to set-theoretic foundations aiming both to provide how mathematicians are able to create or to discover this way (Chihara, 1963; Hale, 2000; Shapiro, 2000; Fine, 2002; Tennant, 2004; Cook & Ebert, 2005; Cook, 2007). "The prevalent view is that abstracts should just be treated as equivalence classes (…) The theory of abstraction thereby becomes a part of the much more comprehensive theory of sets or classes" (Fine, 2002).

Now, equivalence brings us to choice. Choice is incorporated into set theory as an axiom. It states that it is legitimate to construct mathematical objects by endlessly repeating the operation of choosing an element out of a non-empty set. But "because it states the existence of objects about which intuition is uncertain, its usage is not as consistently acknowledged as that of other axioms (ZF), and it is common to use it as little as possible and keep track of it" (Dehornoy, 2006). The axiomatic machine is seemingly unable – without this axiom - to

[18] A repetition of 1 is a collection, the result of 1+1+1 is the unique result 3.
[19] for example: to have same bicycles (same model of bicycle) means a collection of countable and distinct (discernible in space or time) one.
[20] for example: to have same father means a unique father.
[21] In the sense that, "from similarity [equivalence] between elements it is possible to derive another concept to which no name has yet been given. Instead of "the triangles are similar" we say that "the two triangles are of identical shape" or "the shape of one is identical with that of the other" (Frege, 1883).
[22] Hume's principle, Basic Law V, New V, etc. (Shapiro & Weir, 2007)

make certain constructions involving an infinite number of simultaneous choices. It was thus decided to incorporate it and name it ZFC axiomatic set theory. The axiom of choice states that any set of sets has a choice function that selects an element from each nonempty set. This axiom cannot be disproved by the other axioms (Gödel, 1940) or proven (Cohen, 1966).

The issue of the existence of a choice function (meaning axiomatic set theory) is paramount. If to decide is to choose, this question ties in to undecidable problems, which – at least in mathematics – illuminate unpredictabilities established as being caused not by a situation phenomenon or a practical impossibility but by an impossibility in principle: for this particular problem, not only is there no known resolution but there will never be. For example, the problem of program equivalence (do two given computer programs calculate the same thing?) or the problem of the utility of part of a program (Given that a computer program consists of a set of codes, does it contain a useless subset of codes, meaning that it is never used no matter what use of the program is made)? Undecidable questions do not have any algorithm that can decide yes or no through a finite number of steps (it is impossible to come up with a method that systematically processes all cases). This identification and formulation work on the notion of algorithm relates to a constructivist approach to mathematics (Troelstra, 1973). Any function (no matter if it is choice, utility, indifference, etc.) is computable if there exists a finite way to describe it that effectively calculates all its values. A precise definition of the notion of computable function lays down that of algorithm in the process. “Every effectively calculable function is a computable function” (Gandy, 1980). What is feasible by algorithm is feasible by a Turing machine[23] (Turing, 1936; 1948).

- Boundaries and Limits -
Axiomatic Approach To Undecidability And Indiscernibility Between Sets And Collections

Since the early 20th century the axiomatic method has taken hold and any mathematical question can be articulated as: does this property result from this set of axioms? The model of rationality offered by the ZFC axiomatic set theory was outlined above; it is formal in its language and abstract in its independence from the application domain. Early studies and experiments in decision[24] theory are historically characterized by the search for formal structures underlying concrete problems and the use of mathematics and logic as a modeling language (Tsoukiàs, 2004; 2007). “Decision support is an activity performed by an individual drawing on explicit but not necessarily fully-formalized models to help answer some of the questions of a player in a decision process. Those insights help illuminate the decision and make recommendations, or simply facilitate behavior conducive to greater consistency between the evolution of the process and the objectives and value system that this player is working with”[25] (Bernard Roy, 1993). Decision theory models the behavior of a decision-maker (named agent) when coping with choice situations (Raiffa, 1969; Simon, 1979; Cyert & March, 1963). It describes the choices of an agent and links them to a question of preference (Hansson, 1966; von Wright, 1963; Roubens & Vincke, 1985). It shows how preference relationships can be represented by a utility function (Quiggin, 1993; Jaffray, 1989; Fishburn, 1970) and when they imply the existence of beliefs of the agent on the

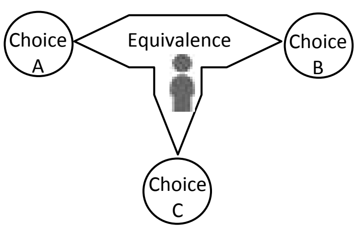
[23] A Turing machine is an idealized mechanism (program, algorithm) intended for computation. It uses an endless tape divided into cells and a head that can read, erase and write on the tape. This read-write head can move over the tape and the machine itself operates the movements. A program must not be ambiguous, that is, never have any elementary instructions simultaneously applicable. When a Turing machine is specified and supplied with a tape, its computation unfolds in a rigorously determined and unique manner.
[24] An act by which a person settles on a solution, decides something; resolution, choice (French dictionary Larousse, 2012).
[25] Translation by the author

various states of the world, which can be represented by probabilities (Ramsey, 1931; Savage, 1972: Schmeidler, 1982; Machina, 1982). As far back as the 18th century (Marquis de Condorcet, 1795), preferences have implicitly displayed rational behavior (or more aptly put, driven by a model of rationality) of an individual or collective making decisions, although it is not always possible to aggregate individual preferences into collective preferences (Arrow, 1951; Granger, 1956). The profusion of choices is thus related to the issue of preferences among possible (decidable) choices that are not always equivalent, rather than to the question of indifference breeding undecidability. "Indifference, by nature, does not translate into choice" (Mongin, 2011).

Indifference may come from a consumer, in which case how to model the choice process and the process by which he/she achieves an equivalence relation between the possible choices? To seek to model those processes is therefore to seek a model for the breaking of indifference to a collection of possible choices, in other words the breaking of a perfectly "symmetric" vision of the (collection of) possibilities, individually discernible in space or in time, or even nameable (A, B, ...) and countable (1, 2, ...) as well as interchangeable and undistinguishable from one another. In fact, the symmetry of a system, in modern language, refers to all the transformations[26] that sustain its invariance (more informally put, shifting things around always lets one see them the same way) and thus help define equivalence classes: "the invariance under a specified group of transformations" (Brading & Castellani, 2008). Indifference applies as well to the maintenance officer having to choose among interchangeable spares and meeting his availability request in a database management system as to the software designer coping with program equivalence or the chemist with molecule equivalence, etc.

In set theory, equivalence is typically a relation defined on a set, specifically in the realm of choices if it is a set of possible or decidable choices [fig.B]. It is conceivable to group together all the entities of the set considered in all possible ways, even when the set is infinite. This is premised[27] by a ZFC axiom (axiom of power set). All possible groupings are not always subsets of the original set considered. For example, in a given set of spare equipments, some subsets or intermediary parts are not listed (they can't be changed in the event of breakdown, the full subset must be changed). Similarly, all combination options are not always catalogued, and so forth. Thus, there is the set from which groupings are made and the set of conceivable sets into which those groupings fall. The latter is always larger (in terms of cardinality) than the former (in fact, it contains it) and not always countable when it is infinite (Cantor, 1883)[28]. As to the equivalence relation, it relates to the decision-maker and those that are involved in it (his/her competing or partner counterparts, etc.), or more broadly, it relates to a common system of knowledge and standards (formal, value systems, etc.). When the entities of a set are connected by equivalence and grouped into subsets called equivalence classes, these are necessarily disjoint as they must otherwise merge. What's more, there is no empty subset because there is at least one entity in an equivalence relation with itself. As a result, if no entity from the original set is left out, the outcome is a set of nonempty and disjoint subsets

Fig. B – Decision-maker indifference vs choice equivalence



[26] Those transformations have a specific algebraic structure designated as « symmetry group »

[27] To make pairs of elements that are in relation, the primitive set has to be replicated (Cartesian product). This operation is done between sets (unicity) and collections (multiplicity).

