Riemannian mathematical morphology

Jesus Angulo, Santiago Velasco-Forero

Riemannian Mathematical Morphology★

Jesús Angulo[a], Santiago Velasco-Forero[b]

[a]*MINES ParisTech, CMM - Centre de Morphologie Mathématique, 35 rue St Honoré 77305 Fontainebleau Cedex, France*
[b]*National University of Singapore, Department of Mathematics*

Abstract

This paper introduces mathematical morphology operators for real-valued images whose support space is a Riemannian manifold. The starting point consists in replacing the Euclidean distance in the canonic quadratic structuring function by the Riemannian distance used for the adjoint dilation/erosion. We then extend the canonic case to a most general framework of Riemannian operators based on the notion of admissible Riemannian structuring function. An alternative paradigm of morphological Riemannian operators involves an external structuring function which is parallel transported to each point on the manifold. Besides the definition of the various Riemannian dilation/erosion and Riemannian opening/closing, their main properties are studied. We show also how recent results on Lasry–Lions regularization can be used for non-smooth image filtering based on morphological Riemannian operators. Theoretical connections with previous works on adaptive morphology and manifold shape morphology are also considered. From a practical viewpoint, various useful image embedding into Riemannian manifolds are formalized, with some illustrative examples of morphological processing real-valued 3D surfaces.

Keywords: mathematical morphology, manifold nonlinear image processing, Riemannian images, Riemannian image embedding, Riemannian structuring function, morphological processing of surfaces

1. Introduction

Pioneered for Boolean random sets (Matheron, 1975), extended latter to grey-level images (Serra, 1982) and more generally formulated in the framework of complete lattices (Serra, 1988; Heijmans, 1994), mathematical morphology is a nonlinear image processing methodology useful for solving efficiently many image analysis tasks (Soille, 1999). Our motivation in this paper is to formulate morphological operators for scalar functions on curved spaces.

Let E be the Euclidean $\mathbb{R}^d$ or discrete space $\mathbb{Z}^d$ (support space) and let $\mathcal{T}$ be a set of grey-levels (space of values). It is assumed that $\mathcal{T} = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. A grey-level image is represented by a function $f : E \to \mathcal{T}$, $f \in \mathcal{F}(E, \mathcal{T})$, i.e., f maps each pixel $x \in E$ into a grey-level value in $\mathcal{T}$. Given a grey-level image, the two basic morphological mappings $\mathcal{F}(E, \mathcal{T}) \to \mathcal{F}(E, \mathcal{T})$ are the dilation and the erosion given respectively by

$$\begin{cases} \delta_b(f)(x) = (f \oplus b)(x) = \sup_{y \in E} \{f(y) + b(y - x)\}, \\ \varepsilon_b(f)(x) = (f \ominus b)(x) = \inf_{y \in E} \{f(y) - b(y + x)\}, \end{cases}$$

where $b \in \mathcal{F}(E, \mathcal{T})$ is the structuring function which determines the effect of the operator. By allowing infinity values, the further convention for ambiguous expressions should be considered: $f(y) + b(x-y) = -\infty$ when $f(y) = -\infty$ or $b(x-y) = -\infty$, and that $f(y) - b(y+x) = +\infty$ when $f(y) = +\infty$ or $b(y+x) = -\infty$. We easily note that both are invariant under translations in the spatial ("horizontal") space E and in the grey-level ("vertical") space $\mathcal{T}$, i.e., $f(x) \mapsto f_{h,\alpha}(x) = f(x - h) + \alpha$, $x \in E$ and $\alpha \in \mathbb{R}$, then $\delta_b(f_{h,\alpha})(x) = \delta_b(f)(x - h) + \alpha$. The other morphological operators, such as the opening and the closing, are obtained by composition of dilation/erosion (Serra, 1982; Heijmans, 1994).

The structuring function is usually a parametric multi-scale family (Jackway and Deriche, 1996) $b_\lambda(x)$, where $\lambda > 0$ is the scale parameter such that $b_\lambda(x) = \lambda b(x/\lambda)$ and which satisfies the semi-group property $(b_\lambda \oplus b_\mu)(x) = b_{\lambda+\mu}(x)$. It is well known in the state-of-the-art of Euclidean morphology that the canonic family of structuring functions is the quadratic (or parabolic) one (Maragos, 1995; van den Boomgaard and Dorst, 1997); i.e.,

$$b_\lambda(x) = q_\lambda(x) = -\frac{\|x\|^2}{2\lambda}.$$

The most commonly studied framework, which additionally presents better properties of invariance, is based on flat structuring functions, called structuring elements. More precisely, let B be a Boolean set defined at the origin, i.e., $B \subseteq E$ or $B \in \mathcal{P}(E)$, which defines the "shape" of the structuring element, the associated structuring function is given by

$$b(x) = \begin{cases} 0 & \text{if } x \in B \\ -\infty & \text{if } x \in B^c \end{cases}$$

where B^c is the complement set of B in $\mathcal{P}(E)$. Hence, the flat dilation and flat erosion can be computed respectively by the moving local maxima and minima filters.

★This is an extended version of a paper that appeared at the 13th International Symposium of Mathematical Morphology held in May 27-29 in Uppsala, Sweden

Email addresses: jesus.angulo@mines-paristech.fr (Jesús Angulo), matsavf@nus.edu.sg (Santiago Velasco-Forero)

Aim of the paper. Let us consider now that the support space is not Euclidean, see Fig. 1(a). This is the case for instance if we deal with a smooth 3D surface, or more generally if the support space is a Riemannian manifold $\mathcal{M}$. In all this paper, we consider that $\mathcal{M}$ is a finite dimensional compact manifold. Starting point of this work is based on a Riemannian sup/inf-convolution where the Euclidean distance in the canonic quadratic structuring function is replaced by the Riemannian distance (Section 3). Besides the definition of Riemannian dilation/erosion and Riemannian opening/closing, we explore their properties and in particular the associated granulometric scale-space. We also show how some theoretical results on Lasry–Lions regularization are useful for image Lipschitz regularization using quadratic Riemannian dilation/erosion. We then extend the canonic case to the most general framework of Riemannian dilation/erosion and subsequent operators in Section 4, by introducing the notion of admissible Riemannian structuring function. Section 5 introduces a different paradigm of morphological operators on Riemannian supported images, where the structuring function is an external datum which is parallel transported to each point on the manifold. We consider theoretically various useful case studies of image manifolds in Section 7, but due to the limited paper length, we only illustrate some cases of real-valued 3D surfaces.

Related work. Generalizations of Euclidean translation-invariant morphology have followed three main directions. On the one hand, adaptive morphology (Debayle and Pinoli, 2005; Lerallut et al., 2007; Welk et al., 2011; Verdú et al., 2011; Ćurić et al., 2012; Angulo, 2013; Landström and Thurley, 2013; Velasco-Forero and Angulo, 2013), where the structuring function becomes dependent on the position or the input image itself. Section 6 explores the connections of our framework with such kind of approaches. On the second hand, group morphology (Roerdink, 2000), where the translation invariance is replaced by other group invariance (similarity, affine, spherical, projective, etc.). Related to that, we have also the morphology for binary manifolds (Roerdink, 1994), whose relationship with our formulation is deeply studied in Section 5. Finally, we should cite also the classical notion of geodesic dilation (Lantuejoul and Beucher, 1981) as the basic operator for (connective) geodesic reconstruction (Soille, 1999), where the marker image is dilated according to the metric yielded by the reference image (see also Section 6).

2. Basics on Riemannian manifold geometry

Let us remind in this section some basics on differential geometry for Riemannian manifolds (Berger and Gostiaux, 1987), see Fig. 1(b) for an explanatory diagram.

The *tangent space* of the manifold $\mathcal{M}$ at a point $p \in \mathcal{M}$, denoted by $T_p\mathcal{M}$, is the set of all vectors tangent to $\mathcal{M}$ at p. The first issue to consider is how to transport vectors from one point of $\mathcal{M}$ to another. Let $p, q \in \mathcal{M}$ and let $\gamma : [a, b] \to \mathcal{M}$ be a parameterized curve (or path) from $\gamma(a) = p$ to $\gamma(b) = q$. For $\mathbf{v} \in T_p\mathcal{M}$, let $\mathbf{V}$ be the unique parallel vector field along γ with $\mathbf{V}(a) = \mathbf{v}$. The map $P_\gamma : T_p\mathcal{M} \to T_q\mathcal{M}$ determined by $P_\gamma(\mathbf{v}) = \mathbf{V}(b)$ is called *parallel transport from p to q along* γ, and $P_\gamma(\mathbf{v})$ the parallel translate of $\mathbf{v}$ along γ to q. Note that parallel transport from p to q is path dependent: the difference between two paths is a rotation around the normal to $\mathcal{M}$ at q.

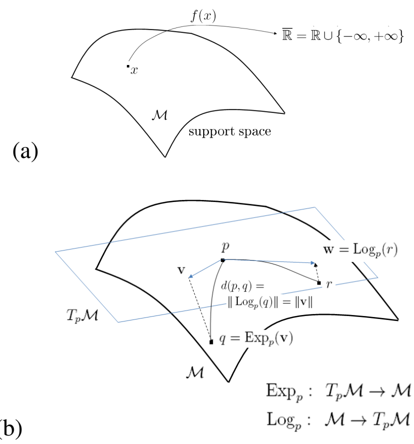

Figure 1: (a) Real-valued Riemannian image. (b) Riemannian manifold at tangent space a given point.

