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PHILIPPE MARTIN, LIONEL ROSIER, AND PIERRE ROUCHON

Abstract. We investigate the internal controllability of the wave equation with structural damping on the one dimensional torus. We assume that the control is acting on a moving point or on a moving small interval with a constant velocity. We prove that the null controllability holds in some suitable Sobolev space and after a fixed positive time independent of the initial conditions.

1. Introduction

In this paper we consider the wave equation with *structural damping*[1]

$$y_{tt} - y_{xx} - \varepsilon y_{txx} = 0 \tag{1.1}$$

where t is time, $x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ is the space variable, and ε is a small positive parameter corresponding to the strength of the structural damping. That equation has been proposed in [21] as an alternative model for the classical spring-mass-damper PDE. We are interested in the control properties of (1.1). The exact controllability of (1.1) with an internal control function supported in the *whole* domain was studied in [12, 14]. With a boundary control, it was proved in [22] that (1.1) is not spectrally controllable (hence not null controllable), but that some approximate controllability may be obtained in some appropriate functional space.

The bad control properties from (1.1) come from the existence of a finite accumulation point in the spectrum. Such a phenomenon was noticed first by D. Russell in [25] for the beam equation with internal damping, by G. Leugering and E. J. P. G. Schmidt in [15] for the plate equation with internal damping, and by S. Micu in [19] for the linearized Benjamin-Bona-Mahony (BBM) equation

$$y_t + y_x - y_{txx} = 0. \tag{1.2}$$

Even if the BBM equation arises in a quite different physical context, its control properties share important common features with (1.1). Remind first that the full BBM equation

$$y_t + y_x - y_{txx} + yy_x = 0 \tag{1.3}$$

is a popular alternative to the Korteweg-de Vries (KdV) equation

$$y_t + y_x + y_{xxx} + yy_x = 0 \tag{1.4}$$

as a model for the propagation of unidirectional small amplitude long water waves in a uniform channel. (1.3) is often obtained from (1.4) in the derivation of the surface equation by noticing

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[1]The terminology *internal damping* is also used by some authors.

that, in the considered regime, $y_x \sim -y_t$, so that $y_{xxx} \sim -y_{txx}$. The dispersive term $-y_{txx}$ has a strong smoothing effect, thanks to which the wellposedness theory of (1.3) is dramatically easier than for (1.4). On the other hand, the control properties of (1.2) or (1.3) are very bad (compared to those of (1.4), see [23]) precisely because of that term. It is by now classical that an "intermediate" equation between (1.3) and (1.4) can be derived from (1.3) by working in a moving frame $x = ct$, $c \in \mathbb{R}$. Indeed, letting

$$z(x,t) = y(x-ct,t) \tag{1.5}$$

we readily see that (1.3) is transformed into the following KdV-BBM equation

$$z_t + (c+1)z_x - cz_{xxx} - z_{txx} + zz_x = 0. \tag{1.6}$$

It is then reasonable to expect the control properties of (1.6) to be better than those of (1.3), thanks to the KdV term $-cz_{xxx}$ in (1.6). In [24], it was proved that the equation (1.6) with a forcing term supported in (any given) subdomain is locally exactly controllable in $H^1(\mathbb{T})$ provided that $T > (2\pi)/c$. Going back to the original variables, it means that the equation

$$y_t + y_x - y_{txx} + yy_x = b(x+ct)h(x,t) \tag{1.7}$$

with a moving distributed control is exactly controllable in $H^1(\mathbb{T})$ in (sufficiently) large time. Actually, this control time has to be chosen in such a way that the support of the control, which is moving at the constant velocity c, can visit all the domain $\mathbb{T}$.

The concept of moving point control was introduced by J. L. Lions in [17] for the wave equation. One important motivation for this kind of control is that the exact controllability of the wave equation with a pointwise control and Dirichlet boundary conditions fails if the point is a zero of some eigenfunction of the Dirichlet Laplacian, while it holds when the point is moving under some (much more stable) conditions easy to check (see e.g. [2]). The controllability of the wave equation (resp. of the heat equation) with a moving point control was investigated in [17, 9, 2] (resp. in [10, 4]). See also [27] for Maxwell's equations.

As the bad control properties of (1.1) come from the BBM term $-\varepsilon y_{txx}$, it is natural to ask whether better control properties for (1.1) could be obtained by using a moving control, as for the BBM equation in [24]. The aim of this paper is to investigate that issue.

Throughout the paper, we will take $\varepsilon = 1$ for the sake of simplicity. All the results can be extended without difficulty to any $\varepsilon > 0$. Let y solve

$$y_{tt} - y_{xx} - y_{txx} = b(x+ct)h(x,t). \tag{1.8}$$

Then $v(x,t) = y(x-ct,t)$ fulfills

$$v_{tt} + (c^2-1)v_{xx} + 2cv_{xt} - v_{txx} - cv_{xxx} = b(x)\tilde{h}(x,t) \tag{1.9}$$

where $\tilde{h}(x,t) = h(x-ct,t)$. Furthermore the new initial condition read

$$v(x,0) = y(x,0), \quad v_t(x,0) = -c\, y_x(x,0) + y_t(x,0). \tag{1.10}$$

As for the KdV-BBM equation, the appearance of a KdV term (namely $-cv_{xxx}$ in (1.9)) results in much better control properties. We shall see that

(i) there is no accumulation point in the spectrum of the free evolution equation ($\tilde{h} = 0$ in (1.9));

(ii) the spectrum splits into one part of "parabolic" type, and another part of "hyperbolic" type.

It follows that one can expect at most a null controllability result in large time. We will see that this is indeed the case. Throughout the paper, we assume that $c = -1$ for the sake of simplicity. Let us now state the main results of the paper. We shall denote by (y_0, ξ_0) an initial condition (taken in some appropriate space) decomposed in Fourier series as

$$y_0(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}, \quad \xi_0(x) = \sum_{k\in\mathbb{Z}} d_k e^{ikx}. \tag{1.11}$$

We shall consider several control problems. The first one reads

$$y_{tt} - y_{xx} - y_{txx} = b(x-t)h(t), \qquad x \in \mathbb{T}, t > 0, \tag{1.12}$$
$$y(x,0) = y_0(x),\ y_t(x,0) = \xi_0(x), \qquad x \in \mathbb{T} \tag{1.13}$$

where h is the scalar control.

Theorem 1.1. *Let $b \in L^2(\mathbb{T})$ be such that*

$$\beta_k = \int_{\mathbb{T}} b(x)e^{-ikx}dx \neq 0 \quad \text{for } k \neq 0, \quad \beta_0 = \int_{\mathbb{T}} b(x)\,dx = 0.$$

For any time $T > 2\pi$ and any $(y_0, \xi_0) \in L^2(\mathbb{T})^2$ decomposed as in (1.11), if

$$\sum_{k\neq 0} |\beta_k|^{-1}(|k|^6|c_k| + |k|^4|d_k|) < \infty \quad \text{and } c_0 = d_0 = 0, \tag{1.14}$$

then there exists a control $h \in L^2(0,T)$ such that the solution of (1.12)-(1.13) satisfies $y(.,T) = y_t(.,T) = 0$.

By Lemma 2.3 (see below) there exist simple functions b such that $|\beta_k|$ decreases like $1/|k|^3$, so that (1.14) holds for $(y_0, \xi_0) \in H^{s+2}(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 15/2$.

The second problem we consider is

$$y_{tt} - y_{xx} - y_{txx} = b(x-t)h(x,t), \qquad x \in \mathbb{T}, t > 0, \tag{1.15}$$
$$y(x,0) = y_0(x),\ y_t(x,0) = \xi_0(x), \qquad x \in \mathbb{T}, \tag{1.16}$$

where the control function h is here allowed to depend also on x. For that internal controllability problem, the following result will be established.

Theorem 1.2. *Let $b = \mathbf{1}_\omega$ with ω a nonempty open subset of $\mathbb{T}$. Then for any time $T > 2\pi$ and any $(y_0, \xi_0) \in H^{s+2}(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 15/2$ there exists a control $h \in L^2(\mathbb{T} \times (0,T))$ such that the solution of (1.15)-(1.16) satisfies $y(.,T) = y_t(.,T) = 0$.*

We now turn our attention to some internal controls acting on a single moving point. The first problem we consider reads

$$y_{tt} - y_{xx} - y_{txx} = h(t)\delta_t, \qquad x \in \mathbb{T},\ t > 0, \tag{1.17}$$
$$y(x,0) = y_0(x),\ y_t(x,0) = \xi_0(x), \qquad x \in \mathbb{T}, \tag{1.18}$$

where δ_{x_0} represents the Dirac measure at $x = x_0$. We can as well replace δ_t by $\frac{d\delta_t}{dx}$ in (1.17), which yields another control problem:

$$y_{tt} - y_{xx} - y_{txx} = h(t)\frac{d\delta_t}{dx}, \qquad x \in \mathbb{T},\ t > 0, \tag{1.19}$$

$$y(x,0) = y_0(x),\ y_t(x,0) = \xi_0(x), \qquad x \in \mathbb{T}. \tag{1.20}$$

Then we will obtain the following results.

Theorem 1.3. *For any time $T > 2\pi$ and any $(y_0, \xi_0) \in H^{s+2}(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 9/2$, there exists a control $h \in L^2(0,T)$ such that the solution of (1.17)-(1.18) satisfies $y(T,.) - [y(T,.)] = y_t(T,.) = 0$, where $[f] = (2\pi)^{-1}\int_0^{2\pi} f(x)dx$ is the mean value of f.*

Theorem 1.4. *For any time $T > 2\pi$ and any $(y_0, \xi_0) \in H^{s+2}(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 7/2$ and such that $\int_{\mathbb{T}} y_0(x)dx = \int_{\mathbb{T}} \xi_0(x)dx = 0$, there exists a control $h \in L^2(0,T)$ such that the solution of (1.19)-(1.20) satisfies $y(T,.) = y_t(T,.) = 0$.*

The paper is organized as follows. Section 2 is devoted to the proofs of the above theorems: in subsection 2.1 we investigate the wellposedness and the spectrum of (1.9) for $c = -1$; in subsection 2.2 the null controllability of (1.12)-(1.13), (1.17)-(1.18) and (1.19)-(1.20) are formulated as moment problems; Theorem 1.1 is proved in subsection 2.4 thanks to a suitable biorthogonal family which is shown to exist in Proposition 2.2; Theorem 1.2 is deduced from Theorem 1.1 in subsection 2.5; finally, the proofs of Theorems 1.3 and 1.4, that are almost identical to the proof of Theorem 1.1, are sketched in subsection 2.6. The rather long proof of Proposition 2.2 is postponed to Section 3. It combines different results of complex analysis about entire functions of exponential type, sine-type functions, atomization of measures, and Paley-Wiener theorem.