[28] Also, see « continuum hypothesis » (Cohen, 1966)

that reflects the decision-maker's vision of the original set from the perspective of the equivalence relation. This set of subsets is called "quotient" in mathematics. It is a particular subset of the set of all conceivable sets possible defined above. "For every equivalence relation there is a natural way to divide the set on which it is defined into mutually exclusive (disjoint) subsets which are called *equivalence classes*." (Borschev & Partee, 2001).

If all entities are deemed equivalent by the decision-maker, it is always possible to consider the set encompassing those entities and the decision-maker himself/herself (this set is nonempty) in order to construct the 'quotient' set with at least two mutually exclusive classes[29] and then move on to a self-reflexive R equivalence relation: "equivalent to itself". This is a self-inclusion process[30] (and, in fact, an involvement[31] process). The decision-maker's vision, which here translates into a 'quotient' set, results from a conception of potentially new subsets, each of which are relational schemes between choices deemed equivalent and having common properties, based on an original set of possible choices. In the process, the decision-maker operates in a relational set larger than the original one. The equivalence relations are specific relations that make sense with respect to the knowledge and models of rationality engaged, because each translates to a "quotient" set. Once the set of possible choices is "quotiented" with an equivalence relation, how does the decision-maker choose from each equivalence class to represent it? What really happens when initially the decision-maker only sees entities as all equivalent?

When indifference reflects the equivalence of possible choices [see fig.B], the term collection should be used instead of sets as in ZFC axiomatic set theory. The choice of a unique entity, individuated and discernible by its property, is easily constructible. By contrast, the choice among multiple individuated, discernible (not commingled) and interchangeable entities but undifferentiated by their property, seemingly requires a self-inclusion process [an abstraction][32] in the sense that the chosen entity must represent all its counterparts including itself. Granted, the axiom of choice can prove the existence of a choice function when no finite reasoning seems capable of doing it. "To choose one sock from each of infinitely many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed" (Darling, 2004)[33]. But this axiom does not provide any modus operandi. Besides, even when the collection of equivalent objects is finite, how can the choice function fulfil its role without building in a mechanism of distinction as a random variable, an order or a designation? How to choose a 'representative' of an equivalence class without such a mechanism?

For example, given a wheel partitioned into 10 equivalent sections that can have four different colors, there are necessarily sections of the same color and the term collection is in order here. Conversely, each color is unique, and thus it is a set. The random variable matches an element from the set of values "colors" with any outcome of the random experience of spinning the wheel until it stops on a section. The choice spans the set of possible distinctive values[34],

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[29] An equivalence relation groups together the elements of a set by mutually exclusive properties

[30] Through a similar infinitely repeatable process of 'abstraction' based on definition and logic, Frege constructed all natural numbers: "what falls under the concept '*non identical to itself*'? If the answer is nothing, then the cardinal that falls into this concept is 0. Now, on to the concept '*identical to 0*': because there is only one object identical to 0, namely 0 itself, the cardinal that falls into this concept is 1. From the definitions of 0 and 1 ($0 \neq 1$), it is possible to use the concept '*identical to 0 or 1*' with 2 as the cardinal.

[31] Which may explain dissonance cognition and subversion of rationality phenomenon (Festinger, 1957; Elster, 1983)

[32] "The class [*Object*] provides an abstraction. An object [*obj*] of the class [*Object*] can represent an arbitrary class. The only operations it provides is to make copies and assignments, so that you can put them in lists and arrays. (…) Therefore, there is no automatic conversion from these classes to [*Object*]. This encapsulation mechanism requires the use of [*assign*] or [*object cast*] to access the encapsulated class" (Brönnimann et Al., 2009).

[33] Citation from B. Russell.

[34] The designation of entities is not warranted by a collection being infinite and uncountable.

colors, numbers or designations (A, B, C, etc.) and is supposed to represent all equivalent choices that might be made instead. The process in which the choice occurs hinges on an equality relation between the designations reflecting the equivalence relation between the entities. Two designations are equal (A=B) if and only if the two designated entities are members of the equivalence class. “Until predicates have been assigned, the two substances remain indiscernible; but they cannot have predicates by which they cease to be indiscernible, unless they are first distinguished as numerically different” (Russel, 1900). Therefore, the entity chosen among the elements of the equivalence class is designated as a representative of this equivalence class. No function can attribute two different designations to one same entity. Hence, it is impossible to describe this self-inclusion process when remaining in ZFC.

The choice on equivalence moves back and forth between collections and sets [fig.C], respectively outside of and within ZFC axiomatization, that is, in and out of a particular model of rationality. Looking at decision, not when it signals the end of a deliberation process in the intentional act of doing or not doing something (for example, after much dithering), but when it signals a process by which a space of decidable choices is designed, requires pondering on the transition points between the collection and the set. Conceptual dualisms such as equivalence and choice, the discernible and the indiscernible, the multiple and the unique, can translate into antisymmetric or non-antisymmetric equality relations.

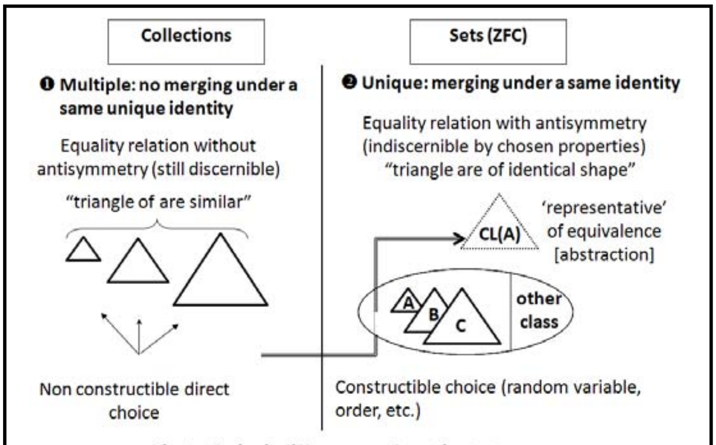

Fig.C - Choice indifference and quotient set: an abstraction between collection and sets

The conception process of a decidable space (described as innovative)[35] is modeled by the Concept-Knowledge (C-K) (Hatchuel & Weil, 2007; Hatchuel & Al; 2010) theory on an interaction principle between spaces respectively structured around ZF axiomatic set theory without and with the axiom of choice (AC); the space of concepts[36] (comprehensible but undecidable as an ‘imaginary number’ or a “flying boat’ can be at a given point) and the space of knowledge (decidable, for example complex numbers, lift on an airfoil and continuous equilibrium, or the principle of buoyancy). More simply put, without the axiom of choice, it would be impossible, according to this theory, to assert that an object exists like an unfinished object in the making. These two spaces mutually expand during the design process so that the development of the knowledge map can ultimately and “in conjunction” highlight initial paradoxes. Arguably, then, the axiom of choice does not function in the imaginary world (that of concepts) yet axiomatized by ZF. Outside of ZFC (which corresponds to the space of knowledge), thus in collections, it is unclear whether ZF axiomatic set theory still holds as is. ZF axiomatic set theory entails the indiscernibility of equivalent entities, which does not seem appropriate to the collections of equivalent and discernible elements. However, the disappearance of the choice axiom allows the construction of nonmeasurable sets and

[35] As is any creative abstraction process, potentially.

[36] This *concept* seems to have been differently designated in the literature: symbolic form, abductive hypothesis (Santanella, 2005), imagination, dreams, etc.

produces the Banach-Tarski paradox[37] (Su, 1990) which pictures the isomorphic collections. However, it is not possible to affirm that others axioms of ZF are preserved, in particular those of foundation and of infinity. The axiom of foundation can be stated as "every nonempty set is disjoint from one of its elements" (Mendelson, 1997) and also as “A set contains no infinitely descending (membership) sequence" (Ciesielski, 1997). Mendelson (1958) proved that the equivalence of these two statements necessarily relies on the axiom of choice. “One of the earliest paradoxes arose out of grappling with the notion of infinity. However, when we restrict the number of pieces to be finite and the allowable transformations to be isometries of the ambient space, any paradox that persists is hightly counterintuitive. Notice that the previous paradoxes [Banach-Tarski] depended on the set of allowable transformations. Hence we shall demand that our definition of paradoxical be dependent on a group whose action on the set produces the transformations” (Su, 1990).