The *Riemannian distance* between two points $p, q \in \mathcal{M}$, denoted $d(p, q)$, is defined as the minimum length over all possible smooth curves between p and q. A *geodesic* $\gamma : [0, 1] \to \mathcal{M}$ connecting two points $p, q \in \mathcal{M}$ is the shortest path on $\mathcal{M}$ having elements p and q as endpoints. The geodesic curve $\gamma(t)$ can be specified in terms of a starting point $p \in \mathcal{M}$ and a tangent vector (initial constant velocity) $\mathbf{v} \in T_p\mathcal{M}$ as it represents the solution of Christoffel differential equation with boundary conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = \mathbf{v}$. The idea behind *exponential map* Exp_p is to parameterize a Riemannian manifold $\mathcal{M}$, locally near any $p \in \mathcal{M}$, in terms of a mapping from the tangent space $T_p\mathcal{M}$ into a point in $\mathcal{M}$. The exponential map is injective on a zero-centered ball B in $T_p\mathcal{M}$ of some non-zero (possibly infinity) radius. Thus for a point q in the image of B under Exp_p there exists a unique vector $\mathbf{v} \in T_p\mathcal{M}$ corresponding to a minimal length path under the exponential map from p to q. Exponential maps may be associated to a manifold by the help of geodesic curves. The exponential map $\mathrm{Exp}_p : T_p\mathcal{M} \to \mathcal{M}$ associated to any geodesic $\gamma_\mathbf{v}$ emanating from p with tangent at the origin $\mathbf{v} \in T_p\mathcal{M}$ is defined as $\mathrm{Exp}_p(\mathbf{v}) = \gamma_\mathbf{v}(1)$, where the geodesic is given by $\gamma_\mathbf{v}(t) = \mathrm{Exp}_p(t\mathbf{v})$. The geodesic has constant speed equal to $\|d\gamma_\mathbf{v}/dt\|(t) = \|\mathbf{v}\|$, and thus the *exponential map preserves distances* for the initial point: $d(p, \mathrm{Exp}_p(\mathbf{v})) = \|\mathbf{v}\|$. A Riemannian manifold is geodesically complete if and only if the exponential map $\mathrm{Exp}_p(\mathbf{v})$ is defined $\forall p \in \mathcal{M}$ and $\forall \mathbf{v} \in T_p\mathcal{M}$. The inverse operator, named *logarithm map*, $\mathrm{Exp}_p^{-1} = \mathrm{Log}_p$ maps a point of $q \in \mathcal{M}$ into to their associated tangent vectors $\mathbf{v} \in T_p\mathcal{M}$. The exponential map is in general only invertible for a sufficient small neighbourhood of the origin in $T_p\mathcal{M}$, although on some manifolds the inverse exists for arbitrary neighbourhoods. For a point q in the domain of Log_p the *geodesic distance* between p and q is given by $d(p, q) = \|\mathrm{Log}_p(q)\|$.

3. Canonic Riemannian dilation and erosion

Let us start by a formal definition of the two basic canonic morphological operators for images supported on a Riemannian manifold.

Definition 1. *Let $\mathcal{M}$ a complete Riemannian manifold and $d_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^{+}$, $(x, y) \mapsto d_{\mathcal{M}}(x, y)$, is the geodesic distance on $\mathcal{M}$, for any image $f : \mathcal{M} \to \overline{\mathbb{R}}$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, so $f \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$ and for $\lambda > 0$ we define for every $x \in \mathcal{M}$ the canonic Riemannian dilation of f of scale parameter λ as*

$$\delta_\lambda(f)(x) = \sup_{y\in\mathcal{M}} \left\{ f(y) - \frac{1}{2\lambda} d_{\mathcal{M}}(x, y)^2 \right\} \tag{1}$$

and the canonic Riemannian erosion of f of parameter λ as

$$\varepsilon_\lambda(f)(x) = \inf_{y\in\mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(x, y)^2 \right\} \tag{2}$$

An obvious property of the canonic Riemannian dilation and erosion is the *duality by the involution* $f(x) \mapsto \complement f(x) = -f(x)$, i.e., $\delta_\lambda(f) = \complement\varepsilon_\lambda(\complement f)$. As in classical Euclidean morphology, the adjunction relationship is fundamental for the construction of the rest of morphological operators.

Proposition 2. *For any two real-valued images defined on the same Riemannian manifold $\mathcal{M}$, i.e., $f, g : \mathcal{M} \to \overline{\mathbb{R}}$, the pair $(\varepsilon_\lambda, \delta_\lambda)$ is called the canonic Riemannian adjunction*

$$\delta_\lambda(f)(x) \le g(x) \Leftrightarrow f(x) \le \varepsilon_\lambda(g)(x) \tag{3}$$

Hence, we have an adjunction if both images f and g are defined on the same Riemannian manifold $\mathcal{M}$, or in other terms, when the same "quadratic geodesic structuring function":

$$\mathfrak{q}_\lambda(x; y) = -\frac{1}{2\lambda} d_{\mathcal{M}}(x, y)^2, \tag{4}$$

is considered for pixel $x \mapsto \mathfrak{q}_\lambda(x; y)$, $y \in \mathcal{M}$ in both f and g. This result implies in particular that the canonic Riemannian dilation commutes with the supremum and the dual erosion with the infimum, i.e., for a given collection of images $f_i \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$, $i \in I$, we have

$$\delta_\lambda\left(\bigvee_{i\in I} f_i\right) = \bigvee_{i\in I} \delta_\lambda(f_i); \quad \varepsilon_\lambda\left(\bigwedge_{i\in I} f_i\right) = \bigwedge_{i\in I} \varepsilon_\lambda(f_i).$$

In addition, using the classical result on adjunctions in complete lattices (Heijmans, 1994), we state that the composition products of the pair $(\varepsilon_\lambda, \delta_\lambda)$ lead to the adjoint opening and adjoint closing if and only the field of geodesic structuring functions is computed on a common manifold $\mathcal{M}$.

Definition 3. *Given an image $f \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$, the canonic Riemannian opening and canonic Riemannian closing of scale parameter λ are respectively given by*

$$\gamma_\lambda(f)(x) = \sup_{z\in\mathcal{M}} \inf_{y\in\mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(z, y)^2 - \frac{1}{2\lambda} d_{\mathcal{M}}(z, x)^2 \right\}, \tag{5}$$

and

$$\varphi_\lambda(f)(x) = \inf_{z\in\mathcal{M}} \sup_{y\in\mathcal{M}} \left\{ f(y) - \frac{1}{2\lambda} d_{\mathcal{M}}(z, y)^2 + \frac{1}{2\lambda} d_{\mathcal{M}}(z, x)^2 \right\}. \tag{6}$$

This technical point is very important since in some image manifold embedding the Riemannian manifold support $\mathcal{M}$ of image f depends itself on f. If $\mathcal{M}$ does not depends on f, the canonic Riemannian opening and closing are respectively given by $\gamma_\lambda(f) = \delta_\lambda(\varepsilon_\lambda(f))$, and $\varphi_\lambda(f) = \varepsilon_\lambda(\delta_\lambda(f))$. We notice that this issue was already considered by Roerdink (2009) for the case of adaptive neighbourhood morphology.

Having the canonic Riemannian opening and closing, all the other morphological filters defined by composition of them are easily obtained.

3.1. Properties of $\delta_\lambda(f)$ and $\varepsilon_\lambda(f)$

Classical properties of Euclidean dilation and erosion have also the equivalent for Riemannian manifold $\mathcal{M}$, and they do not dependent on the geometry of $\mathcal{M}$.

Proposition 4. *Let $\mathcal{M}$ be a Riemannian manifold, and let $f, g \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$ two real valued images $\mathcal{M}$. We have the following properties for the canonic Riemannian operators.*

1. *(Increaseness) If $f(x) \le g(x)$, $\forall x \in \mathcal{M}$ then $\delta_\lambda(f)(x) \le \delta_\lambda(g)(x)$ and $\varepsilon_\lambda(f)(x) \le \varepsilon_\lambda(g)(x)$, $\forall x \in \mathcal{M}$ and $\forall \lambda > 0$.*
2. *(Extensivity and anti-extensivity) $\delta_\lambda(f)(x) \ge f(x)$ and $\varepsilon_\lambda(f)(x) \le f(x)$, $\forall x \in \mathcal{M}$ and $\forall \lambda > 0$.*
3. *(Ordering property) If $0 < \lambda_1 < \lambda_2$ then $\delta_{\lambda_2}(f)(x) \ge \delta_{\lambda_1}(f)(x)$ and $\varepsilon_{\lambda_2}(f)(x) \le \varepsilon_{\lambda_1}(f)(x)$.*
4. *(Invariance under isometry) If $T : \mathcal{M} \to \mathcal{M}$ is an isometry of $\mathcal{M}$ and if f is invariant under T, i.e., $f(Tz) = f(z)$ for all $z \in \mathcal{M}$, then the Riemannian dilation and erosion are also invariant under T, i.e., $\delta_\lambda(f)(Tz) = \delta_\lambda(f)(z)$ and $\varepsilon_\lambda(f)(Tz) = \varepsilon_\lambda(f)(z)$, $\forall z \in \mathcal{M}$ and $\forall \lambda > 0$.*
5. *(Extrema preservation) We have $\sup \delta_\lambda(f) = \sup f$ and $\inf \varepsilon_\lambda(f) = \inf f$, moreover if f is lower (resp. upper) semicontinuous then every minimizer (resp. maximizer) of $\varepsilon_\lambda(f)$ (resp. $\delta_\lambda(f)$) is a minimizer (resp. maximizer) of f, and conversely.*