2. Proof of the main results

2.1. Spectral decomposition.

The free evolution equation associated with (1.9) reads

$$v_{tt} - 2v_{xt} - v_{txx} + v_{xxx} = 0. \tag{2.1}$$

Let v be as in (2.1), and let $w = v_t$. Then (2.1) may be written as

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = A\begin{pmatrix} v \\ w \end{pmatrix} := \begin{pmatrix} w \\ 2w_x + w_{xx} - v_{xxx} \end{pmatrix}. \tag{2.2}$$

The eigenvalues of A are obtained by solving the system

$$\begin{cases} w = \lambda v, \\ 2\lambda v_x + \lambda v_{xx} - v_{xxx} = \lambda^2 v. \end{cases} \tag{2.3}$$

Expanding v as a Fourier series $v = \sum_{k\in\mathbb{Z}} v_k e^{ikx}$, we see that (2.3) is satisfied provided that for each $k \in \mathbb{Z}$

$$(\lambda^2 + (k^2 - 2ik)\lambda - ik^3)v_k = 0. \tag{2.4}$$

For $v_k \neq 0$, the only solution of (2.4) reads

$$\lambda = \lambda_k^{\pm} = \frac{-(k^2 - 2ik) \pm \sqrt{k^4 - 4k^2}}{2}. \tag{2.5}$$

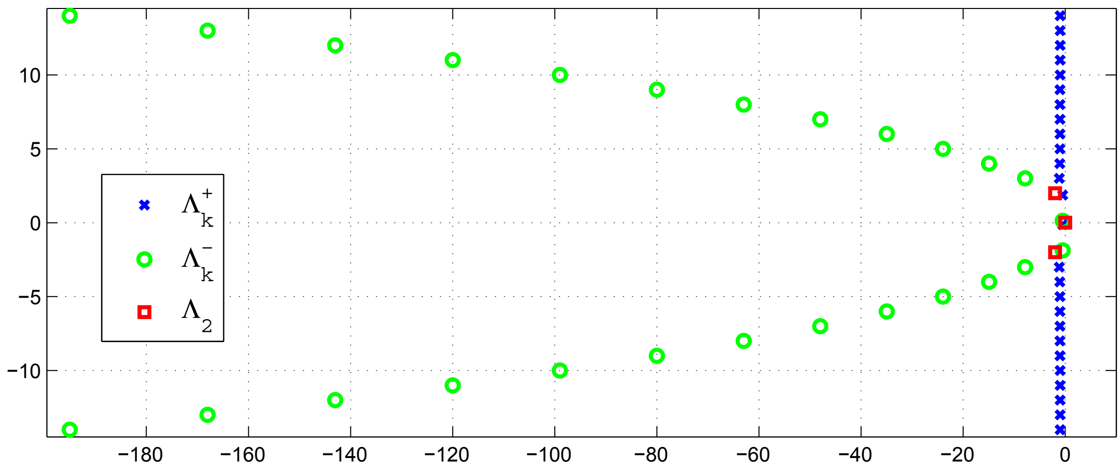

FIGURE 1. Spectrum of (2.1) splits into a hyperbolic part (Λ_k^+ in blue), a parabolic part (Λ_k^- in green) and a finite dimensional part (Λ_2 in red).

Note that

$$\lambda_0^\pm = 0,\ \lambda_2^\pm = -2+2i,\ \lambda_{-2}^\pm = -2-2i$$

while

$$\lambda_k^+ \neq \lambda_l^- \quad \text{for} \quad k,l \in \mathbb{Z}\setminus\{0,\pm 2\} \quad \text{with } k \neq l.$$

For $|k| \geq 3$, $\lambda_k^\pm = \frac{-k^2 \pm k^2(1-2k^{-2}+O(k^{-4}))}{2} + ik$. Hence

$$\lambda_k^+ = -1 + ik + O(k^{-2}) \quad \text{as } |k| \to \infty, \tag{2.6}$$

$$\lambda_k^- = -k^2 + 1 + ik + O(k^{-2}) \quad \text{as } |k| \to \infty. \tag{2.7}$$

The spectrum $\Lambda = \{\lambda_k^\pm;\ k \in \mathbb{Z}\}$ may be split into $\Lambda = \Lambda^+ \cup \Lambda^- \cup \Lambda_2$ where

$$\begin{aligned} \Lambda^+ &= \left\{\lambda_k^+;\ k \in \mathbb{Z}\setminus\{0,\pm 2\}\right\}, \\ \Lambda^- &= \left\{\lambda_k^-;\ k \in \mathbb{Z}\setminus\{0,\pm 2\}\right\}, \\ \Lambda_2 &= \{0, -2 \pm 2i\} \end{aligned}$$

denote the *hyperbolic* part, the *parabolic* part, and the set of double eigenvalues, respectively. It is displayed on Figure 1. (See also [13] for a system whose spectrum may also be decomposed into a hyperbolic part and a parabolic part.)

An eigenvector associated with the eigenvalue $\lambda_k^\pm$, $k \in \mathbb{Z}$, is $\begin{pmatrix} e^{ikx} \\ \lambda_k^\pm e^{ikx} \end{pmatrix}$, and the corresponding exponential solution of (2.1) reads

$$v_k^\pm(x,t) = e^{\lambda_k^\pm t} e^{ikx}.$$

For $k \in \{0, \pm 2\}$, we denote $\lambda_k = \lambda_k^+ = \lambda_k^-$, $v_k(x,t) = e^{\lambda_k t}e^{ikx}$, and introduce

$$\tilde{v}_k(x,t) := te^{\lambda_k t}e^{ikx}.$$

Then we easily check that $\tilde{v}_k$ solves (2.1) and

$$\left(\begin{array}{c}\tilde{v}_k\\ \tilde{v}_{kt}\end{array}\right)(x,0)=\left(\begin{array}{c}0\\ e^{ikx}\end{array}\right).$$

Any solution of (2.1) may be expressed in terms of the $v_k^{\pm}$'s, the v_k's, and the $\tilde{v}_k$'s. Introduce first the Hilbert space $\mathcal{H}=H^1(\mathbb{T})\times L^2(\mathbb{T})$ endowed with the scalar product

$$\langle (v_1,w_1),(v_2,w_2)\rangle_{\mathcal{H}}=\int_{\mathbb{T}}[(v_1\overline{v_2}+v_1'\overline{v_2'})+w_1\overline{w_2}]dx.$$

Pick any

$$\left(\begin{array}{c}v_0\\ w_0\end{array}\right)=\left(\begin{array}{c}\sum_{k\in\mathbb{Z}}c_ke^{ikx}\\ \sum_{k\in\mathbb{Z}}d_ke^{ikx}\end{array}\right)\in\mathcal{H}. \tag{2.8}$$

For $k\in\mathbb{Z}\setminus\{0,\pm 2\}$, we write

$$\left(\begin{array}{c}c_ke^{ikx}\\ d_ke^{ikx}\end{array}\right)=a_k^+\left(\begin{array}{c}e^{ikx}\\ \lambda_k^+e^{ikx}\end{array}\right)+a_k^-\left(\begin{array}{c}e^{ikx}\\ \lambda_k^-e^{ikx}\end{array}\right) \tag{2.9}$$

with

$$a_k^+ = \frac{d_k-\lambda_k^-c_k}{\lambda_k^+-\lambda_k^-}, \tag{2.10}$$

$$a_k^- = \frac{d_k-\lambda_k^+c_k}{\lambda_k^--\lambda_k^+}. \tag{2.11}$$

For $k\in\{0,\pm 2\}$, we write

$$\left(\begin{array}{c}c_ke^{ikx}\\ d_ke^{ikx}\end{array}\right)=a_k\left(\begin{array}{c}e^{ikx}\\ \lambda_ke^{ikx}\end{array}\right)+\tilde{a}_k\left(\begin{array}{c}0\\ e^{ikx}\end{array}\right) \tag{2.12}$$

with

$$a_k=c_k,\quad \tilde{a}_k=d_k-\lambda_kc_k. \tag{2.13}$$

It follows that the solution (v,w) of

$$\left(\begin{array}{c}v\\ w\end{array}\right)_t=A\left(\begin{array}{c}v\\ w\end{array}\right),\quad \left(\begin{array}{c}v\\ w\end{array}\right)(0)=\left(\begin{array}{c}v_0\\ w_0\end{array}\right) \tag{2.14}$$

may be decomposed as

$$\begin{aligned}\left(\begin{array}{c}v(x,t)\\ w(x,t)\end{array}\right)&=\sum_{k\in\mathbb{Z}\setminus\{0,\pm2\}}\{a_k^+e^{\lambda_k^+t}\left(\begin{array}{c}e^{ikx}\\ \lambda_k^+e^{ikx}\end{array}\right)+a_k^-e^{\lambda_k^-t}\left(\begin{array}{c}e^{ikx}\\ \lambda_k^-e^{ikx}\end{array}\right)\}\\ &+\sum_{k\in\{0,\pm2\}}\{a_ke^{\lambda_kt}\left(\begin{array}{c}e^{ikx}\\ \lambda_ke^{ikx}\end{array}\right)+\tilde{a}_ke^{\lambda_kt}\left(\begin{array}{c}te^{ikx}\\ (1+\lambda_kt)e^{ikx}\end{array}\right)\}.\end{aligned} \tag{2.15}$$

Proposition 2.1. *Assume that $(v_0,w_0)\in H^{s+1}(\mathbb{T})\times H^s(\mathbb{T})$ for some $s\geq 0$. Then the solution (v,w) of (2.14) satisfies $(v,w)\in C([0,+\infty);H^{s+1}(\mathbb{T})\times H^s(\mathbb{T}))$.*

Proof. Assume first that $(v_0, w_0) \in C^\infty(\mathbb{T}) \times C^\infty(\mathbb{T})$. Decompose (v_0, w_0) as in (2.8), and let $a_k^\pm$ for $k \in \mathbb{Z} \setminus \{0, \pm 2\}$, and $a_k, \tilde{a}_k$ for $k \in \{0, \pm 2\}$, be as in (2.10)-(2.11) and (2.13), respectively. Then, from the classical Fourier definition of Sobolev spaces, we have that

$$\begin{aligned} ||(v_0, w_0)||_{H^{s+1}(\mathbb{T})\times H^s(\mathbb{T})} &\sim \left(\sum_{k\in\mathbb{Z}} (|k|^2+1)^s \big((|k|^2+1)|c_k|^2 + |d_k|^2 \big) \right)^{\frac{1}{2}} \\ &\sim \left(\sum_{k\in\mathbb{Z}\setminus\{0,\pm2\}} |k|^{2s} (k^2 |a_k^+|^2 + k^4 |a_k^-|^2) + \sum_{k\in\{0,\pm2\}} (|a_k|^2 + |\tilde{a}_k|^2) \right)^{\frac{1}{2}}. \end{aligned}$$