The design of sets from collections has been studied through some concepts like *extension*, “extending knowledge and fruitful concepts” (Frege, 1873; Tappenden, 1995), *pre-sets* (Bishop, 1967): “Constructive mathematics does not postulate a pre-existent universe, with objects lying around waiting to be collected and grouped into sets, like shells on the beach” (Bishop, 1983) and *quasi-sets* (Krause, 1992; da Costa & Krause, 1994). “It seems reasonable (...) to search for a mathematical theory which considers, without dodges, collections of *truly* indistinguishable objects. In characterizing such collections (...) we have (...) developed the theory by posing that the expression x=y is not generally a well-formed formula (and, as a consequence, its negation x≠y is also not a formula). This enables us to consider logico-mathematical systems in which identity and indistinguishability are separated concepts; that is, these concepts do not reduce to one another as in standard set theories” (da Costa, Krause, 2006).

- The Experiment -
The Design Process Of A Space Of Decidable Objects:
Symmetry, Antisymmetry And Asymmetry In Decision-Making

Structuring and formulating a decision problem is a critical issue (Rosenhead, 2001; Stamelos & Al; 2003), one whose addressing hinges on axiomatic set theory. The space of decidable objects results from a design process between collections and sets in order to repartition broken equivalence relations as new knowledge and standards emerge. We now explain – through the example of visual perception that many of our decisions are driven by – how the design process of new quotient sets operates, that is, how a decision-maker substitutes one equivalence relation for a new one. The ultimate purpose is to propose an axiomatic system that complements (in the sense of Gödel) that of ZFC, which only allows an antisymmetric (fusional) equivalence relation. The extraction of invariant aspects from the constantly variable flow of information provided by sense organs, for example, can define structures with properties anticipatable by action as well as discernible shapes out of a tangle of lines.
Hence, our perception mechanisms seem to “hunger for invariance” (Paillard, 1974) in order to contain the uncertainty about the state of the environment. Any adaptation activity would be compromised if we didn’t have a fairly consistent and coherent representation of the environment. By essence, axiomatic set theory maintains through the identity of objects – by their discernibility or uniqueness – consistency in the perceptual construction of objects or beings to anticipate situations. An object that disappears out of view does not stop existing.

[37] A decomposition of the sphere into a finite number of non-overlapping pieces which can then be put back together in a different way to yield *two* identical copies of the original sphere without consideration for the size but only for their shape.

The perception of invariance is a form of learning (Bower, 1966) and invariance is a classification of observations with equivalence relations (Reuchlin, 1966). “The collected data on the perceptual behaviors of congenital blind people regaining vision through a surgical operation performed at adulthood and having long been able to distinguish between a circle and a square through tactile-kinesthetic exploration, do not recognize them upon seeing them. The two figures seem different to them but they cannot single them out as “circles” or “squares”. If they are taught to do it, the learning is compromised as soon as the figures are shifted or replaced by slightly different others. (…). That goes to prove that normal visual perception is the outcome of a long period of learning.” (Reuchlin, 1982). If we look up close at figure D (fig.D) the two images superimpose on one another. If we try to place sideways our forefingers on the end of a pencil standing on a table, with our eyes closed, the distance is difficult to gauge.

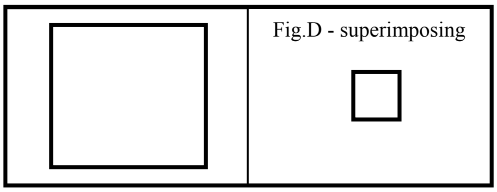
Fig.D - superimposing

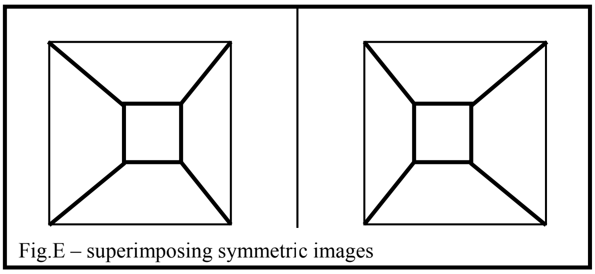
Fig.E – superimposing symmetric images

Finally, if we place a postcard on the line that separates the two images of figure E (fig.E) and put our foreheads or noses on the edge of the card, the two images merge and the frustum of the pyramid appears in three dimensions (as figure F simulates). The two representations of the pyramid frustum are views from two points (6 cm) inches apart, which corresponds to the average distance between the centers of the two corneas. Thus, depth vision (3D space) is generated by two reverse flat images.[38] Each point of the pyramid frustum on the left side of figure E has a counterpart on the right side of that figure. The mismatch along the horizontal axis in 2D space[39] is transformed into depth along a new axis in 3D space. To track down the position of a point in space consists in intersecting the two directions matching each eye with the point considered. By applying this method to all the points, it is possible to position an object. The two directions can focus inside or outside the plane of images (A and B), thus bringing the 3-D object into focus (by squinting or looking into the background). The depth of the 3D object varies according to the AB distance. Each eye solely perceives the image that is intended for it (two options of figure G).

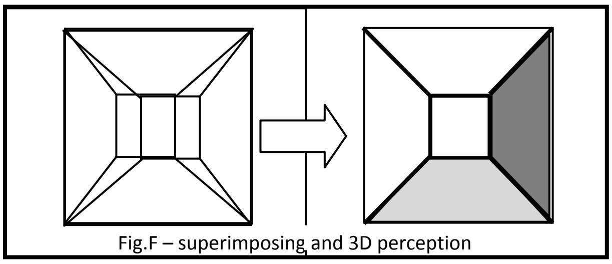
Fig.F – superimposing and 3D perception

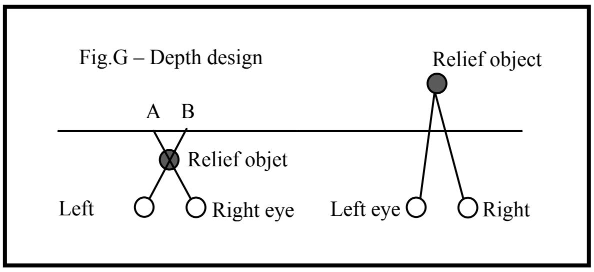

Fig.G – Depth design

Let’s replicate this pair of chiral images[40]: (A ; Sym.A) where ‘Sym. A’ is the abbreviation of “Symmetric to A”. We obtain (fig.H) two strictly identical and perfectly superimposable pairs (property known as achirality): ((A ; Sym.A) ; (A ; Sym.A)).

[38] The images are symmetric but not superimposable (chirality). The image pair is said to be enantiomorph (from the Greek *opposite form*).
[39] The creation of a hologram is based on a similar principle, from the interference between two beams reflecting and building distance.
[40] In physics, such situations can be observed. The electric field generated by a « mirror » electron » is the mirror image of the field generated by the electron, and the magnetic field generated by the movement of the « mirror electron » is reversed.

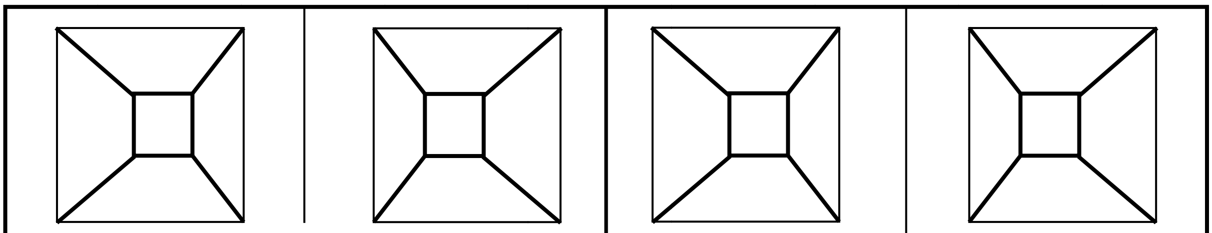

Fig.H - Replicating the couple of images

In set theory, this perfect equality of two sets (each pair is a set of points) should cause them to fuse by virtue of the antisymmetry of equality. Would any information get lost in the process?[41] Would we miss out on a 3D vision more informational than the one obtained with the initial pair alone (fig.E)? These questions address the foundational principle of fusion of equivalent things in set theory. After a "squinting" eye exercise, five pyramids appear, three of which in three dimensions plus one on either side (fig.I).