3.2. Flat isotropic Riemannian dilation and erosion

In order to obtain the counterpart of flat isotropic Euclidean dilation and erosion, we replace the quadratic structuring function $\mathfrak{q}_\lambda(x, y)$ by a flat structuring function given by the geodesic ball of radius r centered at x, i.e.,

$$B_r(x) = \{y : \ d_{\mathcal{M}}(x, y) \le r\}, \ \ r > 0. \tag{7}$$

The corresponding *flat isotropic Riemannian dilation and erosion of size r* are given by:

$$\delta_{B_r}(f)(x) = \sup\left\{f(y) : \ y \in \check{B}_r(x)\right\}, \tag{8}$$

$$\varepsilon_{B_r}(f)(x) = \inf\left\{f(y) : \ y \in B_r(x)\right\}. \tag{9}$$

where $\check{B}_r(x)$ is the transposed shape of ball $B_r(x)$. Corresponding flat isotropic Riemannian opening and closing are obtained by composition of operators (8) and (9):

$$\gamma_{B_r}(f) = \delta_{B_r}\left(\varepsilon_{B_r}(f)\right) ; \ \ \varphi_{B_r}(f) = \varepsilon_{B_r}\left(\delta_{B_r}(f)\right). \tag{10}$$

All the properties formulated for canonic operators hold for flat isotropic ones too. For practical applications, it should be noted that flat operators typically lead to stronger filtering effects than the quadratic ones.

3.3. Riemannian granulometries: scale-space properties of $\gamma_\lambda(f)$ and $\varphi_\lambda(f)$

For the canonic Riemannian opening and closing, we have also the classical properties which are naturally proved as a consequence of the adjunction, see (Heijmans, 1994).

Proposition 5. *Let $\gamma_\lambda(f)$ and $\varphi_\lambda(f)$ be respectively the canonic Riemannian opening and closing of an image $f \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$.*

1. *$\gamma_\lambda(f)$ and $\varphi_\lambda(f)$ are both* increasing *operators.*
2. *$\gamma_\lambda(f)$ is anti-extensive and $\varphi_\lambda(f)$ extensive with the following* ordering relationships*, i.e., for for $0 < \lambda_1 \leq \lambda_2$, we have:*
$$\gamma_{\lambda_2}(f)(x) \leq \gamma_{\lambda_1}(f)(x) \leq f(x) \leq \varphi_{\lambda_1}(f)(x) \leq \varphi_{\lambda_2}(f)(x); \quad (11)$$
3. idempotency of both operators*, $\gamma_\lambda(\gamma_\lambda(f)) = \gamma_\lambda(f)$ and $\varphi_\lambda(\varphi_\lambda(f)) = \varphi_\lambda(f)$*

Property 3 on idempotency together with the increaseness defines a family of so-called algebraic openings/closings (Serra, 1988; Heijmans, 1994) larger than the one associated to the composition of dilation/erosion. Idempotent and increasing operators are also known as ethmomorphisms by Kiselman (2007). Anti-extensivity and extensivity involves that γ_λ is a anoiktomorphism and φ_λ a cleistomorphism. One of the most classical results in morphological operators provided us an example of algebraic opening: given a collection of openings $\{\gamma_i\}$, increasing, idempotent and anti-extensive operators for all i, the supremum of them $\sup_i \gamma_i$ is also an opening (Matheron, 1975). A dual result is obtained for the closing by changing the sup by the inf.

The class of openings (resp. closings) is neither closed under infimum (resp. opening) or a generic composition. There is however a semi-group property leading to a scale-space framework for opening/closing operators, known as granulometries. The notion of granulometry in Euclidean morphology is summarized in the following results (Matheron, 1975; Serra, 1988).

Theorem 6 (Matheron (1975), Serra (1988)). *A parameterized family $\{\gamma_\lambda\}_{\lambda>0}$ of flat openings from $\mathcal{F}(E,\mathcal{T})$ into $\mathcal{F}(E,\mathcal{T})$ is a granulometry (or size ditritribution) when*
$$\gamma_{\lambda_1}\gamma_{\lambda_2} = \gamma_{\lambda_2}\gamma_{\lambda_1} = \gamma_{\sup(\lambda_1,\lambda_2)}; \quad \lambda_1, \lambda_2 > 0. \quad (12)$$
Condition (12) *is equivalent to both*
$$\gamma_{\lambda_1} \leq \gamma_{\lambda_2}; \quad \lambda_1 \geq \lambda_2 > 0; \quad (13)$$
$$\mathcal{B}_{\lambda_1} \subseteq \mathcal{B}_{\lambda_2}; \quad \lambda_1 \geq \lambda_2 > 0$$
where $\mathcal{B}_\lambda$ is the invariance domain of the opening at scale λ; i.e., the family of structuring elements Bs such that $B = \gamma_\lambda(B)$ (Serra, 1988).

By duality, we introduce antisize distributions as the families of closings $\{\varphi_\lambda\}_{\lambda>0}$.

Axiom (12) shows how translation invariant flat openings are composed and highlights their semi-group structure. Equivalent condition (13) emphasizes the monotonicity of the granulometry with respect to λ: the opening becomes more and more active as λ increases. When dealing with Euclidean spaces, Matheron (1975) introduced the notion of Euclidean granulometry as the size distribution being translationally invariant and compatible with homothetics, i.e., $\gamma_\lambda(f(x)) = \lambda\gamma_1(f(\lambda^{-1}x))$, where $f \in \mathcal{F}(E,\mathcal{T})$ is an Euclidean grey-level images. More precisely, a family of mappings γ_λ is an Euclidean granulometry if and only if there exist a class $\mathcal{B}'$ such that
$$\gamma_\lambda(f) = \bigvee_{B\in\mathcal{B}'} \bigvee_{\mu\geq\lambda} \gamma_{\mu B}(f).$$
Then the domain of invariance $\mathcal{B}_\lambda$ are equal to $\lambda\mathcal{B}$, where $\mathcal{B}$ is the class closed under union, translation and homothetics ≥ 1, which is generated by $\mathcal{B}'$. If we reduce the class $\mathcal{B}'$ to a single element B, the associated size distribution becomes
$$\gamma_\lambda(f) = \bigvee_{\mu\geq\lambda} \gamma_{\mu B}(f).$$
The following key result simplifies the situation. The size distribution by a compact structuring element B is equivalent to $\gamma_\lambda(f) = \gamma_{\lambda B}(f)$ if and only if B is convex. The extension of granulometric theory to non-flat structuring functions was deeply studied in (Kraus et al., 1993). In particular, it was proven that one can build grey-level Euclidean granulometries with one structuring function if and only if this function has a convex compact domain and is constant there (flat function).

We can naturally extend Matheron axiomatic to the general case of openings in Riemannian supported images. We start by giving a result which is valid for families of openings $\{\psi_\lambda\}$ (idempotent and anti-extensive operators) more general than the canonic Riemannian openings.

Proposition 7. *Given the set of Riemannian openings $\{\psi_\lambda\}_{\lambda>0}$ indexed according to the positive parameter λ, but not necessary ordered between them, the corresponding Riemannian granulometry on image $f \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$ is the family of multiscale openings $\{\Gamma_\lambda\}_{\lambda>0}$ generated as*
$$\Gamma_\lambda(f) = \bigvee_{\mu\geq\lambda} \psi_\mu(f)$$
such that the granulometric semi-group law holds for any pair of scales:
$$\Gamma_{\lambda_1}(\Gamma_{\lambda_2}(f)) = \Gamma_{\lambda_2}(\Gamma_{\lambda_1}(f)) = \Gamma_{\sup(\lambda_1,\lambda_2)}(f). \quad (14)$$

In the particular case of canonic Riemannian openings, $\{\gamma_\lambda\}_{\lambda>0}$, we always have $\gamma_{\lambda_1} \leq \gamma_{\lambda_2}$ if $\lambda_1 \geq \lambda_2 > 0$. Hence, $\Gamma_\lambda(f) = \gamma_\lambda$ and consequently $\{\gamma_\lambda\}_{\lambda>0}$ is a granulometry. This is also valid for flat isotropic Riemannian openings.

The Riemannian case closest to Matheron's Euclidean granulometries corresponds to the flat isotropic Riemannian openings γ_{B_r} associated to a concave quadratic geodesic structuring function $\mathrm{q}_\lambda(x,y)$. Or in other terms, the case of a Riemannian manifold $\mathcal{M}$ where the Riemannian distance is always a convex function, since this fact involves that $B_r(x)$ as defined in (7) is a convex set for any r at any $x \in \mathcal{M}$. Obviously, the flat convex Riemannian granulometry $\{\gamma_{B_r}\}_{r>0}$ is not translation invariant but we have that $B_{r_1}(x) \subseteq B_{r_2}(x)$, for $r_2 \geq r_1$ and for any $x \in \mathcal{M}$, which involves a natural sieving selection of features in the neighborhood of any point x.

A Riemannian distance function which is convex is not only useful for scale-space properties. As discussed just below, one has powerful results of regularization too.

3.4. Concavity of $\mathfrak{q}_\lambda(x; y)$ and Lipschitz image regularization using $(\varepsilon_\lambda, \delta_\lambda)$

Lasry–Lions regularization (Lasry and Lions, 1986) is a theory of nonsmooth approximation for functions in Hilbert spaces using combinations of Euclidean dilation and erosion with quadratic structuring functions, which leads to the approximation of bounded lower or upper-semicontinuous functions with Lipschitz continuous derivatives which approximate f, without assuming convexity of f. The approach was generalized in (Attouch and Aze, 1993) to semicontinuous, non necessarily bounded, quadratically minorized/majorized functions defined on $\mathbb{R}^n$. More precisely, we have.