For the last equivalence of norms, we used (2.6)-(2.7) and (2.9)-(2.11). Since

$$|e^{\lambda_k^+ t}| + |e^{\lambda_k^- t}| \le C \qquad \text{for } |k| > 2,\ t \ge 0,$$

we infer that

$$\sum_{k\in\mathbb{Z}\setminus\{0,\pm2\}} |k|^{2s} \left(k^2 |a_k^+ e^{\lambda_k^+ t}|^2 + k^4 |a_k^- e^{\lambda_k^- t}|^2 \right) \le C \sum_{k\in\mathbb{Z}\setminus\{0,\pm2\}} |k|^{2s} \left(k^2 |a_k^+|^2 + k^4 |a_k^-|^2 \right) < \infty,$$

hence

$$||(v, w)||_{L^\infty(\mathbb{R}^+,\ H^{s+1}(\mathbb{T})\times H^s(\mathbb{T}))} \le C ||(v_0, w_0)||_{H^{s+1}(\mathbb{T})\times H^s(\mathbb{T})}. \tag{2.16}$$

The result follows from (2.15) and (2.16) by a density argument. □

2.2. **Reduction to moment problems.**

2.2.1. *Internal control.* We investigate the following control problem

$$v_{tt} - 2v_{xt} - v_{txx} + v_{xxx} = b(x)h(t), \tag{2.17}$$

where $b \in L^2(\mathbb{T})$, supp $b \subset \omega \subset \mathbb{T}$ and $h \in L^2(0, T)$. The adjoint equation to (2.17) reads

$$\varphi_{tt} - 2\varphi_{xt} + \varphi_{txx} - \varphi_{xxx} = 0. \tag{2.18}$$

Note that $\varphi(x, t) = v(2\pi - x, T - t)$ is a solution of (2.18) if v is a solution of (2.17) for $h \equiv 0$. Pick any (smooth enough) solutions v of (2.17) and φ of (2.18), respectively. Multiplying each term in (2.17) by $\overline{\varphi}$ and integrating by parts, we obtain

$$\int_{\mathbb{T}} [v_t \overline{\varphi} + v(-\overline{\varphi}_t + 2\overline{\varphi}_x - \overline{\varphi}_{xx})] \Big|_0^T dx = \int_0^T \int_{\mathbb{T}} hb\overline{\varphi}\, dx dt. \tag{2.19}$$

Pick first $\varphi(x, t) = e^{\lambda_{-k}^\pm (T-t)} e^{ikx} = e^{\overline{\lambda_k^\pm} (T-t)} e^{ikx}$ for $k \in \mathbb{Z}$. Then (2.19) may be written

$$\begin{aligned} \langle v_t(T), e^{ikx}\rangle + (\lambda_k^\pm - 2ik + k^2)\langle v(T), e^{ikx}\rangle - e^{\lambda_k^\pm T}\gamma_k^\pm \\ = \int_0^T h(t) e^{\lambda_k^\pm (T-t)} dt \int_{\mathbb{T}} b(x) e^{-ikx} dx, \end{aligned} \tag{2.20}$$

where $\langle ., . \rangle$ stands for the duality pairing $\langle ., . \rangle_{\mathcal{D}'(\mathbb{T}), \mathcal{D}(\mathbb{T})}$, and

$$\gamma_k^\pm = \langle v_t(0), e^{ikx}\rangle + (\lambda_k^\pm - 2ik + k^2)\langle v(0), e^{ikx}\rangle.$$

If we now pick $\varphi(x,t) = (T-t)e^{\overline{\lambda_k}(T-t)}e^{ikx}$ for $k \in \{0, \pm 2\}$, then (2.19) yields

$$\langle v(T), e^{ikx}\rangle - \Big\{Te^{\lambda_k T}\langle v_t(0), e^{ikx}\rangle + [1 + T(\lambda_k - 2ik + k^2)]e^{\lambda_k T}\langle v(0), e^{ikx}\rangle\Big\}$$
$$= \int_0^T (T-t)h(t)e^{\lambda_k(T-t)}dt \int_{\mathbb{T}} b(x)e^{-ikx}dx \quad k \in \{0, \pm 2\}. \tag{2.21}$$

Set $\beta_k = \int_{\mathbb{T}} b(x)e^{-ikx}dx$ for $k \in \mathbb{Z}$. The control problem can be reduced to a moment problem. Assume that there exists some function $h \in L^2(0,T)$ such that

$$\beta_k \int_0^T e^{\lambda_k^\pm (T-t)}h(t)dt = -e^{\lambda_k^\pm T}\gamma_k^\pm \quad \forall k \in \mathbb{Z}, \tag{2.22}$$
$$\beta_k \int_0^T (T-t)e^{\lambda_k(T-t)}h(t)dt$$
$$= -Te^{\lambda_k T}\langle v_t(0), e^{ikx}\rangle - [1 + T(\lambda_k - 2ik + k^2)]e^{\lambda_k T}\langle v(0), e^{ikx}\rangle \quad \forall k \in \{0, \pm 2\}. \tag{2.23}$$

Then it follows from (2.20)-(2.23) that

$$\langle v_t(T), e^{ikx}\rangle + (\lambda_k^\pm - 2ik + k^2)\langle v(T), e^{ikx}\rangle = 0 \quad \forall k \in \mathbb{Z}, \tag{2.24}$$
$$\langle v(T), e^{ikx}\rangle = 0 \quad \forall k \in \{0, \pm 2\}. \tag{2.25}$$

Since $\lambda_k^+ \neq \lambda_k^-$ for $k \in \mathbb{Z} \setminus \{0, \pm 2\}$, this yields

$$v(T) = v_t(T) = 0. \tag{2.26}$$

2.2.2. *Point control.* Let us consider first the control problem

$$v_{tt} - 2v_{xt} - v_{txx} + v_{xxx} = h(t)\frac{d\delta_0}{dx}. \tag{2.27}$$

Then the right hand side of (2.19) is changed into $\int_0^T h(t)\langle \frac{d\delta_0}{dx}, \varphi\rangle dt$. For $\varphi(x,t) = e^{\overline{\lambda_k^\pm}(T-t)}e^{ikx}$, we have

$$\langle \frac{d\delta_0}{dx}, \varphi\rangle = -\langle \delta_0, \frac{\partial \varphi}{\partial x}\rangle = ik\, e^{\lambda_k^\pm (T-t)}$$

hence the right hand sides of (2.20) and (2.21) are changed into $\int_0^T (ik)e^{\lambda_k^\pm (T-t)}h(t)dt$ and $\int_0^T (ik)(T-t)e^{\lambda_k(T-t)}h(t)dt$, respectively. Let $\beta_k = ik$ for $k \in \mathbb{Z}$. Note that $\beta_0 = 0$ and that (2.20)-(2.21) for $k = 0$ read

$$\langle v_t(T), 1\rangle - \langle v_t(0), 1\rangle = 0, \tag{2.28}$$
$$\langle v(T), 1\rangle - T\langle v_t(0), 1\rangle - \langle v(0), 1\rangle = 0. \tag{2.29}$$

Thus, the mean values of v and v_t cannot be controlled. Let us formulate the moment problem to be solved. Assume that

$$\langle v(0), 1\rangle = \langle v_t(0), 1\rangle = 0, \tag{2.30}$$

and that there exists some $h \in L^2(0,T)$ such that

$$ik\int_0^T e^{\lambda_k^\pm (T-t)} h(t)dt = -e^{\lambda_k^\pm T}\gamma_k^\pm \quad \forall k \in \mathbb{Z}\setminus\{0\}, \tag{2.31}$$

$$\begin{aligned} ik\int_0^T (T-t)e^{\lambda_k(T-t)}h(t)dt \\ = -Te^{\lambda_k T}\langle v_t(0), e^{ikx}\rangle - [1+T(\lambda_k - 2ik + k^2)]e^{\lambda_k T}\langle v(0), e^{ikx}\rangle \quad \forall k \in \{\pm 2\}. \end{aligned} \tag{2.32}$$

Then we infer from (2.20)-(2.21) (with the new r.h.s.) and (2.28)-(2.32) that

$$v(T) = v_t(T) = 0.$$

Finally, let us consider the control problem

$$v_{tt} - 2v_{xt} - v_{txx} + v_{xxx} = h(t)\delta_0. \tag{2.33}$$

Then the computations above are valid with the new values of β_k given by

$$\beta_k = \langle \delta_0, e^{ikx}\rangle = 1, \qquad k \in \mathbb{Z}.$$

It will be clear from the proof of Theorem 1.1 that $\langle v_t(T), 1\rangle$ can be controlled, while $\langle v(T), 1\rangle$ cannot. To establish Theorem 1.3, we shall have to find a control function $h \in L^2(0,T)$ such that

$$\int_0^T e^{\lambda_k^\pm (T-t)} h(t)dt = -e^{\lambda_k^\pm T}\gamma_k^\pm \quad \forall k \in \mathbb{Z}, \tag{2.34}$$

$$\begin{aligned} \int_0^T (T-t)e^{\lambda_k(T-t)}h(t)dt \\ = -Te^{\lambda_k T}\langle v_t(0), e^{ikx}\rangle - [1+T(\lambda_k - 2ik + k^2)]e^{\lambda_k T}\langle v(0), e^{ikx}\rangle \quad \forall k \in \{\pm 2\}. \end{aligned} \tag{2.35}$$

2.3. **A Biorthogonal family.** To solve the moments problems in the previous section, we need to construct a biorthogonal family to the functions $e^{\lambda_k^\pm t}$, $k \in \mathbb{Z}$, and $t\,e^{\lambda_k t}$, $k \in \{\pm 2\}$. More precisely, we shall prove the following

Proposition 2.2. *There exists a family* $\{\psi_k^\pm\}_{k\in\mathbb{Z}\setminus\{0,\pm 2\}}\cup\{\psi_k\}_{k\in\{0,\pm 2\}}\cup\{\tilde\psi_k\}_{k\in\{\pm 2\}}$ *of functions in* $L^2(-T/2,T/2)$ *such that*

$$\int_{-T/2}^{T/2}\psi_k^\pm(t)e^{\lambda_l^\pm t}dt=\delta_k^l\delta_+^- \qquad k,l\in\mathbb{Z}\setminus\{0,\pm 2\}, \tag{2.36}$$

$$\int_{-T/2}^{T/2}\psi_k^\pm(t)e^{\lambda_l t}dt=\int_{-T/2}^{T/2}\psi_k^\pm(t)te^{\lambda_p t}dt=0 \qquad k\in\mathbb{Z}\setminus\{0,\pm 2\},\ l\in\{0,\pm 2\},\ p\in\{\pm 2\}, \tag{2.37}$$

$$\int_{-T/2}^{T/2}\psi_l(t)e^{\lambda_k^\pm t}dt=\int_{-T/2}^{T/2}\tilde\psi_p(t)e^{\lambda_k^\pm t}dt=0 \qquad l\in\{0,\pm 2\},\ k\in\mathbb{Z}\setminus\{0,\pm 2\},\ p\in\{\pm 2\}, \tag{2.38}$$

$$\int_{-T/2}^{T/2}\psi_l(t)e^{\lambda_k t}dt=\delta_l^k,\quad \int_{-T/2}^{T/2}\psi_l(t)te^{\lambda_p t}dt=0 \qquad l,k\in\{0,\pm 2\},\ p\in\{\pm 2\}, \tag{2.39}$$

$$\int_{-T/2}^{T/2}\tilde\psi_p(t)e^{\lambda_k t}dt=0,\quad \int_{-T/2}^{T/2}\tilde\psi_p(t)te^{\lambda_q t}dt=\delta_p^q \qquad p,q\in\{\pm 2\},\ k\in\{0,\pm 2\}, \tag{2.40}$$

$$||\psi_k^+||_{L^2(-T/2,T/2)}\le C|k|^4 \qquad k\in\mathbb{Z}\setminus\{0,\pm 2\}, \tag{2.41}$$

$$||\psi_k^-||_{L^2(-T/2,T/2)}\le C|k|^2e^{-\frac{T}{2}k^2+2\sqrt{2}\pi|k|} \qquad k\in\mathbb{Z}\setminus\{0,\pm 2\}, \tag{2.42}$$

where C *denotes some positive constant.*

In Proposition 2.2, δ_k^l and δ_+^- denote Kronecker symbols ($\delta_k^l=1$ if $k=l$, 0 otherwise, while $\delta_+^-=1$ if we have the same signs in the l.h.s of (2.36), 0 otherwise). The proof of Proposition 2.2 is postponed to Section 3. We assume Proposition 2.2 true for the time being and proceed to the proofs of the main results of the paper.