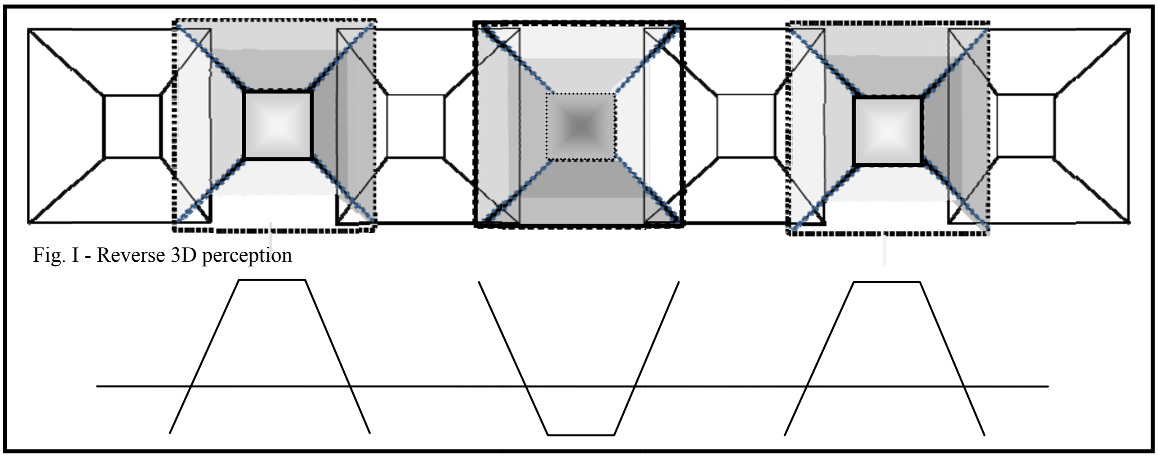

Fig. I - Reverse 3D perception

The middle one is seen in reverse 3D, as if observed from inside the pyramid. The images A and Sym.A of the initial pair of figure E are symmetric, interchangeable but non superimposable. Thus, the resulting pair (A ; Sym.A) presents "asymmetry". The image pairs (A ; Sym.A) and (A ; Sym.A) of figure H are symmetric, interchangeable and superimposable. The pair of pairs ((A ; Sym.A)) is "symmetric". The 3D result[42] of figure H is designed with the collection of two image pairs, namely (A ; Sym.A) and its replication (A ; Sym.A), equivalent and discernible (equality without antisymmetry). The perception of the reverse 3D pyramid stems from the implicit pair (Sym.A ; A) generated by the pair ((A ; Sym.A) ; (A ; Sym.A). Did any "breaking of symmetry" occur?

A system is 'symmetric' when all its elements can be exchanged without changing its structure. The symmetry of geometric shapes is defined in terms of invariance: "crucially, the parts are interchangeable with respect to the whole – they can be exchanged with one another while preserving the original figure" (Brading & Castellani, 2008). Symmetries translate a kind of equality of the system with itself or the uniformity of its structure. The elements A and Sym.A of the system (A ; Sym.A) are not exchangeable without structural change as the order of the pair matters: the pyramid will alternately appear in three dimensions right side up

[41] « Information is what brings us knowledge, alters the way we see the world, reduces our uncertainty (Reix, 2000) « (…) Among all notions considered, informational distance (which stems from algorithmic information theory) seems to be the most inclusive in order to grasp the notion of similarity between objects » (Delahaye, 2003).

[42] It can be seen on figure J if the eye exercise on figure I is unfinished.

or upside down. Conversely, the system ((A ; Sym.A) ; (A ; Sym.A)) presents structural 'symmetry' because the element (A ; Sym.A) can be exchanged with itself without impacting the order of the pair. Does the depth perception generated by this pair 'break' this structural symmetry? It is tempting to say it does by arguing that depth perception is generated (fig.I), and that it brings knowledge that alters our vision of the world" (Reix, 2000; Reix & al; 2011). But some may also note that depth perception is already generated (fig.F) in the initial pair (A ; Sym.A). Ultimately, this may all depend on the difference between the depth perception generated in the pair ((A ; Sym.A) ; (A ; Sym.A)) and the one generated in the pair (A ; Sym.A), namely the upside down pyramid. But then, whether or not the right-side-up or upside-down pyramids are perceived equivalently is yet to be demonstrated. If the answer is yes, there is no breaking of symmetry attributable to the pair ((A ; Sym.A); (A ; Sym.A)). In other words, symmetry would preserve the informational equivalence "right side up / upside down". Then, A and Sym.A images should also be considered as equivalent (as they are obtained by building symmetry). The 3D outcome from these two equivalent images is undoubtedly more informational (2D versus 3D), therefore the symmetry is effectively broken. Of course it could be hypothetically objected that 2D and 3D symmetry operations are not comparable and equivalent operations. In other words, A and Sym.A may not be equivalent through 2D symmetry, hence the depth perception; conversely, the upside-down and right-side-up pyramids may be equivalent through 3D symmetry with no additional information. But this assumption implies that what holds in 2D geometry no longer does in 3D geometry. Mathematically, though, we know that it is not the case because 3D geometry includes 2D geometry (e.g. 3D geometry is an extension of 2D geometry). We only know this because 3D geometry is now known. If it wasn't, it should be designed by abstraction by including 2D geometry to prevent undecidable[43] situations from occurring (Brönnimann & al, 2009; Lee & al; 1992; Brunet, 1991).

All things considered, it can be said that axiomatic set theory and, by extension, its underlying model of rationality, can prevent depth perception as it merges any collection of equivalent images and, in particular, the pair that is yet likely to generate new information by "symmetry breaking" (fig.H). The equivalence relation is relevant to a viewer or group of viewers with similar knowledge standards. The grouping of equivalent entities into a given context can be repartitioned into a new context including and redefining the previous one. This holds true for 3D space, which includes 2D space by redefining all geometric objects more broadly. New equivalence relations preserving existing ones and generalizing them are generated by creating a new dimension[44] independent from those that already serve to spawn the initial space and describe its elements. "This means that any dimension needs to be studied using the ideas of spaces/sets or sub-spaces/sub-sets or partitions/cuts. The idea of dimension is complex and rather deep for the human mind to penetrate. This is because we need to often perform abstractions and parameterizations of time and space of any geometrical object that cannot be visualized in the real world (Inselberg & Dimsdale,1994) (…) According to this book, the term "dimension" can be defined "as the unique mega-space that is built by infinite general-spaces, subspaces and micro-spaces that are systematically interconnected" (Estrada, 2011). This process reflects a meta-restructuring in knowledge standards. The solution to a problem appears as a construction rather than the outcome of research in a given space (Bateson, 1972), highlighting the fundamental distinction between classic and new

[43] Such an indecidibility seems to be formulated so: "(…) For many properties undecided by the ZFC system (…) it is possible to construct an extension in which the considered property is true and (…) construct another extension in which the property is false (…). At this point it seems extremely difficult to distinguish the property from its negation and break the symmetry by focusing on one over the other" (Dehornoy, 2003).

[44] A dimension is defined as the number of independent variables that help define a state, an event or a system.

approaches to decision theory (Tsoukiàs, 2004). From this standpoint, depending on whether the axiom system permits an antisymmetric equivalence alone or is also open to an antisymmetry-free equivalence, the possible conceptions of the decidable space are just as varied as 2D and 3D geometry. Tversky shows that intuitive ideas about the properties of preference relations are not so much a behavior of real decision-makers as a theoretical requirement (Tversky, 1967). He also shows that the indifference relation can be not symmetric (Tversky, 1977).