Theorem 8 (Lasry and Lions(1986), Attouch and Aze(1993)). *For all $0 < \mu < \lambda$, let us define for a given image f the Lasry-Lions regularizers based on Euclidean dilation and erosion by a quadratic structuring function q_λ as:*

$$
\begin{aligned}
(f_\lambda)^\mu(x) &= \Big((f \ominus q_\lambda) \oplus q_\mu\Big)(x),\\
(f^\lambda)_\mu(x) &= \Big((f \oplus q_\lambda) \ominus q_\mu\Big)(x).
\end{aligned}
$$

- *Let f be a bounded uniformly continuous scalar functions in $\mathbb{R}^n$. Then the functions $(f_\lambda)^\mu$ and $(f^\lambda)_\mu$ converge uniformly to f when $\lambda, \mu \to 0$, and belong to the class $C_b^{1,1}(\mathbb{R}^n)$ (i.e., bounded continuously differentiable with a Lipschitz continuous gradient), namely $|\nabla(f_\lambda)^\mu(x) - \nabla(f_\lambda)^\mu(y)| \leq M_{\lambda,\mu}\|x-y\|$ and $|\nabla(f^\lambda)_\mu(x) - \nabla(f^\lambda)_\mu(y)| \leq M_{\lambda,\mu}\|x-y\|$, where $M_{\lambda,\mu} = (\mu^{-1}, (\lambda-\mu)^{-1})$.*

- *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower-semicontinuous function and $g : E \subseteq \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ an upper-semicontinuous. We assume the growing conditions $f(x) \geq -\frac{c}{2}(1 + \|x\|^2)$, $c \geq 0$ (quadratically minorized), then $g(x) \leq \frac{c'}{2}(1 + \|x\|^2)$, $c' \geq 0$ (quadratically majorized). Then for $0 < \mu < \lambda < c^{-1}$ and $0 < \mu < \lambda < c'^{-1}$ the regularizes $(f_\lambda)^\mu$ and $(g^\lambda)_\mu$ are $C_b^{1,1}(\mathbb{R}^n)$ functions, whose gradient is Lipschitz continuous with constant* $\max(\mu^{-1}, (1-\lambda c)^{-1}c)$. *In addition they converge point-wise respectively to f and g when $\lambda, \mu \to 0$.*

Hence, we can replace the bounded and uniformly continuous assumptions by rather general growing conditions. The idea is that given a quadratically majorized function g of parameter c', the quadratic dilation $f \oplus q_\lambda$ with $\lambda < c'^{-1}$ produces a λ-weakly convex function. Then for any $\mu < \lambda$ (strictly smaller than the dilation scale), the corresponding quadratic erosion $(f \oplus q_\lambda) \ominus q_\mu$ produces a function belongings to the class of bounded C^1, with has Lipschitz continuous gradient. Note that the key element of this approximation is the transfer of the regularity of the quadratic kernel associated to its concavity and smoothness of q_λ to the function f.

Lasry–Lions regularization has been recently generalized to finite dimensional compact manifolds Bernard (2010); Bernard and Zavidovique (2013), and consequently these results can be used to show how Riemannian morphological operators are appropriate for image regularization. More precisely, let us focus on the case where $\mathcal{M}$ is finite dimensional compact Cartan–Hadamard manifold, hence every two points can be connected by a minimizing geodesic. We remind that a Cartan–Hadamard manifold is a simply connected Riemannian manifold $\mathcal{M}$ with sectional curvature $K \leq 0$ (Lang, 1999). Let A be a closed convex subset of $\mathcal{M}$. Then the distance function to A, $x \mapsto d_{\mathcal{M}}(x, A)$, where $d_{\mathcal{M}}(x, A) = \inf\{d_{\mathcal{M}}(x, y) : y \in A\}$ is C^1 smooth on $\mathcal{M} \setminus A$ and, moreover, the square of the distance function $x \mapsto d_{\mathcal{M}}(x, A)^2$ is C^1 smooth and convex on all of M (Azagra and Ferrera, 2006). Consequently, if $\mathcal{M}$ is a Cartan–Hadamard manifold, the structuring function $x \mapsto \mathfrak{q}(x, y)$, $\forall y \in \mathcal{M}$, is always a concave function; or equivalently, $-\mathfrak{q}(x, y)$ is a convex function.

Theorem 9. *Let $\mathcal{M}$ be a compact finite dimensional Cartan–Hadamard manifold. Let $\Omega \subset \mathcal{M}$ be a bounded set of $\mathcal{M}$. Given a image $f \in \mathcal{F}(\Omega, \mathbb{R})$, for all $0 < \mu < \lambda$ let us define the Riemannian Lasry–Lions regularizers:*

$$
\begin{aligned}
(f_\lambda)^\mu(x) &= \sup_{z\in\mathcal{M}} \inf_{y\in\mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(z, y)^2 - \frac{1}{2\mu} d_{\mathcal{M}}(z, x)^2 \right\}\\
(f^\lambda)_\mu(x) &= \inf_{z\in\mathcal{M}} \sup_{y\in\mathcal{M}} \left\{ f(y) - \frac{1}{2\lambda} d_{\mathcal{M}}(z, y)^2 + \frac{1}{2\mu} d_{\mathcal{M}}(z, x)^2 \right\}
\end{aligned}
$$

We have $(f_\lambda)^\mu \leq f$ and $(f^\lambda)_\mu \geq f$.

- *Let f be a bounded uniformly continuous image in Ω. Then the images $(f_\lambda)^\mu$ and $(f^\lambda)_\mu$ belong to the class $C_b^{1,1}(\Omega)$ and converges uniformly to f on Ω.*

- *Assume that there exists $c, c' > 0$, such that we have the following growing conditions for semicontinuous functions $\mathcal{F}(\Omega, \mathbb{R})$:*

$$
f(x) \geq -\frac{c}{2}(1 + d(x, x_0)^2),\; g(x) \leq \frac{c'}{2}(1 + d(x, x_0)^2),\; x_0 \in \mathcal{M}.
$$

  Then, for all $0 < \mu < \lambda < c^{-1}$ the pseudo-opened image $(f_\lambda)^\mu$ and for all $0 < \mu < \lambda < c'^{-1}$ the pseudo-closed image $(g^\lambda)_\mu$ are of class $C_b^{1,1}(\Omega)$. In addition, they converge point-wise respectively to f and g.

We remark that this result is theoretically valid only for bounded images supported on bounded subsets on manifolds of nonpositive sectional curvature. However, in practice we observe that it works for bounded images on bounded surfaces of positive and negative curvature. By the way, one should note that our result conjectured in (Angulo and Velasco-Forero, 2013) was too general since the support space of the image should be a bounded set Ω. As discussed in Bernard (2010) and Bernard and Zavidovique (2013), more general versions of

Lasry-Lions regularization can be obtained in Riemannian manifolds. In particular the case of compact nonnegative curvature manifolds is relevant for optimal transport problems (Villani, 2009).

4. Generalized Riemannian morphological operators

We have discussed the canonic case on Riemannian mathematical morphology associated to the structuring function $\mathfrak{q}_\lambda(x,y)$. Let consider now the most general family of Riemannian operators. We start by introducing the minimal properties that a Riemannian structuring function should verify.

Definition 10. *Let $\mathcal{M}$ be a Riemannian manifold. A mapping $\mathfrak{b} : \mathcal{M} \times \mathcal{M} \to \overline{\mathbb{R}}$ defined for any pair of points in $\mathcal{M}$ is said an admissible Riemannian structuring function in $\mathcal{M}$ if and only if*

1. $\mathfrak{b}(x,y) \leq 0$, $\forall x,y \in \mathcal{M}$ *(non-positivity);*
2. $\mathfrak{b}(x,x) = 0$, $\forall x \in \mathcal{M}$ *(maximality at the diagonal).*

Now, we can introduce the pair of dilation and erosion for any image f according to $\mathfrak{b}$.

Definition 11. *Given an admissible Riemannian structuring function $\mathfrak{b}$ in a Riemannian manifold $\mathcal{M}$, the Riemannian dilation and Riemannian erosion of an image $f \in \mathcal{F}(\mathcal{M},\overline{\mathbb{R}})$ by $\mathfrak{b}$ are given respectively by*

$$\delta_{\mathfrak{b}}(f)(x) = \sup_{y\in\mathcal{M}} \{f(y) + \mathfrak{b}(x,y)\}, \quad (15)$$

$$\varepsilon_{\mathfrak{b}}(f)(x) = \inf_{y\in\mathcal{M}} \{f(y) - \mathfrak{b}(y,x)\}. \quad (16)$$

Note that this formulation has been considered recently in the framework of adaptive morphology (Ćurić and Luengo-Hendriks, 2013). Both are increasing operators which, by the maximality at the diagonal, preserves the extrema. By the non-positivity, Riemannian dilation is extensive and erosion is anti-extensive. In addition, we can easily check that the pair $(\varepsilon_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ forms an adjunction as in Proposition 3. Consequently, their composition leads to the *Riemannian opening and closing according to the admissible Riemannian structuring function* $\mathfrak{b}$ given respectively by:

$$\gamma_{\mathfrak{b}}(f)(x) = \sup_{z\in\mathcal{M}} \inf_{y\in\mathcal{M}} \{f(y) - \mathfrak{b}(y,z) + \mathfrak{b}(z,x)\}, \quad (17)$$

$$\varphi_{\mathfrak{b}}(f)(x) = \inf_{z\in\mathcal{M}} \sup_{y\in\mathcal{M}} \{f(y) + \mathfrak{b}(z,y) - \mathfrak{b}(x,z)\}. \quad (18)$$

Remarkably, the symmetry of $\mathfrak{b}$ is not a necessary condition for the adjunction. Examples of such asymmetric structuring functions have recently appeared in the context of stochastic morphology (Angulo and Velasco-Forero, 2013), non-local morphology (Velasco-Forero and Angulo, 2013) and saliency-based adaptive morphology (Ćurić and Luengo-Hendriks, 2013).