2.4. **Proof of Theorem 1.1.** Pick any pair $(y_0,\xi_0)\in L^2(\mathbb{T})^2$ fulfilling (1.14). From (1.10) with $c=-1$, we have that $v(0)=y_0$, $v_t(0)=\frac{dy_0}{dx}+\xi_0$, so that

$$\begin{aligned}\gamma_k^\pm &= \langle\frac{dy_0}{dx}+\xi_0,e^{ikx}\rangle+(\lambda_k^\pm-2ik+k^2)\langle y_0,e^{ikx}\rangle,\\ &= \langle\xi_0,e^{ikx}\rangle+(\lambda_k^\pm-ik+k^2)\langle y_0,e^{ikx}\rangle, \qquad k\in\mathbb{Z}.\end{aligned}$$

Let

$$\gamma_k=\gamma_k^\pm \quad \text{for } k\in\{0,\pm 2\}.$$

The result will be proved if we can construct a control function $h\in L^2(0,T)$ fulfilling (2.22)-(2.23). Let us introduce the numbers

$$\begin{aligned}\alpha_k^\pm &= -\beta_k^{-1}e^{\lambda_k^\pm\frac{T}{2}}\gamma_k^\pm, \qquad k\in\mathbb{Z}\setminus\{0,\pm 2\},\\ \alpha_k &= -\beta_k^{-1}e^{\lambda_k\frac{T}{2}}\gamma_k, \qquad k\in\{\pm 2\},\\ \tilde\alpha_k &= -\beta_k^{-1}\left(\frac{T}{2}e^{\lambda_k\frac{T}{2}}\gamma_k+e^{\lambda_k\frac{T}{2}}\langle y_0,e^{ikx}\rangle\right), \qquad k\in\{\pm 2\},\end{aligned}$$

and

$$\psi(t)=\sum_{k\in\mathbb{Z}\setminus\{0,\pm 2\}}\alpha_k^+\psi_k^+(t)+\sum_{k\in\mathbb{Z}\setminus\{0,\pm 2\}}\alpha_k^-\psi_k^-(t)+\sum_{k\in\{\pm 2\}}[\alpha_k\psi_k(t)+\tilde\alpha_k\tilde\psi_k(t)].$$

Finally let $h(t) = \psi(\frac{T}{2} - t)$. Note that $h \in L^2(0,T)$ with

$$\begin{aligned} ||h||_{L^2(0,T)} &= ||\psi||_{L^2(-\frac{T}{2},\frac{T}{2})} \\ &\le C \Bigg(\sum_{k\in\{\pm 2\}} (|d_k| + |c_k|) + \sum_{k\in\mathbb{Z}\setminus\{0,\pm 2\}} |\beta_k|^{-1}(|d_k| + |k|^2|c_k|)|k|^4 \\ &\qquad + \sum_{k\in\mathbb{Z}\setminus\{0,\pm 2\}} |\beta_k|^{-1}(|d_k| + |c_k|)|k|^2 e^{-T|k|^2+2\sqrt{2}\pi|k|} \Bigg) \\ &< \infty, \end{aligned}$$

by (1.14). Then it follows from (1.14) and (2.36)-(2.40) that for $k \in \mathbb{Z} \setminus \{0, \pm 2\}$

$$\beta_k \int_0^T e^{\lambda_k^\pm (T-t)} h(t)dt = \beta_k e^{\lambda_k^\pm \frac{T}{2}} \int_{-T/2}^{T/2} e^{\lambda_k^\pm \tau} \psi(\tau)\, d\tau = \beta_k e^{\lambda_k^\pm \frac{T}{2}} \alpha_k^\pm = -e^{\lambda_k^\pm T} \gamma_k^\pm .$$

and also that

$$\begin{aligned} \beta_k \int_0^T e^{\lambda_k (T-t)} h(t)dt &= -e^{\lambda_k T}\gamma_k \qquad \text{for } k \in \{0,\pm 2\}, \\ \beta_k \int_0^T (T-t) e^{\lambda_k (T-t)} h(t)dt &= -Te^{\lambda_k T}\gamma_k - e^{\lambda_k T}\langle y_0, e^{ikx}\rangle \qquad \text{for } k \in \{0,\pm 2\}, \end{aligned}$$

as desired. □

2.5. **Proof of Theorem 1.2.** Set $\epsilon = (T - 2\pi)/2$, $v(x,t) = y(x+t,t)$ and $\xi(x,t) = y_t(x,t)$. We first steer to 0 the components of v and v_t along the mode associated to the double eigenvalue $\lambda_0 = 0$. Denote $\gamma(t) = \int_{\mathbb{T}} v(x,t)\, dx$ and $\eta(t) = \int_{\mathbb{T}} v_t(x,t)\, dx$. According to (1.11), $\gamma(0) = 2\pi c_0$, $\eta(0) = 2\pi d_0$ and

$$\frac{d\gamma}{dt} = \eta, \quad \frac{d\eta}{dt} = \int_\omega \tilde{h}(x,t)dx.$$

Take a C^∞ scalar function $\varpi(t)$ on $[0,\epsilon]$ with $\varpi(0) = 1$ and $\varpi(\epsilon) = 0$ and such that the support of $d\varpi/dt$ lies inside $[0,\epsilon]$. Consider another C^∞ function of x, $\bar{b}(x)$ with support inside ω and such that $\int_\omega \bar{b}(x)\, dx = 1$. Then the C^∞ control

$$\tilde{h}(x,t) = \bar{b}(x)\bar{h}(t) \quad \text{with} \quad \bar{h}(t) = \frac{d^2}{dt^2}\left((c_0 + d_0 t)\varpi(t)\right)$$

steers (γ,η) from (c_0,d_0) at time $t = 0$ to $(0,0)$ at time $t = \epsilon$. Its support lies inside $[0,\epsilon]$. Since $\gamma(\epsilon) = \int_{\mathbb{T}} y(x,\epsilon)\, dx$ and $\eta(\epsilon) = \int_{\mathbb{T}} \xi(x,\epsilon)\, dx$, we can assume that $c_0 = d_0 = 0$ up to a time shift of ϵ.

Since ω is open and nonempty, it contains a small interval $[a, a + 2\sigma\pi]$ where $\sigma > 0$ is a *quadratic irrational*; i.e., an irrational number which is a root of a quadratic equation with integral coefficients. Set for $t \in [\epsilon, T]$

$$h(x,t) = \left(\mathbf{1}_{[a,a+\sigma\pi]}(x-t) - \mathbf{1}_{[a+\sigma\pi,a+2\sigma\pi]}(x-t)\right)\tilde{h}(t)$$

where $\widetilde{h}$ denotes a control input independent of x. Then $b(x-t)h(x,t) = \widetilde{b}(x-t)\widetilde{h}(t)$ where

$$\widetilde{b}(x) = \mathbf{1}_{[a,a+\sigma\pi]}(x) - \mathbf{1}_{[a+\sigma\pi,a+2\sigma\pi]}(x)$$

satisfies $\int_{\mathbb{T}} \widetilde{b}(x)\, dx = 0$. Moreover there exists by Lemma 2.3 (see below) a number $C > 0$ such that for all $k \in \mathbb{Z}^*$

$$\widetilde{\beta}_k = \int_{\mathbb{T}} \widetilde{b}(x) e^{-ikx} dx \geq \frac{C}{|k|^3}.$$

According to Theorem 1.1 we can find $\widetilde{h} \in L^2(\epsilon, T)$ steering $y(.,\epsilon)$ and $\xi(.,\epsilon)$ to $y(.,T) = \xi(.,T) = 0$ as soon as

$$\sum_{k\neq 0} \frac{k^6|\widetilde{c}_k| + k^4|\widetilde{d}_k|}{|\widetilde{\beta}_k|} < \infty,$$

with

$$y(x,\epsilon) = \sum_{k\in\mathbb{Z}} \widetilde{c}_k e^{ikx}, \quad \xi(x,\epsilon) = \sum_{k\in\mathbb{Z}} \widetilde{d}_k e^{ikx}.$$

Let W denote the space of the couples $(\hat{y}, \hat{\xi}) \in L^2(\mathbb{T})^2$ such that $||(\hat{y},\hat{\xi})||_W := |\hat{c}_0| + |\hat{d}_0| + \sum_{k\neq 0}(|k|^9|\hat{c}_k| + |k|^7|\hat{d}_k|) < \infty$, where $\hat{y}(x) = \sum_{k\in\mathbb{Z}} \hat{c}_k e^{ikx}$ and $\hat{\xi}(x) = \sum_{k\in\mathbb{Z}} \hat{d}_k e^{ikx}$. Clearly, W endowed with the norm $||\cdot||_W$, is a Banach space. Standard estimations based on the spectral decomposition used to prove Proposition 2.1 show that if the initial value (y_0, ξ_0) lies in W, then the solution of (1.12)-(1.13) (with $h \equiv 0$) remains in W. Therefore, since $\sum_{k\neq 0}(|k|^9|c_k| + |k|^7|d_k|) < \infty$ and since the control is C^∞ with respect to $x \in \mathbb{T}$ and $t \in [0,\epsilon]$, we also have $\sum_{k\neq 0} |k|^9(|\widetilde{c}_k| + |k|^7|\widetilde{d}_k|) < \infty$ (see e.g. [5, 20]). Since $(y_0, \xi_0) \in H^{s+2}(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 15/2$, we have by Cauchy-Schwarz inequality for $\varsigma = 2s - 15 > 0$ that

$$\sum_{k\neq 0}(|k|^9|c_k| + |k|^7|d_k|) \leq 2(\sum_{k\neq 0}|k|^{-1-\varsigma})^{\frac{1}{2}}(\sum_{k\neq 0}|k|^{19+\varsigma}|c_k|^2 + |k|^{15+\varsigma}|d_k|^2)^{\frac{1}{2}} < \infty. \quad \square$$

Lemma 2.3. *Let $\sigma \in (0,1)$ be a quadratic irrational, and let $\tilde{b}$, $\tilde{\beta}_k$ be defined as above. Then $\tilde{\beta}_0 = 0$ and there exists $C > 0$ such that for all $k \in \mathbb{Z}^*$, $|\tilde{\beta}_k| \geq \frac{C}{|k|^3}$.*

Proof. Being a quadratic irrational, σ is approximable by rational numbers to order 2 and to no higher order [8, Theorem 188]); i.e., there exists $C_0 > 0$ such that for any integers p and q, $q \neq 0$, $\left|\sigma - \frac{p}{q}\right| \geq \frac{C_0}{q^2}$. On the other hand, $|\tilde{\beta}_k| = \frac{4}{|k|}\sin^2(\frac{\pi}{2}k\sigma)$ for $k \neq 0$. Pick any $k \neq 0$, take $p \in \mathbb{Z}$ such that $0 \leq \frac{\pi}{2}k\sigma - p\pi < \pi$ and use the elementary inequality $\sin^2\theta \geq \frac{4\theta^2}{\pi^2}$ valid for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then two cases occur.