- DISCUSSION -
Decisional Conception, Abstraction And Imaginary: Complementing The ZFC Axiomatic Of Set Theory

Depth perception arises from a mechanism that:
[1]. "symmetrizes" to create "non antisymmetric" (unmerged) pairs of entities connected by an equivalence relation with respect to a knowledge standard.
[2]. Repeats this "symmetrizing" to obtain a structurally symmetric system (pair of pairs)[45], that is, indifferent to the permutations of its elements.
[3]. "asymmetrizes" (breaks the symmetry of the previous system) from the differences that show through and bring new knowledge (or engage implicit or even tacit knowledge) to repartition.

This mechanism brings to mind[46] von Neumann's *The Self-Replicating Machines* which have spawned recent developments in the field of artificial life and genetic algorithms (Lipson and Pollack, 2000; Nakhla & Moisdon, 2010). The machine is described as follows: "Machine replication works in three steps:
[1]. Machine P (parent) reads the blueprint and makes a copy of itself, machine C (child);
[2]. Machine P now puts its blueprint in the photocopier [the machine contains a photocopier], making a copy of the blueprint;
[3]. Machine P hands the copy of the blueprint to machine C. Note the blueprint is used in two ways, as active instructions and as passive data" (von Neumann, 1966).

Of all the pairs of pyramids constructed with symmetry (fig.E; fig.H), only the pair with a 6 centimeter distance (between corneas), in both figures E and H, gives a sense of perspective and depth in space. The other pairs (fig.E) or pairs of pairs (fig.H) look partly or totally fuzzy and the "breaking of symmetry" does not occur. A "fuzzy" logic developed with set theory to address uncertainty, ambiguity and linguistic variables was introduced in the early 60s (Bellman & Zadeh, 1970). A "measurement" of the membership of an element in a set has helped increase the expressiveness and flexibility of formal languages and thus decision-support models.[47] Accordingly, each pair of 2D pyramids (discernible unlike the pair of pairs in figure H and thus making it possible to use set theory reasoning) can always be related to a value that graduates "fuzziness" from 0 to 1 (fig.J).

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[45] A pair of pairs is a relation of order 2 (connections between connections). An ordered pair is a relation of order 1.

[46] This mechanism also brings to mind an ancient symbolic formulation: a symmetric and imaginary world through the mirror, from which one comes back with a form of asymmetry in all mythologies (limp of Jason, Oedipus, Hephaestus, etc.)

[47] "More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership (...) however, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. (...) The notion of a fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respects the framework used in the case of ordinary sets, but is more general than the latter and potentially, may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing." (Zadeh,1965).

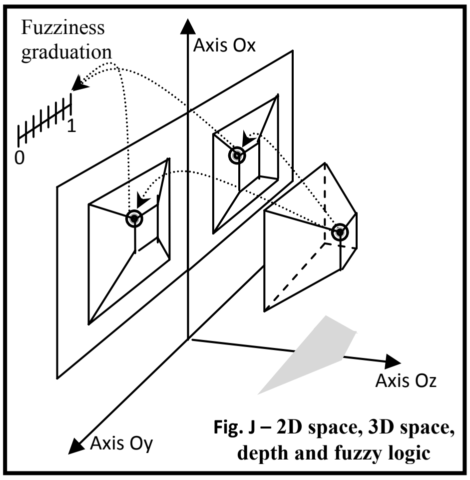


Fig. J – 2D space, 3D space, depth and fuzzy logic

The 3D pyramid is a member of 3D space but each of its points splits into two corresponding points in 2D space (it has two corresponding points, one on each pyramid of the 2D pair). And yet, "A function is a binary relation, which is one-one or many-one, not one-many" (Codd, 1970). It is easy to functionally match many points of the 3D pyramid with one single point of the 2D pyramid, for example when projecting 3D space onto a 2D space plane. By contrast, it is not possible to build a split correspondence of 3D space in 2D space using a function. It is thus relevant to represent each pair of points in 2D space by a value ranging from 0 to 1. The pairs of points in 2D space formed from a point and its counterpart on a 2D pyramid and its 2D symmetric respectively, are assigned a 1 value as they relate to a point of the 3D pyramid in 3D space. But this relation (creation of a third dimension) is not a geometric membership relation. None of these pairs of points of the plane (xOy) (fig.J) is geometrically a member of the 3D pyramid. And the translation into "fuzzy set (class)" requires ranking the degree of membership of the sets: 0 (no membership) or 1 (membership). Therefore, the depth conception relation should be interpreted as a "hyper n-ary relation" of gradual membership of 2D space within 3D space (fig.K).

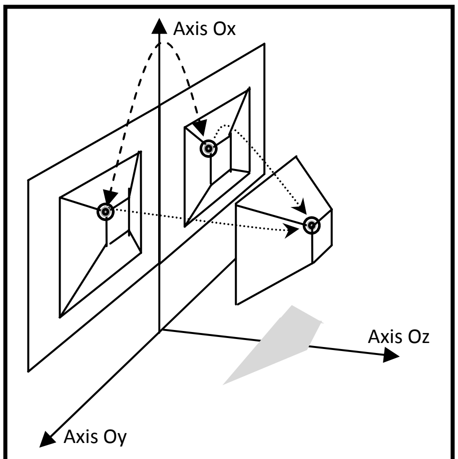


Fig. K – Designing depth
- Connections between one point and its counterpart in 2D space (broken line - - - -)
- Hyper connections between 2D and 3D space (dotted line ……)

The translation into "fuzzy sets" must attribute a value ranging from 0 to 1 to the pairs of 2D space during the depth construction process. This "many-one" correspondence stems from the inability of a function to translate the "one-many" correspondence" (Codd, 1970) in set theory. Using figure H rather than figure E, translating into "fuzzy sets" would be theoretically impossible as both pairs of 2D pyramids of that figure are strictly identical and cannot coexist in set theory. Evidently, there is a lack of a formal construction (see §. 2) in collections that permits an antisymmetry-free equivalence (discernibility). It would then be possible to define very simply a "one-many" and 'n-ary' relation with a type of 'hyper'-function between a set and a collection. However, if we keep to set theory as a reasoning constraint, only one pair of pyramids must be considered. But how should the other pair be considered? "A fine and wonderful recourse to the human spirit, almost an amphibian between being and non being" (Leibnitz, 1989). The second pair can be represented in an imaginary dimension as imaginary numbers usually are: "Just as one can think of the realm of all real quantities as (represented by) an infinite straight line, so one can make sense of the realm of all quantities, real and imaginary, as an infinite plane in which each point, determines by abscissa a and ordinate b, represents the quantity *a + i b* (…)" (Gauss, 1811). With one point of the pyramid (fig.E) acting as the real number "a" on a real dimension and the point counterpart on the symmetric pyramid (fig.E) acting as the imaginary number "*i*.b", just as the two numbers "a" and "*i*.b" combine under the addition a + *i*.b, the points and their counterparts (fig.E) combine by the law of depth construction.[48] It is possible through this translation of the imaginary to define the 3D object just as a complex

[48] Which converts distance into depth to create perspective.

number z = a + i.b, partly real “a” and partly imaginary “i.b” is defined. The same reasoning is applicable to figure H. The pairs of point pairs are constructed like complex numbers pairs designated by mathematicians as hypercomplex numbers or “quaternions” (Lygeros, 2003)[49]: (A + i.Sym.A) + j.(A + i.Sym.A). In fact, they constitute a useful tool for space processing in computer graphics or 3D digital animation (fractal images, representations of hierarchical objects, etc.).

The resort to the formalism[50] of complex and hypercomplex numbers to expose the imaginary part necessary for designing a space of decidable objects is related to the abstraction process (self-inclusion) previously described (see §-3). In the present case, self-inclusion exemplifies that duality because the imaginary element springs from the decision-maker himself. He/she will be unable to construct a quotient set as described [fig.C] in a situation of total indifference if they consider themselves to be absent and in this case forced to be represented by the empty set (symbol Ø) [in fig.C when there is no ‘other class’ the collection can’t be transformed into a set].