In our framework, we propose a general form of any admissible Riemannian structuring function $\mathfrak{b}(x,y)$, $\forall x,y \in \mathcal{M}$, which should be decomposable into the sum of two terms:

$$\mathfrak{b}(x,y) = \alpha\mathfrak{b}^{sym}(x,y) + \beta\mathfrak{b}^{asym}(x,y), \;\; \alpha,\beta \geq 0. \quad (19)$$

Symmetric structuring function. The symmetric term of the structuring function will be a scaled p-norm shaped function depending exclusively on the Riemannian distance, i.e., $\mathfrak{b}^{sym}(x,y) = \mathfrak{b}^{sym}(y,x) = k_{\lambda,p}\left(d_{\mathcal{M}}(x,y)\right)$ such that

$$k_{\lambda,p}(\eta) = -C_p \frac{\eta^{\frac{p}{p-1}}}{\lambda^{\frac{1}{p-1}}}; \;\; \lambda > 0, \;\; p > 1,$$

where the normalization factor is given by $C_p = (p-1)p^{-\frac{p}{p-1}}$. We note that with the shape parameter $p = 2$ we recover the canonic quadratic structuring function. In fact, this generalization of the quadratic structuring is inspired from the solution of a generalized morphological PDE (Lions et al., 1987): $u_t(t,x) + \|u_x(t,x)\|^p = 0$, $(t,x) \in (0,+\infty) \times E$; $u(0,x) = f(x)$, $x \in E$, since the quadratic one is the solution of the classical (Hamilton-Jacobi) morphological PDE (Bardi et al., 1984; Crandall et al., 1992): $u_t(t,x) + \|u_x(t,x)\|^2 = 0$. Asymptotically, one is dealing with almost flat shapes over $\mathcal{M}$ as $p \to 1$; as $p > 2$ increases and $p \to \infty$ the shape of $k_{\lambda,p}(\eta)$ evolves from a parabolic shape $p = 2$, i.e., term on $d_{\mathcal{M}}(x,y)^2$, to the limit case, which is a conic shape, i.e., term on $d_{\mathcal{M}}(x,y)$.

We note that if $\mathcal{M}$ is a Cartan–Hadamard manifold, the symmetric part $\mathfrak{b}^{sym}(x,y)$ is a concave function for any $\lambda > 0$ and any $p > 1$.

Asymmetric structuring function. Relevant forms of the asymmetric term is an open issue on Riemannian morphology, which will probably allows to introduce more advanced morphological operators. For instance, we can fix a reference point $o \in \mathcal{M}$ and define, for $x,y \in \mathcal{M}$, $y \neq o$, the function

$$\mathfrak{b}^{asym}_{\lambda,o}(x,y) = -\frac{1}{2\lambda}\frac{d_{\mathcal{M}}(x,y)^2}{d_{\mathcal{M}}(y,o)^2}.$$

The assignment $x \mapsto \mathfrak{b}^{asym}_{\lambda,o}(x,y)$ involves a shape strongly deformed near the reference point. One can also replace the reference point by a set $O \subset \mathcal{M}$, hence changing $d_{\mathcal{M}}(y,o)$ by the distance function $d_{\mathcal{M}}(y,O)$.

An alternative asymmetric function could be based on the notion of Busemann function (Ballmann et al., 1985). Given a point $x \in \mathcal{M}$ and a ray γ starting at x in the direction of the tangent vector $\mathbf{v}$, i.e., a unit-speed geodesic line $\gamma : [0,\infty) \to \mathcal{M}$ such that $d_{\mathcal{M}}(\gamma(0),\gamma(t)) = t$ for all $t \geq 0$, one defines its Busemann function $b_{\gamma_{x,\mathbf{v}}}$ by the formula

$$\begin{aligned} b_{\gamma_{x,\mathbf{v}}}(y) &= \lim_{t\to\infty}\left[d_{\mathcal{M}}\left(x,\gamma_{x,\mathbf{v}}(t)\right) - d_{\mathcal{M}}\left(y,\gamma_{x,\mathbf{v}}(t)\right)\right] \\ &= \lim_{t\to\infty}\left[t - d_{\mathcal{M}}\left(y,\gamma_{x,\mathbf{v}}(t)\right)\right]. \end{aligned}$$

Since $t - d_{\mathcal{M}}\left(y,\gamma_{x,\mathbf{v}}(t)\right)$ is bounded above by $d_{\mathcal{M}}\left(x,\gamma_{x,\mathbf{v}}(0)\right)$ and is monotone non-decreasing in t, the limit always exists. It follows that $|b_{\gamma_{x,\mathbf{v}}}(y) - b_{\gamma_{x,\mathbf{v}}}(z)| \leq d_{\mathcal{M}}(y,z)$, i.e., Busemann function is Lipschitz with constant 1. If $\mathcal{M}$ has non-negative sectional curvature $b_{\gamma_{x,\mathbf{v}}}(y)$ is convex. If $\mathcal{M}$ is Cartan–Hadamard manifold, it is concave. Consequently, we can define our asymmetric structuring function as

$$\mathfrak{b}^{asym}_{\lambda,\mathbf{v}}(x,y) = \begin{cases} -(2\lambda)^{-1}\, b_{\gamma_{x,\mathbf{v}}}(y) & \text{if sect. curvature of } \mathcal{M} \geq 0 \\ (2\lambda)^{-1}\, b_{\gamma_{x,\mathbf{v}}}(y) & \text{if sect. curvature of } \mathcal{M} < 0 \end{cases}$$

From a practical viewpoint, asymmetric structuring functions obtained by Busemann functions allow to introduce a shape which depends on the distance between the point x and a kind of orthogonal projection of point y on the geodesic along the direction $\mathbf{v}$. Hence, it could be a way to introduce directional Riemannian operators.

5. Parallel transport of a fixed external structuring function

Previous Riemannian morphological operators are based on geodesic structuring functions $\mathfrak{b}(x;y)$ which are defined by the geodesic distance function on $\mathcal{M}$. Let us consider now the case where a prior (semi-continuous) structuring function b external to $\mathcal{M}$ is given and it should be adapted to each point $x\in\mathcal{M}$. Our approach is inspired from Roerdink (1994) formulation of dilation/erosion for binary images on smooth surfaces.

5.1. Manifold morphology

The idea behind the binary Riemannian morphology on smooth surfaces introduced in (Roerdink, 1994) is to replace the translation invariance by the parallel transport (the transformations are referred to as "covariant" operations). Let $\mathcal{M}$ be a (geodesically complete) Riemannian manifold and $\mathcal{P}(\mathcal{M})$ denotes the set of all subsets of M. A binary image X on the manifold is just $X\in\mathcal{P}(\mathcal{M})$. Let $A\subset\mathcal{M}$ be the basic structuring, a subset which is defined on the tangent space at a given point ω of $\mathcal{M}$ by $\tilde{A}=\mathrm{Log}_{\omega}(A)\subset T_{\omega}\mathcal{M}$. Let $\gamma=\gamma_{[p,q]}$ be a path from p to q, then the operator

$$\tau_{\gamma}(A)=\mathrm{Exp_q}\,P_{\gamma}\,\mathrm{Log_p}(A)=B,$$

transports the subset A of p to the set B of q. As the image of the set X under parallel translation from p to q will depend in general on which path is taken; the solution proposed in (Roerdink, 1994), denoted by $\delta_A^{Roerdink}$, is to consider all possible paths from p to q. The mapping $\delta_A^{Roerdink}:\mathcal{P}(\mathcal{M})\to\mathcal{P}(\mathcal{M})$ given by

$$\delta_A^{Roerdink}(X)=\bigcup_{x\in\mathcal{M}}\bigcup_{\gamma}\tau_{\gamma}(A)=\bigcup_{x\in\mathcal{M}}\bigcup_{\gamma}\mathrm{Exp_x}\,P_{\gamma_{[\omega,x]}}\,\mathrm{Log}_{\omega}(A), \tag{20}$$

is a dilation of image X according to the structuring element A. Using the symmetry group morphology (Roerdink, 2000), this operator can be rewritten as

$$\delta_A^{Roerdink}(X)=\bigcup_{x\in\mathcal{M}}\mathop{\mathrm{Exp}}_{x}P_{\gamma_{[\omega,x]}}\mathop{\mathrm{Log}}_{\omega}(\bar{A}),$$

where $\bar{A}=\bigcup_{s\in\Sigma}sA$, with Σ being the holonomy group around the normal at ω. For instance, if $\tilde{A}=\mathrm{Log}_{\omega}(A)$ is a line segment of length r starting at ω then $\bar{A}$ is a disk of radius r centered at ω.