(i) If $0 \leq \frac{\pi}{2}k\sigma - p\pi \leq \frac{\pi}{2}$, then

$$\sin^2(\tfrac{\pi}{2}k\sigma) = \sin^2(\tfrac{\pi}{2}k\sigma - p\pi) \geq \tfrac{4}{\pi^2}(\tfrac{\pi}{2}k\sigma - p\pi)^2 = k^2\left(\sigma - \tfrac{2p}{k}\right)^2 \geq \tfrac{C_0^2}{k^2};$$

(ii) If $-\frac{\pi}{2} \leq \frac{\pi}{2}k\sigma - (p+1)\pi \leq 0$, then

$$\sin^2(\tfrac{\pi}{2}k\sigma - (p+1)\pi) \geq \tfrac{4}{\pi^2}(\tfrac{\pi}{2}k\sigma - (p+1)\pi)^2 = k^2\left(\sigma - \tfrac{2(p+1)}{k}\right)^2 \geq \tfrac{C_0^2}{k^2}.$$

The lemma follows with $C = 4C_0^2$. $\square$

2.6. **Proofs of Theorem 1.3 and Theorem 1.4.** The proofs are the same as for Theorem 1.1, with the obvious estimate

$$\sum_{|k|>2} |k|^p(|k|^2|c_k| + |d_k|) \le C_\varepsilon \left(\sum_{|k|>2} \{|k|^{2p+5+\varepsilon}|c_k|^2 + |k|^{2p+1+\varepsilon}|d_k|^2 \right)^{\frac{1}{2}}$$

for $p \in \{3, 4\}$, $\varepsilon > 0$. □

3. Proof of Proposition 2.2

This section is devoted to the proof of Proposition 2.2. The method of proof is inspired from the one in [6, 7, 18]. We first introduce an entire function vanishing precisely at the $i\lambda_k^\pm$'s, namely the canonical product

$$P(z) = z(1 - \frac{z}{i\lambda_2})(1 - \frac{z}{i\lambda_{-2}}) \prod_{k\in\mathbb{Z}\setminus\{0,\pm2\}} (1 - \frac{z}{i\lambda_k^+}) \prod_{k\in\mathbb{Z}\setminus\{0,\pm2\}} (1 - \frac{z}{i\lambda_k^-}) \cdot \tag{3.1}$$

Next, following [1, 7], we construct a multiplier m which is an entire function that does not vanish at the $\lambda_k^\pm$'s, such that $P(z)m(z)$ is bounded for z real while $P(z)m(z)$ has (at most) a polynomial growth in z as $|z| \to \infty$ on each line Im $z = const$. Next, for $k \in \mathbb{Z} \setminus \{0, \pm 2\}$ we construct a function $I_k^\pm$ from $P(z)$ and $m(z)$ and we define $\psi_k^\pm$ as the inverse Fourier transform of $I_k^\pm$. The other ψ_k's are constructed in a quite similar way. The fact that $\psi_k^\pm$ is compactly supported in time is a consequence of Paley-Wiener theorem.

3.1. **Functions of type sine.** To estimate carefully $P(z)$, we use the theory of functions of type sine (see e.g. [16, pp. 163–168] and [26, pp. 171–179]).

Definition 3.1. *An entire function $f(z)$ of exponential type π is said to be of* type sine *if*

(i) *The zeros μ_k of $f(z)$ are separated; i.e., there exists $\eta > 0$ such that* $|\mu_k - \mu_l| \ge \eta \qquad k \ne l;$
(ii) *There exist positive constants A, B and H such that*

$$Ae^{\pi|y|} \le |f(x + iy)| \le Be^{\pi|y|} \quad \forall x \in \mathbb{R},\ \forall y \in \mathbb{R} \quad \text{with } |y| \ge H. \tag{3.2}$$

Some of the most important properties of an entire function of type sine are gathered in the following

Proposition 3.2. *(see* [16, Remark and Lemma 2 p. 164], [26, Lemma 2 p. 172]*) Let $f(z)$ be an entire function of type sine, and let $\{\mu_k\}_{k\in J}$ be the sequence of its zeros, where $J \subset \mathbb{Z}$. Then*

(1) *For any $\varepsilon > 0$, there exist some constants $C_\varepsilon, C'_\varepsilon > 0$ such that*

$$C_\varepsilon e^{\pi|Im\ z|} \le |f(z)| \le C'_\varepsilon e^{\pi|Im\ z|} \qquad \text{if } dist\{z, \{\mu_k\}\} > \varepsilon.$$

(2) *There exist some constants C_1, C_2 such that*

$$0 < C_1 < |f'(\mu_k)| < C_2 < \infty \quad \forall k \in J.$$

Finally, we shall need the following result.

Theorem 3.3. *(see [16, Corollary p. 168 and Theorem 2 p. 157] Let $\mu_k = k + d_k$ for $k \in \mathbb{Z}$, with $\mu_0 = 0$, $\mu_k \neq 0$ for $k \neq 0$, and $(d_k)_{k\in\mathbb{Z}}$ bounded, and let*

$$f(z) = z \prod_{k\in\mathbb{Z}\setminus\{0\}} (1 - \frac{z}{\mu_k}) = \lim_{K\to\infty} z \prod_{k\in\{-K,\dots,K\}\setminus\{0\}} (1 - \frac{z}{\mu_k}).$$

Then f is a function of type sine if, and only if, the following three properties are satisfied:

1. $\inf_{k\neq l} |\mu_k - \mu_l| > 0$;
2. *There exists some constant $M > 0$ such that*
$$\left|\sum_{k\in\mathbb{Z}} (d_{k+\tau} - d_k)\frac{k}{k^2+1}\right| \leq M \quad \forall \tau \in \mathbb{Z};$$
3. $\limsup_{y\to+\infty} \dfrac{\log|f(iy)|}{y} = \pi, \quad \limsup_{y\to-\infty} \dfrac{\log|f(-iy)|}{|y|} = \pi.$

Corollary 3.4. *Assume that $\mu_k = k + d_k$, where $d_0 = 0$ and $d_k = d + O(k^{-1})$ as $|k| \to \infty$ for some constant $d \in \mathbb{C}$, and that $\mu_k \neq \mu_l$ for $k \neq l$. Then $f(z) = z\prod_{k\in\mathbb{Z}\setminus\{0\}}(1 - \frac{z}{\mu_k})$ is an entire function of type sine.*

Proof. We check that the conditions (1), (2) and (3) in Theorem 3.3 are fulfilled.
(1) From $\mu_k - \mu_l = k - l + O(k^{-1}, l^{-1})$ and the fact that $\mu_k - \mu_l \neq 0$ for $k \neq l$, we infer that (1) holds.
(2) Let us write $d_k = d + e_k$ with $e_k = O(k^{-1})$. Then for all $\tau \in \mathbb{Z}$

$$\left(\sum_{k\in\mathbb{Z}} (d_{k+\tau} - d_k)^2\right)^{\frac{1}{2}} \leq 2\left(\sum_{k\in\mathbb{Z}} |e_k|^2\right)^{\frac{1}{2}} < \infty.$$

Therefore, for any $\tau \in \mathbb{Z}$, by Cauchy-Schwarz inequality

$$\begin{aligned}\left|\sum_{k\in\mathbb{Z}} (d_{k+\tau} - d_k)\frac{k}{k^2+1}\right| &\leq \left(\sum_{k\in\mathbb{Z}} (d_{k+\tau} - d_k)^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} (\frac{k}{k^2+1})^2\right)^{\frac{1}{2}} \\ &\leq 2\left(\sum_{k\in\mathbb{Z}} |e_k|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} (\frac{k}{k^2+1})^2\right)^{\frac{1}{2}} =: M < \infty.\end{aligned}$$

(3) We first notice that

$$f(z) = z\prod_{k=1}^{\infty} (1 - \frac{z}{\mu_k})(1 - \frac{z}{\mu_{-k}}).$$

Let $z = iy$, with $y \in \mathbb{R}$. Then

$$(1 - \frac{z}{\mu_k})(1 - \frac{z}{\mu_{-k}}) = 1 - \frac{y^2 + i(\mu_k + \mu_{-k})y}{\mu_k\mu_{-k}}$$

with

$$\mu_k\mu_{-k} = -k^2 + O(k), \quad \mu_k + \mu_{-k} = 2d + O(k^{-1}).$$

It follows that for any given $\varepsilon \in (0,1)$, there exist $k_0 \in \mathbb{N}^*$ and some numbers $C_1, C_2 > 0$ such that

$$1 + \frac{(1-\varepsilon)y^2 - C_1|y|}{|\mu_k \mu_{-k}|} \le \left|(1-\frac{z}{\mu_k})(1-\frac{z}{\mu_{-k}})\right| \le 1 + \frac{y^2 + C_2|y|}{|\mu_k\mu_{-k}|} \tag{3.3}$$

for $y \in \mathbb{R}$ and $k \ge k_0 > 0$. Let

$$n(r) := \#\{k \in \mathbb{N}^*;\ |\mu_k\mu_{-k}| \le r\}.$$

Since $|\mu_k\mu_{-k}| \sim k^2$ as $k \to \infty$ and $\mu_k \ne 0$ for $k \ne 0$, we obtain that

$$\sqrt{r} - C_3 \le n(r) \le \sqrt{r} + C_3 \qquad \text{for } r > 0, \tag{3.4}$$

$$n(r) = 0 \qquad \text{for } 0 < r < r_0, \tag{3.5}$$

for some constants $C_3 > 0$, $r_0 > 0$. It follows that

$$\begin{aligned}
\limsup_{|y|\to\infty} \frac{\log|f(iy)|}{|y|} &\le \limsup_{|y|\to+\infty} |y|^{-1} \sum_{k=1}^{\infty} \log\left|(1-\frac{iy}{\mu_k})(1-\frac{iy}{\mu_{-k}})\right| \\
&\le \limsup_{|y|\to\infty} |y|^{-1} \sum_{k=1}^{\infty} \log(1 + \frac{y^2 + C_2|y|}{|\mu_k\mu_{-k}|}),
\end{aligned}$$

where we used the fact that

$$\lim_{|y|\to\infty} |y|^{-1} \log\left|1 - \frac{iy}{\mu_{\pm k}}\right| = 0 \qquad \text{for } 1 \le k \le k_0.$$