The empty set Ø that is at the core of set theory to disconnect a set of a new subset and to avoid paradoxes of self-inclusion while designing the latter, may be superseded by the ‘imaginary’ (symbol i) in collections. The empty set Ø may then symbolize a state being ‘hollowed out of any imaginary content’ and characterizing the abstraction-making that gives the designed subset (representing an object, a decision, etc.) its independent existence from the designer’s imaginary (fig.L_a). From a methodological point of view, this interpretation may solve the contradiction raised by Russel about Meinong’s theory and ground a new axiomatic system expanding ZFC axiomatic set theory without contradicting it. “Mr. MacColl regards individuals as of two sorts, real and unreal; hence he defines the null-class as the class consisting of all unreal individuals (…) This is essentially Meinong’s theory, which we have seen reason to reject because it conflicts with the law of contradiction. With our theory of denoting, we are able to hold that there are no unreal individuals; so that the null-class is the class containing no members, not the class containing as members all unreal individuals” (Russel, 1905).

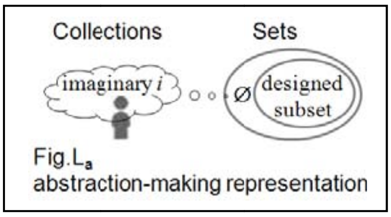


Fig.L_a
abstraction-making representation

The same way the empty set Ø fits in between all entities in the universe of sets, to make each of them distinguishable and unique, preventing this way any self-inclusion paradoxes, the ‘imaginary’ would fit in between imaginary individuated entities, yet undistinguishable through their properties in the universe of collections. This dualism (fig.L_b) would be established between their ‘individual’ description (as a set) and their ‘plural’ description (as a collection) by a hyper-function[51] f of the type ‘one-many’.

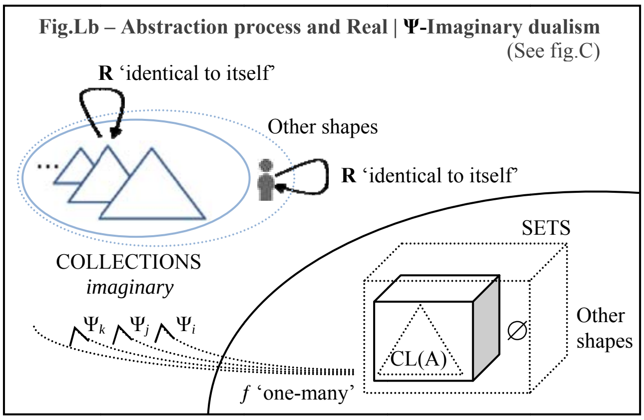


Fig.Lb – Abstraction process and Real | Ψ-Imaginary dualism

[49] i, j being imaginary numbers

[50] Which would make it easier to consider (hyper)complex numbers as a particular iterate application of the collection - set duality to the universe of numbers.

[51] ‘Hyper’ allows the possibility for a function to be of the type a one-many

The 'imaginary' may thus permit the existence of an infinite collection of equivalent copies (the 'plural' description) being each generated by isomorphism Ψ_i operating from sets to collections and preserving the relational structure[52] of the original set. f: set e $\rightarrow$ collection c {Ψ_i (e); Ψ_j (e); etc.}. The 'imaginary' representation is already known when it comes to numbers under the symbol *i* and is also well adapted to formalize variational phenomena. The isomorphism plays a key role in mathematics as in management. Mathematical objects are considered to be essentially the same, from the point of view of their algebraic properties, when they are isomorphic, the usual expression being 'up to isomorphism' (Manalo, 2001). Hawley (1968) defined isomorphism as a constraining process that forces one unit in a population to resemble other units that face the same set of environmental conditions. More recently, Thornton (2011) recounted that "Clearer evidence of isomorphism is found within the world system literature, where the unit of analysis is better defined and highly aggregated, prompting the question of how observer distance and level of abstraction contribute to findings".

The new axiomatic system expanding ZFC, in the same way as mathematicians agree with the existence of models[53] of ZFC, should permit antisymmetric equality (merging to make the unity from indiscernibles) in the universe of sets governed with ZFC and non-antisymmetric equality (no merging to preserve the discernibility) in the universe of collections. Such new axiomatic system is potentially more powerful than ZFC. For example, it can account for the design process of a space of decidable objects such as depth, as well as describe polymorphous situations (several equivalent and discernible forms) such as the equivalence of contextual choice in a highly changing environment, including algorithmic geometry[54] (Brönnimann & al, 2009; Mitchell, 2003), relational databases (Codd, 1970, 1990), aeronautics or software engineering (Giacomoni & Sardas, 2010).

In this view, when the decision-maker performs a representation of a real situation without perfect knowledge of the states of the world (Simon, 1956, 1969, 1976), she/he ought to consider the possible construction of a space of decidable objects supplemented with a 'Ψ-imaginary' dimension (§-4), one that may radically change her/his vision of the world just like depth creation in 3D space from 2D space. Decidable objects in such a dual space, partly real and partly imaginary, would be formalized with two dimensions [linked by Ψ], a real one for possible preference-making and an imaginary one for possible abstraction-making (thus undecidability-unmaking). Most concepts and relations, valid in the usual real space, would be extendable in the new one, according to Hankel's principle (Crowe, 1990): "definition of an operation should be extended from a restricted domain to a wider one in such a way as to conserve the crucial algebraic properties of the operation".

Principle of 'Real | Ψ-Imaginary' spaces based formalism:

Decidable space |Quotiented with R $\oplus$ ***i*** • **Undecidable space** |Generated with Ψ and quotiented with R* by abstraction-making [R and R* being relations of equivalence]

[52] More precisely, the group-theoretic structure: "The importance of group theory is relevant to every branch of Mathematics where symmetry is studied. Every symmetrical object is associated with a group. It is in this association why groups arise in many different areas like in Quantum Mechanics, in Crystallography, in Biology, and even in Computer Science. There is no such easy definition of symmetry among mathematical objects without leading its way to the theory of groups" (Manolo, 2001).

[53] Model means a collection M of sets with the property that the axioms of ZFC are satisfied under the interpretation that "sets" are only the sets belonging to M (Jech, 2008).

[54] Computational Geometry Algorithms Library

Zimmer (2003) recounted a story (Solso, 1996) concerning the abstraction applied in forming our perception of the world and in manipulating concepts to achieve goals: "The soldier took Picasso to task for not producing realistic pictures and, to illustrate the ideal from which Picasso has fallen so far short, he pulled out a photograph of his fiancee back home saying: 'This is what a picture should look like'. Picasso looked carefully at the photograph and said: 'Your girlfriend is rather small, isn't she?' This story reminds us that all artworks, all pictorial representations, are abstractions. Picasso makes the point about size, but similar points can be made about the soldier's fiancee's stillness, lack of a third dimension, and so on; (…) The story may be a critique of science as a way of describing the world or it could be read as saying that exact sciences must be built on inexact representations. In either case, there is a sure insistence on the need to abstract away from exactitude if we are to arrive at meaningful representations".

Now comes the question of how to get the relational key R* to designing a new decidable quotient set (like depth with flat views 6 centimeters apart). This is finally the strategic question of the origin of the 'eureka'[55] (Rivkin & Gavetti, 2007; Bilton & al., 2003). But the answer is usually sophisticated: "The multidimensional coordinate spaces can open the possibility to offer an alternative graphical modeling to visualize unknown dimensions in the same graphical space and time" (Estrada, 2011). "We establish the correspondence between C-K theory and Forcing, a method of Set theory developed by Paul Cohen in 1963 for the 'invention' of new sets" (Hatchuel & Weil, 2007). The theories of abstraction operators (Tennant, 2004) and also the standard modern formula based on equivalence classes, that is: $\forall\alpha\forall\beta\ (\S\alpha = \S\beta \leftrightarrow \alpha \approx \beta)$ where α and β are expressions, § is an operator forming singular terms, and $\approx$ is equivalence (Dummett, 1992; Fine, 2002; Wright, 1983). A relation between the referents of two separate expressions has resulted in a single abstract object, defined by an identity between the referents of the singular terms.