5.2. b_{ω}-transported Riemannian dilation and erosion

Coming back to our framework of real-valued images on a geodesically complete Riemannian manifold $\mathcal{M}$. From our viewpoint, it seems more appropriate to fix the reference structuring element as a Boolean set S on the tangent space at the reference point $\omega\in\mathcal{M}$, i.e., $S_{\omega}\subset T_{\omega}\mathcal{M}$. More precisely, let S_{ω} be a compact set which contains the origin of $T_{\omega}\mathcal{M}$. We can now formulate the *S_{ω}-transported flat Riemannian dilation and erosion* as

$$\breve{\delta}_{S_{\omega}}(f)(x) = \sup\left\{f(y):\ y\in\mathrm{Exp_x}\,P_{\gamma^{geo}_{[\omega,x]}}\check{S}_{\omega}\right\}, \tag{21}$$

$$\breve{\varepsilon}_{S_{\omega}}(f)(x) = \inf\left\{f(y):\ y\in\mathrm{Exp_x}\,P_{\gamma^{geo}_{[\omega,x]}}S_{\omega}\right\}. \tag{22}$$

Thus, in comparison to dilation (20), we prefer to consider in our case that the parallel transport from ω to x is done exclusively along the geodesic path $\gamma^{geo}_{[\omega,x]}$ between ω and x, i.e., if S_{ω} is a line in ω then it will be also at x a line, but rotated.

This idea leads to a natural extension to the case where the fixed datum is an upper-semicontinuous structuring function $b_{\omega}(\mathbf{v})$, defined in the Euclidean tangent space at ω, i.e., $b_{\omega}:T_{\omega}\mathcal{M}\to[-\infty,0]$. Let consider now the upper level sets (or cross-section) of b_{ω} obtained by thresholding at a value l:

$$X_l(b_{\omega})=\{\mathbf{v}\in T_{\omega}\mathcal{M}\ :\ b_{\omega}(\mathbf{v})\geq l\},\ \ \forall l\in[-\infty,0]. \tag{23}$$

The set of upper level sets constitutes a family of decreasing closed sets: $l\geq m\Rightarrow X_l\subseteq X_m$ and $X_l=\cap\{X_m,m<l\}$. Any function $b_{\omega}(\mathbf{v})$ can be now viewed as a unique stack of its cross-sections, which leads to the following reconstruction property:

$$b_{\omega}(\mathbf{v})=\sup\{l\in[-\infty,0]\ :\ \mathbf{v}\in X_l(b_{\omega})\},\ \ \forall\mathbf{v}\in T_{\omega}\mathcal{M}. \tag{24}$$

Using this representation, the corresponding Riemannian structuring function at ω is given by $\mathfrak{b}_{\omega}(\omega,y)=\sup\{l\in[-\infty,0]:z\in\mathrm{Exp}_{\omega}X_l(b_{\omega})\}$. In the case of a different point $x\in\mathcal{M}$, the cross-section should be transported to the tangent space of x before mapping back to $\mathcal{M}$, i.e.,

$$\mathfrak{b}_{\omega}(x,y)=\sup\left\{l\in[-\infty,0]:\ z\in\mathrm{Exp_x}\,P_{\gamma^{geo}_{[\omega,x]}}X_l(b_{\omega})\right\}.$$

Finally, the *b_{ω}-transported Riemannian dilation and erosion* of image f are given respectively by

$$\breve{\delta}_{b_{\omega}}(f)(x) = \sup_{y\in\mathcal{M}}\{f(y)+\mathfrak{b}_{\omega}(x,y)\}, \tag{25}$$

$$\breve{\varepsilon}_{b_{\omega}}(f)(x) = \inf_{y\in\mathcal{M}}\{f(y)-\mathfrak{b}_{\omega}(y,x)\}. \tag{26}$$

Obviously, the case of a concave structuring function b_{ω} is particularly well defined since in such a case, its cross-sections are convex sets. In addition, if $\mathcal{M}$ is a Cartan–Hadamard manifold, the corresponding Riemannian structuring function $\mathfrak{b}_{\omega}(x,y)$ is also a concave function.

A typical useful case consists in taking at reference ω the structuring function:

$$b_{\omega}(\mathbf{v})=-\frac{\mathbf{v}^TQ\mathbf{v}}{2}$$

where Q is a $d \times d$ symmetric positive definite matrix, d being the dimension of manifold $\mathcal{M}$. It corresponds just to a generalized quadratic function such that the eigenvectors of Q define the principal directions of the concentric ellipsoids and the eigenvalues their eccentricity. Therefore, we can introduce by means of Q an anisotropic/directional shape on $\mathfrak{b}_\omega(x, y)$. We can easily check that $Q = \frac{1}{\lambda} I$, I being the identity matrix of dimension d, corresponds just to the canonic Riemannian dilation and erosion (1) and (2).

Without an explicit expression of the exponential map, we cannot compute straightforwardly the b_ω-transported Riemannian dilation and erosion on a Riemannian manifold $\mathcal{M}$. This is for instance the situation when is f is an image on a 3D smooth surface. Hence, in the case of applications to valued surfaces, manifold learning techniques as LOGMAP (Brun et al., 2005) can be used to numerically obtain the transported cross-sections on $\mathcal{M}$.

6. Connections with classical Euclidean morphology

6.1. Spatially-invariant operators

First of all, it is obvious that the Riemannian dilation/erosion naturally extends the quadratic Euclidean dilation/erosion for images $\mathcal{F}(\mathbb{R}^d, \overline{\mathbb{R}})$ by considering that the intrinsic distance is the Euclidean one (or the discrete one for $\mathbb{Z}^d$), i.e., $d_\mathcal{M}(x, y) = \|x - y\| = d_{space}(x, y)$.

By the way, we note also that definition of the Riemannian flat dilation and erosion of size r given in (8) and (9) are compatible with the formulation of the classical geodesic dilation and erosion (Lantuejoul and Beucher, 1981) of size r of image f (marker) constrained by the image g (reference or mask), $\delta_{g,\lambda}(f)$ and $\varepsilon_{g,\lambda}(f)$, which underly the operators by reconstruction (Soille, 1999), where the upper-level sets of the reference image g are considered as the manifold $\mathcal{M}$ where the geodesic distance is defined.

6.2. Adaptive (spatially-variant) operators

From (Kimmel et al., 1997), the idea of embedding a 2D grey-level image $f \in \mathcal{F}(\mathbb{R}^2, \overline{\mathbb{R}})$, $x = (x_1, x_2)$, into a surface embedded in $\mathbb{R}^3$, i.e.,

$$f(x) \mapsto \xi_x = (x_1, x_2, \alpha f(x_1, x_2)),\ \alpha > 0,$$

where α is a scaling parameter useful for controlling intensity distances, has become popular in differential geometry inspired image processing. This embedded Riemannian manifold $\mathcal{M} = \mathbb{R}^2 \times \overline{\mathbb{R}}$ has a product metric of type $ds^2_\mathcal{M} = ds^2_{space} + \alpha ds^2_f$, where $ds^2_{space} = dx_1^2 + dx_2^2$ and $ds^2_f = df^2$. The geodesic distance between two points $\xi_x, \xi_y \in \mathcal{M}$ is the length of the shortest path between the points, i.e., $d_\mathcal{M}(\xi_x, \xi_y) = \min_{\gamma=\gamma_{[\xi_x,\xi_y]}} \int_\gamma ds_\mathcal{M}$.

As shown in (Welk et al., 2011), this is essentially the framework behind the morphological amoebas (Lerallut et al., 2007), which are flat spatially adaptive structuring functions centered in a point x, $A_\lambda(x)$, computed by thresholding the geodesic distance at radius $\lambda > 0$, i.e., $A_\lambda(x) = \left\{ y \in E : d_\mathcal{M}(\xi_x, \xi_y) < \lambda \right\}$. In the discrete setting, the geodesic distance is given by

$$d_\mathcal{M}(\xi_x, \xi_y) = \min_{\{\xi^1=\xi_x, \xi^2, \cdots, \xi^N=\xi_y\}} \sum_{i=1}^{N} \alpha |f(x^i) - f(x^{i+1})| + \sqrt{(x_1^i - x_1^{i+1})^2 + (x_2^i - x_2^{i+1})^2}. \quad (27)$$

We should remark that for $x \to y$ and assuming a smooth manifold, the geodesic distance is asymptotically equivalent to the corresponding distance in the Euclidean product space, i.e.,

$$d_\mathcal{M}(\xi_x, \xi_y)^2 \approx d_{space}(x, y)^2 + \alpha^2 |f(x) - f(y)|^2, \quad (28)$$

which is the distance appearing in the bilateral structuring functions (Angulo, 2013). We can also see that the salience maps behind the salience adaptive structuring elements (Ćurić et al., 2012) can be approached in a Riemannian formulation by choosing the appropriate metric.