On the other hand, setting $\rho = y^2 + C_2|y| \ge 0$, we have that

$$\begin{aligned}
\sum_{k=1}^{\infty} \log(1 + \frac{\rho}{|\mu_k\mu_{-k}|}) &= \int_0^\infty \log(1+\frac{\rho}{t})\, dn(t) \\
&= \rho \int_0^\infty \frac{n(t)}{t(t+\rho)}\, dt \\
&= \int_0^\infty \frac{n(\rho s)}{s(s+1)}\, ds \\
&\le \sqrt{\rho} \int_0^\infty \frac{ds}{\sqrt{s}(s+1)} + C_3 \int_{r_0/\rho}^\infty \frac{ds}{s(s+1)} \\
&\le |y|\sqrt{1 + C_2|y|^{-1}}\pi + C_3 \log\left(1 + r_0^{-1}(y^2 + C_2|y|)\right).
\end{aligned}$$

Thus

$$\limsup_{|y|\to\infty} \frac{\log|f(iy)|}{|y|} \le \pi.$$

Using again (3.3), we obtain by the same computations that

$$\limsup_{y\to+\infty} \frac{\log|f(iy)|}{y} \ge \pi, \quad \text{and} \quad \limsup_{y\to-\infty} \frac{\log|f(iy)|}{|y|} \ge \pi.$$

The proof of (3) is completed. □

In what follows, $\arg z$ denotes the principal argument of any complex number $z \in \mathbb{C}\setminus\mathbb{R}^-$; i.e., $\arg z \in (-\pi,\pi)$, and

$$\log z = \log|z| + i\arg z, \quad \sqrt{z} = \sqrt{|z|}\, e^{i\frac{\arg z}{2}}.$$

We introduce, for $k \in \mathbb{Z}\setminus\{0\}$,

$$\mu_k = \mathrm{sgn}(k)\sqrt{-\lambda_k^-} = k\sqrt{\frac{1+\sqrt{1-4k^{-2}}}{2} - ik^{-1}} =: k + d_k,\ k\in\mathbb{Z}$$

with

$$d_k = -\frac{i}{2} + O(k^{-1}).$$

and $\mu_0 = 0$. Let

$$P_1(z) = z\prod_{k\in\mathbb{Z}\setminus\{0\}}\left(1+\frac{z}{i\lambda_k^+}\right), \tag{3.6}$$

$$P_2(z) = z\prod_{k\in\mathbb{Z}\setminus\{0\}}\left(1+\frac{z}{i\lambda_k^-}\right), \tag{3.7}$$

$$P_3(z) = z^2\prod_{k\in\mathbb{Z}\setminus\{0\}}\left(1+\frac{z^2}{\lambda_k^-}\right), \tag{3.8}$$

and

$$P_4(z) = z\prod_{k\in\mathbb{Z}\setminus\{0\}}\left(1-\frac{z}{\mu_k}\right). \tag{3.9}$$

It follows from (2.7) that the convergence in (3.7) is uniform in z on each compact set of $\mathbb{C}$, so that P_2 is an entire function. Note also that

$$P_2(z) = iP_3(e^{-i\frac{\pi}{4}}\sqrt{z}), \tag{3.10}$$

$$P_3(z) = -P_4(z)P_4(-z), \tag{3.11}$$

$$P(z) = \frac{P_1(-z)P_2(-z)}{z(1-\frac{z}{i\lambda_2})(1-\frac{z}{i\lambda_{-2}})}. \tag{3.12}$$

Applying Corollary 3.4 to P_1, noticing that

$$-i\lambda_k^+ = k + i + O(k^{-2})$$

with $\lambda_k^+ \neq \lambda_l^-$ for $k\neq l$, and $\lambda_0^+ = 0$, we infer that $P_1(z)$ is an entire function of sine type. Thus, for given $\varepsilon > 0$ there are some positive constants C_4, C_5, C_6 such that

$$C_4 e^{\pi|y|} \le |P_1(x+iy)| \le C_5 e^{\pi|y|}, \quad \mathrm{dist}\,(x+iy, \{-i\lambda_k^+\}) > \varepsilon \tag{3.13}$$

$$|P_1'(-i\lambda_k^+)| \ge C_6, \quad k\in\mathbb{Z}. \tag{3.14}$$

Next, applying Corollary 3.4 to P_4, noticing that

$$\mu_k = k - \frac{i}{2} + O(k^{-1})$$

with $\mu_k \neq \mu_l$ if $k \neq l$ and $\mu_0 = 0$, we infer that $P_4(z)$ is also an entire function of sine type. In particular, it is of exponential type π

$$|P_4(z)| \le C e^{\pi |z|}, \qquad z \in \mathbb{C}. \tag{3.15}$$

Therefore, we have for any $\varepsilon > 0$ and for some positive constants C_7, C_8, C_9

$$C_7 e^{\pi|y|} \le |P_4(x+iy)| \le C_8 e^{\pi |y|}, \quad \text{dist } (x+iy, \{\mu_k\}) > \varepsilon \tag{3.16}$$

$$|P_4'(\mu_k)| \ge C_9, \quad k \in \mathbb{Z}. \tag{3.17}$$

In particular, P_3 is an entire function of exponential type 2π with

$$C_7^2 e^{2\pi |y|} \le |P_3(x+iy)| \le C_8^2 e^{2\pi|y|} \qquad \text{dist } (\pm(x+iy), \{\mu_k\}) > \varepsilon. \tag{3.18}$$

Combined to (3.10), this yields

$$|P_2(z)| \le C e^{2\pi\sqrt{|z|}} \qquad z\in\mathbb{C}. \tag{3.19}$$

Substituting $e^{-i\frac{\pi}{4}}\sqrt{z}$ to $x+iy$ in (3.18) yields

$$C_7^2 \exp(2\pi |\text{Im}(e^{-i\frac{\pi}{4}}\sqrt{z})|) \le |P_2(z)| \le C_8^2 \exp(2\pi|\text{Im}(e^{-i\frac{\pi}{4}}\sqrt{z})|) \quad \text{dist}(\pm e^{-i\frac{\pi}{4}}\sqrt{z}, \{\mu_k\}) > \varepsilon. \tag{3.20}$$

From (3.20) (applied for x large enough) and the continuity of P_2 on $\mathbb{C}$, we obtain that

$$|P_2(x)| \le C e^{\sqrt{2}\pi\sqrt{|x|}}. \tag{3.21}$$

We are now in a position to give bounds for the canonical product P in (3.1).

Proposition 3.5. *The canonical product P in (3.1) is an entire function of exponential type at most π. Moreover, we have for some constant $C > 0$*

$$|P(x)| \le C(1+|x|)^{-3} e^{\sqrt{2}\pi\sqrt{|x|}}, \qquad x \in \mathbb{R}, \tag{3.22}$$

$$|P'(i\lambda_k^+)| \ge C^{-1}|k|^{-3} e^{\sqrt{2}\pi\sqrt{|k|}} \qquad k \in \mathbb{Z}\setminus\{0,\pm 2\}, \tag{3.23}$$

$$|P'(i\lambda_k^-)| \ge C^{-1}|k|^{-7} e^{\pi k^2} \qquad k\in\mathbb{Z}\setminus\{0,\pm 2\}. \tag{3.24}$$

Proof. Note first that $\text{dist}(\mathbb{R}, \{-i\lambda_k^+;\ k\neq 0\}) > 0$ from (2.5). Since $(1+\frac{is}{z})P_1(z)$ is also an entire function of sine type for $s \gg 1$, with $\text{dist}(\mathbb{R}, \{-i\lambda_k^+;\ k \neq 0\} \cup \{is\}) > 0$, we infer from Proposition 3.2 that for some constant $C>0$

$$|P_1(x)| \le C \qquad \forall x \in \mathbb{R}.$$

Combined to (3.12) and (3.21), this yields (3.22). Let us turn to (3.23). Note first that for $k\in\mathbb{Z}\setminus\{0,\pm 2\}$

$$P'(i\lambda_k^+) = P_1'(-i\lambda_k^+)\frac{P_2(-i\lambda_k^+)}{(-i\lambda_k^+)(1-\frac{\lambda_k^+}{\lambda_2})(1-\frac{\lambda_k^+}{\lambda_{-2}})}\cdot \tag{3.25}$$

Clearly, for some $\delta>0$, $|\lambda_k^+ - \lambda_l^-| > \delta$ for all $k\in\mathbb{Z}\setminus\{0,\pm 2\}$, $l\in\mathbb{Z}$, and

$$|\text{Im }(e^{-i\frac{\pi}{4}}\sqrt{-i\lambda_k^+})| = |\text{Im }\big(\frac{1-i}{\sqrt{2}}\sqrt{k+i+O(k^{-2})}\big)| = \sqrt{\frac{|k|}{2}} + O(|k|^{-\frac{1}{2}}).$$

With (3.20), this gives

$$|P_2(-i\lambda_k^+)| \geq Ce^{\sqrt{2}\pi\sqrt{|k|}}. \tag{3.26}$$

It follows then from (3.14), (3.25), and (3.26) that

$$|P'(i\lambda_k^+)| \geq C\frac{e^{\sqrt{2}\pi\sqrt{|k|}}}{|k|^3}$$

for some constant $C > 0$ independent of $k \in \mathbb{Z} \setminus \{0, \pm 2\}$. On the other hand

$$P'(i\lambda_k^-) = P_2'(-i\lambda_k^-)\frac{P_1(-i\lambda_k^-)}{(-i\lambda_k^-)(1 - \frac{\lambda_k^-}{\lambda_2})(1 - \frac{\lambda_k^-}{\lambda_{-2}})}. \tag{3.27}$$

By (2.7) and (3.13), we have that

$$|P_1(-i\lambda_k^-)| \geq Ce^{\pi k^2}, \qquad k \in \mathbb{Z} \setminus \{0, \pm 2\}.$$