We argue that the process to design R* from R is based on a meta-restructuring knowledge which is possible by abstraction-making, precisely by enclosing a representation of the 'missing' knowledge[56] which was at the origin considered as irrelevant when designing the relation R. Lewis said "incomplete descriptions" of concrete entities (Lewis, 1986). This 'missing' knowledge has the high value of being obviously independent of R and thus works for generating a new independent dimension (with attached degrees of freedom). The perpetual selective search of new invariances when exploring the unknown requires a relevant well-timed understanding of the environment to anticipate the future. This is the role of consistent theories to accounting past and present experiments. But it is well known that « To obtain the coherence of a system T, we need "more than T" » (Girard, 2006, 2007). Indeed, Gödel's theorem (Cori & Lascar, 1993; Nagel & Al, 1989) states that for any (proof) formal system powerful enough, it is possible algorithmically, to find an undecidable statement in the system based on its precise definition. Hence, no formalizing method can succeed completely. But an undecidable Gödel system is undecidable in relation to a given proof system. No Gödel undecidable statement is absolute (meaning undecidable in any proof system). Therefore, if we take an undecidable Gödel statement from a proof system and add it to the axioms of that system, we obtain a new proof system in which the undecidable becomes provable because it is an axiom. This method "slots together" (Delahaye, 1995, 2002) ever more powerful systems (consisting of fewer and fewer undecidable statements).

[55] Archimedes (287 BC – c. 212 BC)
[56] e.g. missing information according to Reix's understanding

Using set theory as a proof system, the decision maker knows that there is a more powerful system based on Gödel's theorem. We previously modeled the passage to this powerful system through the design of an imaginary dimension with isomorphic collections. The relation R works between real elements (deemed so) and also but separately, between imaginary ones according to isomorphism[57] Ψi. The relation R* works between a real element 'e' of the original structured set and an imaginary element '*i* e' of an isomorphic structure, such that R* is to be similar to R. This similarity is obviously significant for an outside observer (the decider) only. Indeed, none element 'e' of the original structure can co-exist with its isomorphic double '*i* e' in an imaginary structure according to the unicity principle of set theory (and according to the exclusive middle)[58]. 'R* similar to R' means that a common representation is possible which is an abstraction according to Frege's understanding and which encapsulates the 'missing knowledge'. It also means that an isomorphism can be defined between R and R*, such isomorphism being applied between relations (order 2) applied themselves between elements (order 1). Real elements 'e' and imaginary ones '*i* e' become commutable when R* is similar to R. Thus, R becomes independent from the status of elements, indifferently 'real' or 'imaginary'. The relation R* then appears as an extension of the relation R in a new space populated with elements e* partly real and partly imaginary, describable by the expression:

$$\mathbf{e^* = e \oplus \mathit{i}\, \Psi_{(R^*=R)}\,(e)}$$

Applied to the universe of numbers it gives complex numbers formula $z = a + i.b$. For numbers, "*i*" is generally confused with the corresponding isomorphism Ψ (rotations angle of $\pi/2$ applied to the straight line representing real numbers). But in fact, among all "imaginary" isomorphic straight lines generated from the original one representing reals, that corresponding in particular to a rotation angle of $\pi/2$ answers the question of finding the unknown imaginary number *i* such that $i^2 = -1$.

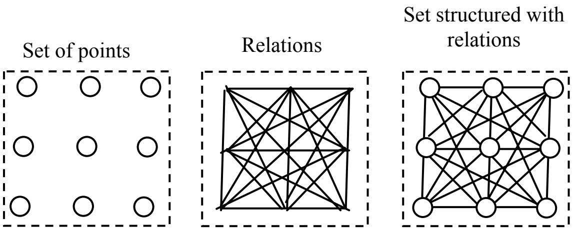
An easy way to interpret the isomorphism $\Psi_{(R^*=R)}$ [working from sets to collections and preserving the relation R between all elements, while making it independent from their real or imaginary state] is the definition of a digital image, by formats (set of pixels[59]) or by vectorization (structure). The latter allows infinity of sizes preserving details and superimposing formats coming together without affecting the velocity (and the capacity of the memory) in a substantial way because all information resides in the structure. But to be rendered (displayed or printed) a conversion into a format is required. This interpretation gives a possible explanation for the dual form of human reasoning 'deductive – abstractive' as an adaptation of the mental capacity to 'structuring – designing' processes.

We suggest the proposed formalism to be now applied to the so called *Thinking Outside the Box* problem [known first as the *continuous drawn line problem* (Loyd, 1919)] which consists in linking points (nine) with lines (four) without raising the pencil. *Thinking Outside the Box* means an original and creative way of thinking. This problem[60] is supposed to be quite impossible to solve (difficulty to get out from the mental square of thinking). Points can be understood as elements (representing properties, objects, knowledge pockets, etc.). Links

[57] Ψ preserves the theoretic-group structure: two elements of the original structure that are together in relation with R have corresponding elements in the isomorphic structure that are also in relation with R.
[58] A proposition is true or false (no third option). So is it for an element being real or imaginary.
[59] Smallest controllable subdivision of a digital image
[60] It is taken from the "shapes" Theory (*Gestalt* – perceptive and structural field) in psychoanalysis (Reuchlin, 1986). This discipline is interested in shapes containing an ambiguity, a paradoxical, and generating an insight through a new square of thinking.

between points can be then considered as interactions (dependences, relations, etc.) and links between links as possible interactions between N elements of a set (constraints, network, graphs…).

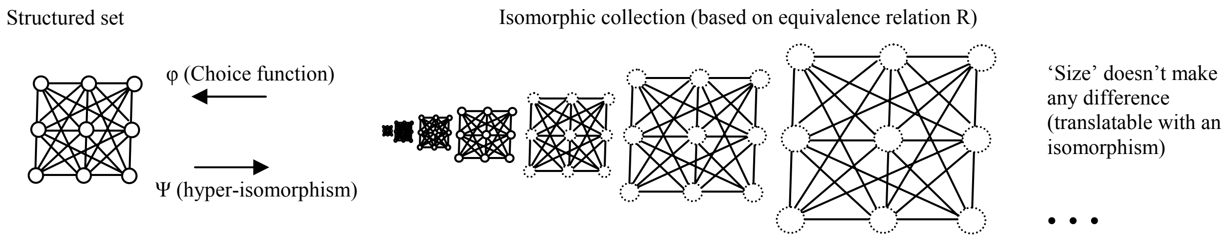
- Step 1 (formalization)



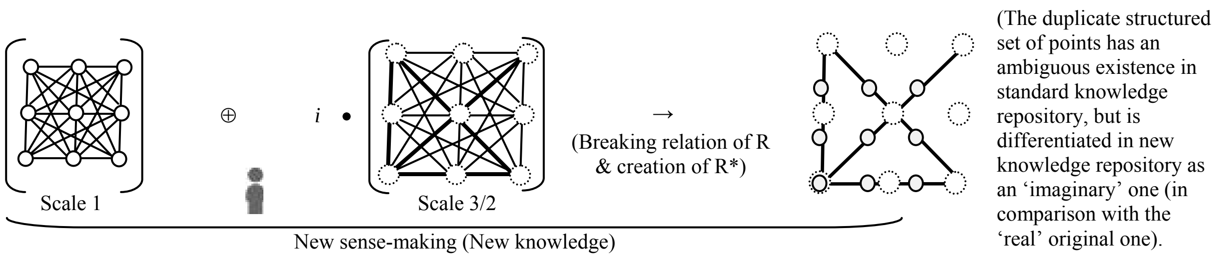
- Step 2 (Standard problem solving)

If E means the set of all possible decisions to link 4 lines (e.g. E is the ‘decidable space’), E contains no satisfying decision to solve the problem.