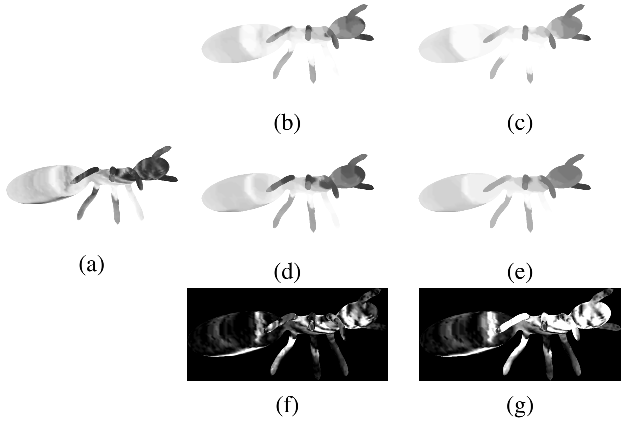

Figure 2: Morphological processing of real valued 3D surface: (a) original image on a surface $\mathcal{S} \subset \mathbb{R}^3$, $f(x) \in \mathcal{F}(\mathcal{S}, \mathbb{R}_+)$; (b) and (c) Riemannian dilation $\delta_\lambda(f)(x)$ with respectively $\lambda = 4$ and $\lambda = 8$; (d) and (e) Riemannian closing $\varphi_\lambda(f)(x)$ with respectively $\lambda = 4$ and $\lambda = 8$; (f) and (g) residue between the original surface and the Riemannian closings $\varphi_\lambda(f)(x) - f(x)$, $\lambda = 4$ and $\lambda = 8$.

7. Various useful case studies

7.1. Hyperbolic embedding of an Euclidean positive image into Poincaré half-space $\mathcal{H}^3$

Shortest path distance (27) is not invariant to scaling of image intensity, i.e., $f \mapsto f' = \beta f$, $\beta > 0$ involves that $|f'(x^i) - f'(x^{i+1})| = \beta |f(x^i) - f(x^{i+1})|$ and hence the shape of the corresponding Riemannian structuring function for f and f' will be different. This lack of contrast invariance can be easily solved by using a logarithmic metric in the intensities. Hence, if we assume positive intensities, $f(x) > 0$, for all $x \in \mathcal{M}$, we can consider the distance $d_\mathcal{M}(\xi_x, \xi_y) = \min_{\gamma_{\xi_x,\xi_y}} \sum_{i=1}^{N} d_{space}(x^i, x^{i+1}) + \alpha |\log f(x^i) - \log f(x^{i+1})|$. This metric can be connected to the logarithmic image processing (LIP) model (Jourlin and Pinoli, 1988). This geometry can be also justified from a human perception viewpoint. The classical Weber-Fechner law states that human sensation is proportional to the logarithm of the stimulus intensity. In the case of vision, the eye senses brightness approximately according to the Weber-Fechner law over a moderate range.

Following the same assumption of positive intensities, we can also consider that a 2D image can be embedded into the hyperbolic space $\mathcal{H}^3$ (Cannon et al., 1997). More particularly the (Poincaré) upper half-space model is the domain $\mathcal{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ with the Riemannian metric $ds^2_{\mathcal{H}^3} = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}$. This space has constant negative sectional curvature. If we consider the image embedding $f(x) \mapsto \xi_x = (x_1, x_2, f(x_1, x_2)) \in \mathcal{H}^3$, the Riemannian distance needed for morphological operators will be given by

$$d_{\mathcal{M}}(\xi_x, \xi_y) = \min_{\gamma_{\xi_x,\xi_y}} \sum_{i=1}^{N} \cosh^{-1}\left(1 + \frac{(x_1^i - x_1^{i+1})^2 + (x_2^i - x_2^{i+1})^2 + (f(x^i) - f(x^{i+1}))^2}{2f(x^i)f(x^{i+1})}\right). \tag{29}$$

The geometry of this space is extremely rich in particular concerning the invariance and isometric symmetry. Hence, distance (29) is for instance invariant to translations $\xi = (x_1, x_2, x_3) \mapsto \xi + \alpha$, $\alpha \in \mathbb{R}$, scaling $\xi \mapsto \beta\xi$, $\beta > 0$. A specific theory on granulometric scale-space properties in this embedding can be intended.

7.2. *Embedding an Euclidean image into the structure tensor manifold*

Besides the space×intensity embeddings discussed just above, we can consider other more alternative non-Euclidean geometric embedding of scalar images, using for instance the local structure.

More precisely, given a 2D Euclidean image $f(x) = f(x_1, x_2) \in \mathcal{F}(\mathbb{R}^2, \overline{\mathbb{R}})$, the structure tensor representing the local orientation and edge information (Förstner and Gülch, 1987) is obtained by Gaussian smoothing of the dyadic product $\nabla f \nabla f^T$:

$$S(f)(x) = G_\sigma * \left(\nabla f(x_1, x_2) \nabla f(x_1, x_2)^T\right) = \begin{pmatrix} s_{x_1 x_1}(x_1, x_2) & s_{x_1 x_2}(x_1, x_2) \\ s_{x_1 x_2}(x_1, x_2) & s_{x_2 x_2}(x_1, x_2) \end{pmatrix}$$

where $\nabla f(x_1, x_2) = \left(\frac{\partial f(x_1,x_2)}{\partial x_1}, \frac{\partial f(x_1,x_2)}{\partial x_2}\right)^T$ is the 2D spatial intensity gradient and G_σ stands for a Gaussian smoothing with a standard deviation of σ. From a mathematical viewpoint, $S(f)(x) : E \rightarrow \mathrm{SPD}(2)$ is an image where at each pixel we have a symmetric positive (semi-)definite matrix 2×2. The differential geometry in the manifold $\mathrm{SPD}(n)$ is very well-known (Bhatia, 2007). Namely, the metric is given by $ds^2_{SPD(n)} = \mathrm{tr}(M^{-1}dMM^{-1}dM)$ and the Riemannian distance is defined as $d_{SPD(n)}(M_1, M_2) = \| \log\left(M_1^{-1/2} M_2 M_1^{-1/2}\right) \|_F$, $\forall M_1, M_2 \in \mathrm{SPD}(n)$. Let consider now the embedding $f(x) \mapsto \xi_x = (x_1, x_2, \alpha S(f)(x_1, x_2))$, $\alpha > 0$, in the product manifold $\mathcal{M} = \mathbb{R}^2 \times \mathrm{SPD}(2)$, which has the product metric $ds^2_{\mathcal{M}} = ds^2_{space} + \alpha ds^2_{SPD(2)}$. It is a (complete, not compact, negative sectional curved) Riemannian manifold of geodesic distance given by $d_{\mathcal{M}}(\xi_x, \xi_y) = \min_{\gamma_{\xi_x,\xi_y}} \sum_{i=1}^{N} d_{space}(x^i, x^{i+1}) + \alpha d_{SPD(n)}(S(f)(x^i), S(f)(x^{i+1}))$, which is asymptotically equal to $d_{\mathcal{M}}(\xi_x, \xi_y)^2 \approx d_{space}(x, y)^2 + \alpha d_{SPD(2)}(S(f)(x), S(f)(y))^2$.

By means of this embedding, we can compute anisotropic morphological operators following the flow coherence of image structures. This embedding is related to previous adaptive approaches such as (Verdú et al., 2011) and (Landström and Thurley, 2013).

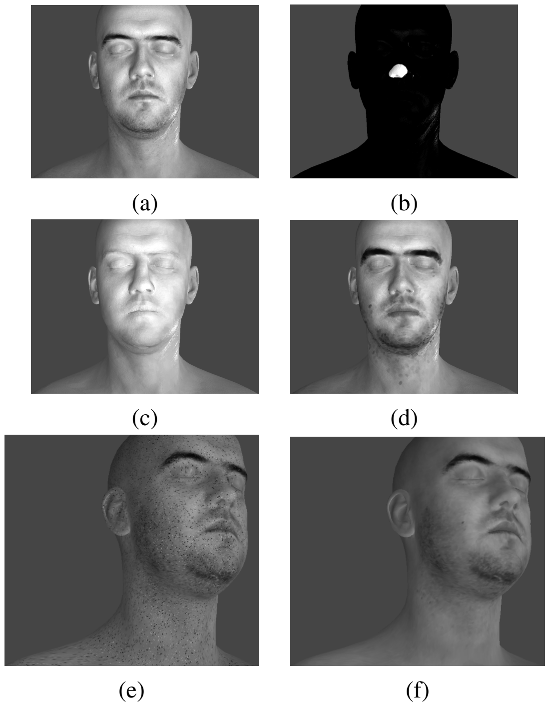

Figure 3: Morphological processing of real valued 3D surface of a face: (a) original image on a surface $\mathcal{S} \subset \mathbb{R}^3$, $f(x) \in \mathcal{F}(\mathcal{S}, \mathbb{R}_+)$; (b) example of geodesic ball $B_r(x)$ at a given point $x \in \mathcal{S}$; (d) and (e) Riemannian dilation $\delta_\lambda(f)(x)$ and Riemannian erosion $\varepsilon_\lambda(f)(x)$ with $\lambda = 0.5$; (e) nonsmooth version of surface (added impulse noise); (f) filtered surface obtained by Lasry–Lions regularizers.

7.3. *Morphological processing of real valued 3D surfaces*

In Fig. 2(a) is given an example of real-valued 3D surface, i.e., the image to be processed is $f : \mathcal{S} \subset \mathbb{R}^3 \rightarrow \overline{\mathbb{R}}$. In practice, the 3D surface is represented by a mesh (i.e., triangulated manifold with a discrete structure composed of vertices, edges and faces). In our example, the grey-level intensities are supported on the vertices. In the case of a discrete approximation of a manifold based on mesh representation, the geodesic distance $d_{\mathcal{S}}(x, y)$ can be calculated by the Floyd–Warshall algorithm for finding shortest path in the weighted graph of vertices of the mesh. Efficient algorithms are based on Fast Marching generalized to arbitrary triangulations Kimmel and Sethian (1998). Fig. 2 depicts examples of Riemannian dilation $\delta_\lambda(f)$ and Riemannian closing $\varphi_\lambda(f)$, for two different scales ($\lambda = 4$ and $\lambda = 8$) and the corresponding dual top-hats.