From (3.10)-(3.11), we have that

$$P_2'(z) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{z}}\left[P_4'(e^{-i\frac{\pi}{4}}\sqrt{z})P_4(-e^{-i\frac{\pi}{4}}\sqrt{z}) - P_4(e^{-i\frac{\pi}{4}}\sqrt{z})P_4'(-e^{-i\frac{\pi}{4}}\sqrt{z})\right].$$

For $z = -i\lambda_k^-$, $e^{-i\frac{\pi}{4}}\sqrt{z} = \sqrt{-\lambda_k^-} = \operatorname{sgn}(k)\mu_k$, hence

$$P_2'(-i\lambda_k^-) = \frac{1}{2\mu_k}P_4'(\mu_k)P_4(-\mu_k).$$

Since $|\mu_k + \mu_l| > \delta > 0$ for $k \in \mathbb{Z} \setminus \{0\}$ and $l \in \mathbb{Z}$, we have from (3.16) that $|P_4(-\mu_k)| \geq c$ while, by (3.17), $|P_4'(\mu_k)| > c > 0$. It follows that for some constant $C > 0$

$$|P_2'(-i\lambda_k^-)| \geq \frac{C}{|k|} \qquad \forall k \in \mathbb{Z} \setminus \{0\}.$$

Therefore,

$$|P'(i\lambda_k^-)| \geq C\frac{e^{\pi k^2}}{|k|^7}, \qquad k \in \mathbb{Z} \setminus \{0\}.$$

□

We seek for an entire function m (the so-called *multiplier*) such that

$$\begin{aligned}
|m(x)| &\leq C(1 + |x|)e^{-\sqrt{2}\pi\sqrt{|x|}}, && x \in \mathbb{R}, \\
|m(i\lambda_k^+)| &\geq C^{-1}|k|^{-3}e^{-\sqrt{2}\pi\sqrt{|k|}}, && k \in \mathbb{Z} \setminus \{0\}, \\
|m(i\lambda_k^-)| &\geq C^{-1}e^{a\pi k^2 - 2\sqrt{2}\pi\sqrt{|k|}}, && k \in \mathbb{Z} \setminus \{0\}.
\end{aligned}$$

We shall use the same multiplier as in [7], providing additional estimates required to evaluate it at the points $i\lambda_k^-$ for $k \in \mathbb{Z}$. Let

$$s(t) = at - b\sqrt{t}, \qquad t > 0 \tag{3.28}$$

where the constants $a > 0$ and $b > 0$ will be chosen later. Note that s is increasing for $t > \left(\frac{b}{2a}\right)^2$ and that $s(B) = 0$ where $B = (b/a)^2$. Let

$$\nu(t) = \begin{cases} 0 & t \leq B, \\ s(t) & t \geq B. \end{cases} \tag{3.29}$$

Introduce first

$$g(z) = \int_0^\infty \log(1 - \frac{z^2}{t^2}) d\nu(t) = \int_B^\infty \log(1 - \frac{z^2}{t^2}) ds(t) \qquad z \in \mathbb{C} \setminus \mathbb{R}, \tag{3.30}$$

$$U(z) = \int_0^\infty \log|1 - \frac{z^2}{t^2}| d\nu(t) = \int_B^\infty \log|1 - \frac{z^2}{t^2}| ds(t) \qquad z \in \mathbb{C}. \tag{3.31}$$

Note that g is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and U is continuous on $\mathbb{C}$, with $U(z) = \text{Re } g(z)$. Next we atomize the measure μ in the above integrals, setting

$$\tilde{g}(z) = \int_0^\infty \log(1 - \frac{z^2}{t^2}) d[\nu(t)] \qquad z \in \mathbb{C} \setminus \mathbb{R}, \tag{3.32}$$

$$\tilde{U}(z) = \int_0^\infty \log|1 - \frac{z^2}{t^2}| d[\nu(t)] \qquad z \in \mathbb{C}, \tag{3.33}$$

where $[x]$ denotes the integral part of x. Again, $\tilde{g}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and $\tilde{U}$ is continuous on $\mathbb{C}$ with $\tilde{U}(z) = \text{Re } g(z)$. Actually, $\exp \tilde{g}$ is an entire function. Indeed, if $\{\tau_k\}_{k \geq 0}$ denotes the sequence of discontinuity points for $t \mapsto [\nu(t)]$, then $\tau_k \sim k/a$ as $k \to \infty$ and

$$\tilde{g}(z) = \sum_{k \geq 0} \log(1 - \frac{z^2}{\tau_k^2}), \quad z \in \mathbb{C} \setminus \mathbb{R}\cdot \tag{3.34}$$

Therefore,

$$e^{\tilde{g}(z)} = \prod_{k \geq 0} (1 - \frac{z^2}{\tau_k^2}), \tag{3.35}$$

the product being uniformly convergent on any compact set in $\mathbb{C}$. We shall pick later $m(z) = \exp(\tilde{g}(z - i))$ with $a = \frac{T}{2\pi} - 1$ and $b = \sqrt{2}$. The strategy, which goes back to [1], consists in estimating carefully U, and next $U - \tilde{U}$. Let for $x > 0$

$$w(x) = -\pi\sqrt{x} + x \log\left|\frac{x+1}{x-1}\right| - \sqrt{x} \log\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}\right| + 2\sqrt{x}\arctan(\sqrt{x}). \tag{3.36}$$

Note that $w \in L^\infty(\mathbb{R}^+)$, for $\lim_{x \to \infty} w(x) = -2$ and $w(0^+) = 0$.

Lemma 3.6. [7] *It holds*

$$U(x) + b\pi\sqrt{|x|} = -aBw(|x|) \qquad \forall x \in \mathbb{R} \tag{3.37}$$

Our first aim is to extend that estimate to the whole domain $\mathbb{C}$.

Lemma 3.7. *There exists some positive constant $C = C(a, b)$ such that*

$$-C - b\pi(1 + \frac{1}{\sqrt{2}})\sqrt{|y|} \leq U(z) + b\pi\sqrt{|x|} - a\pi|y| \leq C, \qquad z = x + iy \in \mathbb{C}. \tag{3.38}$$

Proof. We follow the same approach as in [7]. We first use the following identity from [7, (36)] (note that U is even)

$$U(z) = |\mathrm{Im}\ z|(\pi a + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{U(t)}{|z-t|^2}dt). \tag{3.39}$$

To derive (3.38), it remains to estimate the integral term in (3.39) for $z = x + iy \in \mathbb{C}$. We may assume without loss of generality that $y > 0$. From Lemma 3.6, we can write

$$U(t) = -b\pi\sqrt{|t|} - aBw(|t|)$$

where $w \in L^\infty(\mathbb{R}^+)$. Then, with $t = ys$,

$$\begin{aligned} \left|\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{aBw(|t|)}{(x-t)^2+y^2}\,dt\right| &\le ||w||_{L^\infty(\mathbb{R}^+)}\frac{aBy}{\pi}\int_{-\infty}^{\infty}\frac{ds}{y((\frac{x}{y}-s)^2+1)} \\ &= aB||w||_{L^\infty(\mathbb{R}^+)} =: C. \end{aligned} \tag{3.40}$$

On the other hand, still with $t = ys$, and using explicit computations in [7] of some integral terms,

$$\begin{aligned} \frac{y}{\pi}\int_{-\infty}^{\infty}(-b\pi)\frac{\sqrt{|t|}}{(x-t)^2+y^2}dt &= -b\sqrt{y}\int_{-\infty}^{\infty}\frac{\sqrt{|s|}}{(\frac{x}{y}-s)^2+1}ds \\ &= -b\sqrt{y}\left(\frac{\pi}{\sqrt{2\sqrt{1+\frac{x^2}{y^2}}-\frac{2x}{y}}}+\frac{\pi}{\sqrt{2\sqrt{1+\frac{x^2}{y^2}}+\frac{2x}{y}}}\right) \\ &= -b\frac{\pi}{\sqrt{2}}(\sqrt{\sqrt{x^2+y^2}-x}+\sqrt{\sqrt{x^2+y^2}+x}). \end{aligned}$$

Routine computations give

$$\sqrt{2|x|} \le \sqrt{\sqrt{x^2+y^2}-x}+\sqrt{\sqrt{x^2+y^2}+x} \le \sqrt{2|x|}+(\sqrt{2}+1)\sqrt{y} \quad \forall x\in\mathbb{R},\ \forall y>0.$$

Therefore

$$-b\pi\sqrt{|x|} - b\pi(1+\frac{1}{\sqrt{2}})\sqrt{y} \le \frac{y}{\pi}\int_{-\infty}^{\infty}(-b\pi)\frac{\sqrt{|t|}}{(x-t)^2+y^2}dt \le -b\pi\sqrt{|x|}.$$

Combined to (3.39) and (3.40), this yields (3.38). □

In order to obtain estimates for $\tilde{U}(z)$, we need to give bounds from above and below for

$$\tilde{U}(z) - U(z) = \int_0^\infty \log|1-\frac{z^2}{t^2}|\,d([\nu](t)-\nu(t)).$$

We need the following lemma, which is inspired from [11, Vol. 2, Lemma p. 162]

Lemma 3.8. *Let $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing and null on $(0, B)$. Then for $z = x + iy$ with $y \neq 0$, we have*

$$-\log^+\frac{|x|}{|y|} - \log^+\frac{x^2+y^2}{B^2} - \log 2 \le I = \int_0^\infty \log|1-\frac{z^2}{t^2}|d([\nu](t)-\nu(t)) \le \log^+\frac{|x|}{|y|}. \tag{3.41}$$

Proof. The proof of the upper bound is the same as in [11]. It is sketched here just for the sake of completeness. Pick any $z = x + iy$ with $y \neq 0$. Integrate by part in I to get

$$I = \int_0^\infty (\nu(t) - [\nu(t)]) \frac{\partial}{\partial t} \log |1 - \frac{z^2}{t^2}| dt.$$

Let $\zeta = \frac{z^2}{t^2}$. If Re $z^2 \leq 0$ (i.e. if $|x| \leq |y|$), then the distance $|1-\zeta|$ is decreasing w.r.t. t ($t \in (0, +\infty)$), so that $I \leq 0$. If Re $z^2 > 0$, then $|1-\zeta|$ decreases to the minimal value $\frac{|\text{Im } z^2|}{|z^2|}$ taken at $t = t^* := \frac{|z|^2}{|x|}$, and then it increases. Since $0 \leq \nu(t) - [\nu(t)] \leq 1$, we have that

$$I \leq \int_{t^*}^\infty \frac{\partial}{\partial t} \log |1 - \frac{z^2}{t^2}| dt = \log \frac{|z^2|}{|\text{Im } z^2|} = \log(\frac{|x|}{2|y|} + \frac{|y|}{2|x|}) \leq \log \frac{|x|}{|y|}\cdot$$

Let us pass to the lower bound. If Re $z^2 \leq 0$,

$$I \geq \int_B^\infty \frac{\partial}{\partial t} \log |1 - \frac{z^2}{t^2}| dt = -\log |1 - \frac{z^2}{B^2}|\cdot$$