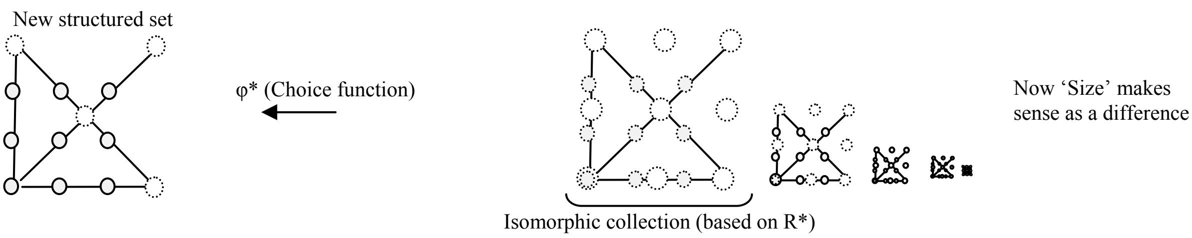
- Step 3 (Set [real status E] | Collections [imaginary status *i* • E])



- Step 4 (Abstraction & symmetry-breaking of R])



- Step 5 (New ‘decidable space’ E*)



Conclusion

ZFC axiomatic set theory was founded on the set concept of *unity* that makes parts *indiscernible*. It provides real or thought-based objects with a kind of identity form along with a model of rationality for validating human formal constructions. The undecidability of this axiomatic system may be simply stated as A=B in the sense that A and B are both equal and indiscernible and yet differently designated. Thus any equality relation is an abstraction process as it requires, according to Frege (1883), the design of a new and unnamed equivalence class. The study aimed to find a more powerful axiomatic system that can provide unequivocally a dual identity form in the universe of collections with the concept of *discernible multiplicity of equivalent parts* (such meaning an equality relation up to isomorphism). A concept of 'hyper' function is introduced to deal with a 'n-ary' relation of the type "one-many" between a set and the corresponding isomorphic collection. Choice as axiomatized by ZFC is not possible in the dual identity form until a return in the decidable space. It is intentionally proposed to resort to the 'real (for preference-making) | Ψ-imaginary (for abstraction-making)' formalism, that has been widely used in the theory of numbers (complex and hypercomplex, etc) to translate this duality. This formalism extends most concepts and relations valid in the usual real space according to Hankel's principle. With the imaginary space and a concept of 'hyper' function (n-ary relation of the type "one-many"), the abstraction process can manage through self-inclusion under visible terms and meta-restructure knowledge to find a new relational key and to design the new decidable while extended space where real and imaginary entities have become commutable.

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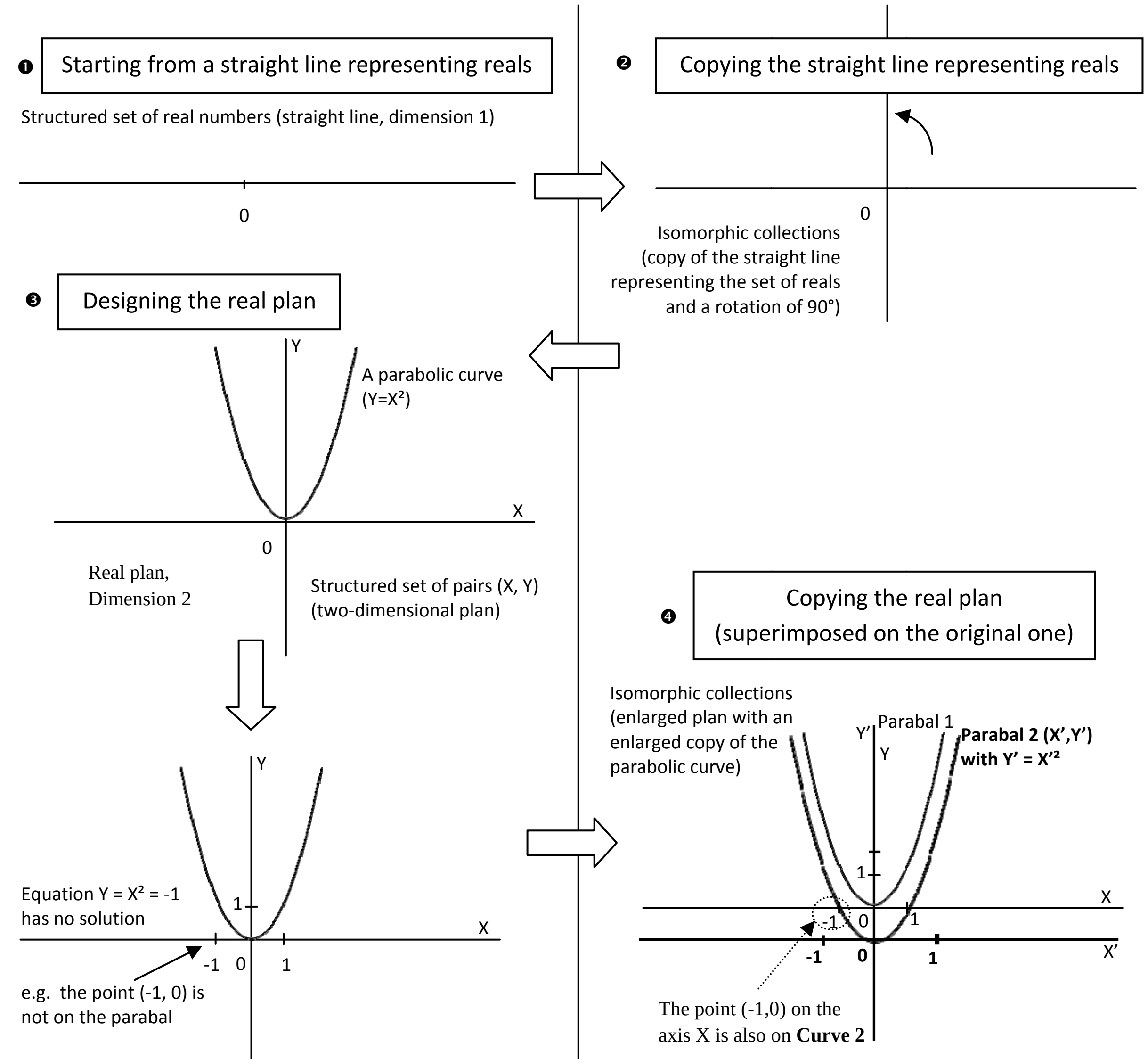
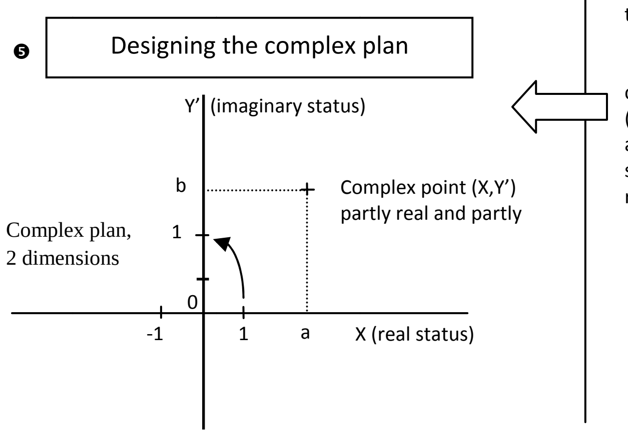
Appendix (designing complex numbers through isomorphic collections)



The size of the parabolic curve must increase such that the relation R "having a common point" is transposable between the point (X=-1,Y=0) on axis X in the original real plan (X,Y) and the point (X',Y'=X'²) on the parabolic curve 2 drawn in the isomorphic imaginary plan (X',Y')

This means the unity on the axis Y' has increased in comparison with the unity on the axis X. Such homothety (increase) goes with a rotation of 90° (angle between axis X and axis Y'). This is exactly what appears on a logarithmic spiral (rotation equals homothety). The set of reals can be represented by a straight line or by a logarithmic spiral[1].



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