Another example of real valued surface is given in Fig. 3. It corresponds to the 3D acquisition of a face. We observe how the canonic Riemannian dilation and erosion are able to locally process the face details taking into the geometry of the surface. In Fig. 3 is also given an example of image filtering using the composition our Lasry–Lions regularizers (15) (with $\lambda = 4$ and $\mu = 2$), where the original surface is a nonsmooth version obtained by adding impulse noise.

8. Conclusions

We have introduced in this paper a general theory for the formulation of mathematical morphology operators for images valued on Riemannian manifolds. We have defined the main operators and studied their fundamental properties. We have considered two main families of operators. On the one hand, morphological operators based on an admissible Riemannian structuring function which is adaptively obtained for each point x according to the geometry of the manifold. On the other hand, morphological operators founded on an external Euclidean structuring function which is parallel transported to the tangent space at each point x and then mapped to the manifold. We have also discussed some original Riemannian embedding of Euclidean images onto Cartan–Hadamard manifolds. This is the case of the Poincaré half-space $\mathcal{H}^3$ as well as the structure tensor manifold. Riemannian structuring functions defined on Cartan–Hadamard manifolds are particular rich in terms of scale-space properties as well as in Lipschitz regularization.

Acknowledgment. The authors would like to thank the anonymous reviewer who pointed out the problem of the result on Lasry-Lions regularization for the case of unbounded support space or unbounded functions.

Remark on related work. In the last stages of writing this paper, we learned of the work (Azagra and Ferrera, 2014) where it is provided a complete analysis of the generalization of Lasry-Lions regularization for bounded functions in manifolds of bounded sectional curvature.

References

J. Angulo. Morphological Bilateral Filtering. *SIAM Journal on Imaging Sciences,* 6(3):1790–1822, 2013.

J. Angulo and S. Velasco-Forero. Mathematical morphology for real-valued images on Riemannian manifolds. In *Proc. of ISMM'13 (11th International Symposium on Mathematical Morphology)*, Springer LNCS 7883, p. 279–291, 2013.

J. Angulo and S. Velasco-Forero. Stochastic Morphological Filtering and Bellman-Maslov Chains. In *Proc. of ISMM'13 (11th International Symposium on Mathematical Morphology)*, Springer LNCS 7883, p. 171–182, 2013.

D. Attouch, D. Aze. Approximation and regularization of arbitray functions in Hilbert spaces by the Lasry-Lions method. *Annales de l'I.H.P., section C,* 10(3): 289–312, 1993.

D. Azagra, J. Ferrera. Inf-Convolution and Regularization of Convex Functions on Riemannian Manifols of Nonpositive Curvature. *Rev. Mat. Complut.* 19(2): 323–345, 2006.

D. Azagra, J. Ferrera. Regularization by sup-inf convolutions on Riemannian manifolds: an extension of Lasry-Lions theorem to manifolds of bounded curvature. arXiv preprint arXiv:1401.5053, 2014.

W. Ballmann, M. Gromov, V. Schroeder. *Manifolds of nonpositive curvature.* Progr. Math., 61, Birkhäuser, 1985.

M. Bardi, L.C. Evans. On Hopf's formulas for solutions of Hamilton- Jacobi equations. *Nonlinear Analysis, Theory, Methods and Applications,* 8(11):1373–1381, 1984.

R. Bhatia. *Positive Definite Matrices.* Princeton University Press, 2007.

P. Bernard. Lasry-Lions regularization and a lemma of Ilmanen. *Rend. Semin. Mat. Univ. Padova,* 124:221–229, 2010.

P. Bernard, M. Zavidovique. Regularization of Subsolutions in Discrete Weak KAM Theory. Canadian Journal of Mathematics, 65:740–756, 2013.

M. Berger, B. Gostiaux. *Differential Geometry: Manifolds, Curves, and Surfaces*. Springer, 1987.

R. van den Boomgaard, L. Dorst. The morphological equivalent of Gaussian scale-space. In *Proc. of Gaussian Scale-Space Theory,* 203–220, Kluwer, 1997.

A. Brun, C.-F. Westin, M. Herberthson, H. Knutsson. Fast Manifold Learning Based on Riemannian Normal Coordinates. In *Proc. of 14th Scandinavian Conference (SCIA'05),* Springer LNCS 3540, 920–929, 2005.

J.W. Cannon, W.J. Floyd, R. Kenyon, W.R. Parry. *Hyperbolic Geometry.* Flavors of Geometry, MSRI Publications, Vol. 31, 1997.

M.G. Crandall, H. Ishii, P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society,* 27(1):1–67, 1992.

V. Ćurić, C.L. Luengo Hendriks, G. Borgefors. Salience adaptive structuring elements. *IEEE Journal of Selected Topics in Signal Processing,* 6(7): 809–819, 2012.

V. Ćurić and C.L. Luengo-Hendriks. Salience-Based Parabolic Structuring Functions. In *Proc. of ISMM'13 (11th International Symposium on Mathematical Morphology)*, Springer LNCS 7883, p. 183–194, 2013.

J. Debayle and J. C. Pinoli. Spatially Adaptive Morphological Image Filtering using Intrinsic Structuring Elements. *Image Analysis and Stereology,* 24(3):145–158, 2005.

W. Förstner, E. Gülch. A fast operator for detection and precise location of distinct points, corners and centres of circular features. In *Proc. of ISPRS Intercommission Conference on Fast Processing of Photogrammetric Data,* p. 281–304, 1987.

H.J.A.M. Heijmans. *Morphological image operators.* Academic Press, Boston, 1994.

P.T. Jackway, M. Deriche. Scale-Space Properties of the Multiscale Morphological Dilation-Erosion. *IEEE Trans. Pattern Anal. Mach. Intell.,* 18(1): 38–51, 1996.

M. Jourlin, J.C. Pinoli. A model for logarithmic image processing. *Journal of Microscopy,* 149(1):21–35, 1988.

R. Kimmel, N. Sochen, R. Malladi. Images as embedding maps and minimal surfaces: movies, color, and volumetric medical images. In *Proc. of IEEE CVPR'97,* pp. 350–355, 1997.

R. Kimmel, J.A. Sethian. Computing geodesic paths on manifolds. *Proc. of National Academy of Sci.,* 95(15): 8431–8435, 1998.

C. Kiselman. Division of mappings between complete lattices. In *Proc. of the 8th International Symposium on Mathematical Morphology (ISMM'07),* Rio de Janeiro, Brazil, MCT/INPE, vol. 1, p. 27–38.

E.J. Kraus, H.J.A.M. Heijmans, E.R. Dougherty. Gray-scale granulometries compatible with spatial scalings. Signal Processing, 34(1): 1–17, 1993.

A. Landström, M.J. Thurley. Adaptive morphology using tensor-based elliptical structuring elements. *Pattern Recognition Letters,* 34(12): 1416–1422, 2013.

S. Lang. *Fundamentals of differential geometry.* Springer-Verlag, 1999.

C. Lantuejoul, S. Beucher. On the use of the geodesic metric in image analysis. *Journal of Microscopy,* 121(1): 39–49, 1981.

J.M. Lasry, P.-L. Lions. A remark on regularization in Hilbert spaces. *Israel Journal of Mathematics,* 55: 257–266, 1986

R. Lerallut, E. Decencière, F. Meyer. Image filtering using morphological amoebas. *Image and Vision Computing,* 25(4): 395–404, 2007.

P.-L. Lions, P.E. Souganidis, J.L. Vásquez. The Relation Between the Porous Medium and the Eikonal Equations in Several Space Dimensions. *Revista Matemática Iberoamericana,* 3 : 275–340, 1987.

P. Maragos. Slope Transforms: Theory and Application to Nonlinear Signal Processing. *IEEE Trans. on Signal Processing,* 43(4): 864–877, 1995.

G. Matheron. *Random sets and integral geometry.* John Wiley & Sons, 1975.

J.B.T.M. Roerdink. Manifold shape: from differential geometry to mathematical morphology. In *Shape in Picture,* NATO ASI F 126, pp. 209–223, Springer, 1994.

J.B.T.M. Roerdink. Group morphology. *Pattern Recognition,* 33: 877–895, 2000.

J.B.T.M. Roerdink. Adaptivity and group invariance in mathematical morphology. In *Proc. of ICIP'09,* 2009.

J. Serra. *Image Analysis and Mathematical Morphology,* Academic Press, London, 1988.

J. Serra. *Image Analysis and Mathematical Morphology. Vol II: Theoretical Advances,* Academic Press, London, 1988.

P. Soille. *Morphological Image Analysis,* Springer-Verlag, Berlin, 1999.

S. Velasco-Forero and J. Angulo. On Nonlocal Mathematical Morphology. In *Proc. of ISMM'13 (11th International Symposium on Mathematical Morphology)*, Springer LNCS 7883, p. 219–230, 2013.

R. Verdú, J. Angulo and J. Serra.Anisotropic morphological filters with spatially-variant structuring elements based on image-dependent gradient

fields. *IEEE Trans. on Image Processing*, 20(1): 200–212, 2011.
C. Villani. *Optimal transport, old and new,* Grundlehren der mathematischen Wissenschaften, Vol.338, Springer-Verlag, 2009.
M. Welk, M. Breuß, O. Vogel. Morphological amoebas are self-snakes. *Journal of Mathematical Imaging and Vision,* 39(2):87–99, 2011.