Assume now that Re $z^2 > 0$. If $t^* = \frac{|z^2|}{|x|} \leq B$, $I \geq 0$. If $t^* > B$, then

$$I \geq \int_B^{t^*} \frac{\partial}{\partial t} \log |1 - \frac{z^2}{t^2}| dt = -\log \frac{|z^2|}{|\text{Im } z^2|} - \log |1 - \frac{z^2}{B^2}|\cdot$$

Note that

$$\log |1 - \frac{z^2}{B^2}| \leq \log(1 + |\frac{z}{B}|^2) \leq \log^+ \frac{x^2+y^2}{B^2} + \log 2.$$

Therefore

$$I \geq -\log^+ \frac{|x|}{|y|} - \log^+ \frac{x^2+y^2}{B^2} - \log 2.$$

□

Gathering together Lemma 3.7 and Lemma 3.8, we obtain the

Proposition 3.9. *There exists some positive constant $C = C(a, b)$ such that for any complex number $z = x + iy$ with $y \neq 0$,*

$$-C - b\pi(1 + \frac{1}{\sqrt{2}})\sqrt{|y|} - \log^+ \frac{|x|}{|y|} - \log^+(\frac{x^2+y^2}{B^2}) - \log 2 \leq \tilde{U}(z) + b\pi\sqrt{|x|} - a\pi|y| \leq C + \log^+ \frac{|x|}{|y|}. \tag{3.42}$$

Pick now

$$a = \frac{T}{2\pi} - 1 > 0, \quad b = \sqrt{2}, \quad \text{and } m(z) = \exp \tilde{g}(z - i) \tag{3.43}$$

Note that $|m(z)| = \exp \tilde{U}(z - i)$. The needed estimates for the multiplier m are collected in the following

Proposition 3.10. *m is an entire function on $\mathbb{C}$ of exponential type at most $a\pi$. Furthermore, the following estimates hold for some constant $C>0$:*

$$|m(x)| \le C(1+|x|)e^{-\sqrt{2\pi}\sqrt{|x|}}, \qquad x\in\mathbb{R} \tag{3.44}$$

$$|m(i\lambda_k^+)| \ge C^{-1}|k|^{-3}e^{-\sqrt{2\pi}\sqrt{|k|}}, \qquad k\in\mathbb{Z}\setminus\{0\} \tag{3.45}$$

$$|m(i\lambda_k^-)| \ge C^{-1}e^{a\pi k^2-2\sqrt{2}\pi|k|}, \qquad k\in\mathbb{Z}\setminus\{0\}. \tag{3.46}$$

Proof. (3.44) follows at once from (3.42) (with $y=-1$). We infer from (2.5) that for $k\in\mathbb{Z}\setminus\{0\}$

$$\mathrm{Im}\ (i\lambda_k^\pm) \le -\frac{1}{2}. \tag{3.47}$$

It follows then from (2.6) and (3.42) that

$$|m(i\lambda_k^+)| = \exp\tilde{U}(-k-2i(1+O(k^{-2}))) \ge C|k|^{-3}e^{-\sqrt{2\pi}\sqrt{|k|}} \quad (k\neq 0).$$

Finally, from (2.7) and (3.42), we infer that

$$\begin{aligned} |m(i\lambda_k^-)| &= \exp\tilde{U}(-k-i(k^2+O(k^{-2}))) \\ &\ge C\exp(-\sqrt{2}\pi\sqrt{|k|}+a\pi k^2-(\sqrt{2}+1)\pi|k|-4\log|k|) \\ &\ge C\exp(a\pi k^2-2\sqrt{2}\pi|k|). \end{aligned}$$

□

We are in a position to define the functions in the biorthogonal family. Pick first any $k\in\mathbb{Z}\setminus\{0,\pm 2\}$, and set

$$I_k^\pm(z) = \frac{P(z)}{P'(i\lambda_k^\pm)(z-i\lambda_k^\pm)}\cdot\frac{m(z)}{m(i\lambda_k^\pm)}\cdot\frac{(1-\frac{z}{i\lambda_2})(1-\frac{z}{i\lambda_{-2}})}{(1-\frac{\lambda_k^\pm}{\lambda_2})(1-\frac{\lambda_k^\pm}{\lambda_{-2}})}.$$

Clearly, $I_k^\pm$ is an entire function of exponential type at most $\pi(1+a)=T/2$. Furthermore, we have that

$$I_k^\pm(i\lambda_l^\pm) = \delta_k^l\delta_+^- \qquad \forall l\in\mathbb{Z}, \tag{3.48}$$

where δ_+^- is 1 if the two signs in the l.h.s. are the same, and 0 otherwise. Moreover,

$$(I_k^\pm)'(i\lambda_{\pm 2}) = 0. \tag{3.49}$$

On the other hand, by (2.6), (3.22), (3.23), (3.44) and (3.45), we have that

$$|I_k^+(x)| \le C\frac{|k|^4}{|x-i\lambda_k^+|} \le C\frac{|k|^4}{1+|k+x|}.$$

Thus $I_k^+\in L^2(\mathbb{R})$ with

$$||I_k^+||_{L^2(\mathbb{R})} \le C|k|^4. \tag{3.50}$$

Finally, by (2.7), (3.22), (3.24), (3.44) , and (3.46), we have that

$$|I_k^-(x)| \le C\frac{|k|^3}{|x-i\lambda_k^-|}e^{-(a+1)\pi k^2+2\sqrt{2}\pi|k|} \le C\frac{|k|^3}{|x+k|+k^2}e^{-\frac{T}{2}k^2+2\sqrt{2}\pi|k|}.$$

Thus

$$||I_k^-||_{L^2(\mathbb{R})} \leq C|k|^2 e^{-\frac{T}{2}k^2+2\sqrt{2}\pi|k|}. \tag{3.51}$$

It remains to introduce the functions $I_0(z), I_2(z), I_{-2}(z), \tilde{I}_2(z)$, and $\tilde{I}_{-2}(z)$. We set

$$\begin{aligned}
I_0(z) &= \frac{P(z)}{P'(0)z}\cdot\frac{m(z)}{m(0)}\cdot(1-\frac{z}{i\lambda_2})(1-\frac{z}{i\lambda_{-2}}),\\
\tilde{I}_2(z) &= -i\frac{P(z)}{P'(i\lambda_2)}\cdot\frac{m(z)}{m(i\lambda_2)}\cdot\frac{1-\frac{z}{i\lambda_{-2}}}{1-\frac{\lambda_2}{\lambda_{-2}}},\\
\tilde{I}_{-2}(z) &= -i\frac{P(z)}{P'(i\lambda_{-2})}\cdot\frac{m(z)}{m(i\lambda_{-2})}\cdot\frac{1-\frac{z}{i\lambda_2}}{1-\frac{\lambda_{-2}}{\lambda_2}},\\
K_2(z) &= i\frac{\tilde{I}_2(z)}{z-i\lambda_2},\quad I_2(z)=K_2(z)-iK_2'(i\lambda_2)\tilde{I}_2(z),\\
K_{-2}(z) &= i\frac{\tilde{I}_{-2}(z)}{z-i\lambda_{-2}},\quad I_{-2}(z)=K_{-2}(z)-iK_2'(i\lambda_{-2})\tilde{I}_{-2}(z).
\end{aligned}$$

Then we have that

$$I_0(0)=1,\quad I_0(i\lambda_k^{\pm})=0\quad k\in\mathbb{Z}\setminus\{0\},\quad I_0'(i\lambda_{\pm 2})=0, \tag{3.52}$$

$$\tilde{I}_2(i\lambda_k^{\pm})=0\quad k\in\mathbb{Z},\quad \tilde{I}_2'(i\lambda_2)=-i,\quad \tilde{I}_2'(i\lambda_{-2})=0, \tag{3.53}$$

$$\tilde{I}_{-2}(i\lambda_k^{\pm})=0\quad k\in\mathbb{Z},\quad \tilde{I}'_{-2}(i\lambda_{-2})=-i,\quad \tilde{I}'_{-2}(i\lambda_2)=0, \tag{3.54}$$

$$I_2(i\lambda_k^{\pm})=0\quad k\in\mathbb{Z}\setminus\{2\},\quad I_2(i\lambda_2)=1,\quad I_2'(i\lambda_{\pm 2})=0, \tag{3.55}$$

$$I_{-2}(i\lambda_k^{\pm})=0\quad k\in\mathbb{Z}\setminus\{-2\},\quad I_{-2}(i\lambda_{-2})=1,\quad I'_{-2}(i\lambda_{\pm 2})=0. \tag{3.56}$$

Moreover, I_0, $\tilde{I}_2$, $\tilde{I}_{-2}$, I_2, and I_{-2} are entire functions of exponential type at most $\pi(1+a)$ and they belong all to $L^2(\mathbb{R})$.

Let $\psi_k^{\pm}$, ψ_k, and $\tilde{\psi}_k$ denote the inverse Fourier transform of $I_k^{\pm}$, I_k, and $\tilde{I}_k$ for $k\in\mathbb{Z}\setminus\{0,\pm 2\}$, $k\in\{0,\pm 2\}$ and $k\in\{\pm 2\}$, respectively. Then, by Paley-Wiener theorem, the functions $\psi_k^{\pm}$, ψ_k and $\tilde{\psi}_k$ belong to $L^2(\mathbb{R})$, and are supported in $[-T/2,T/2]$. On the other hand, if $I(z)=\hat{\psi}(z)=\int_{-\infty}^{\infty}\psi(t)e^{-itz}dt$ with $\psi\in L^2(\mathbb{R})$, supp $\psi\subset[-T/2,T/2]$, then

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\psi(t)e^{\lambda t}dt = I(i\lambda)\quad \text{and}\quad -i\int_{-\frac{T}{2}}^{\frac{T}{2}}t\psi(t)e^{\lambda t}dt = I'(i\lambda).$$

Thus (2.36)-(2.40) follow from (3.48)-(3.49) and (3.52)-(3.56), while (2.41)-(2.42) follow from (3.50)-(3.51). The proof of Proposition 2.2 is complete.

4. Concluding remark

In this paper, the equation $y_{tt}-y_{xx}-y_{txx}=b(x-u(t))h(t)$ is proved to be null controllable on the torus (i.e. with periodic boundary conditions) when the support of the scalar control $h(t)$ moves at a constant velocity c ($u(t)=ct$). What happens for a domain with boundary?

More precisely, we may wonder under which assumptions on the initial conditions, the control time T, the support of the controller b and its pulsations ω the null controllability of the system

$$\begin{aligned}
&y_{tt} - y_{xx} - y_{txx} = b(x - \cos(\omega t))h(t), \qquad x \in (-1,1),\ t \in (0,T),\\
&y(-1,t) = y(1,t) = 0, \qquad t \in (0,T)
\end{aligned}$$

holds.

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