Lipschitz Regularization of Images supported on Surfaces using Riemannian Morphological

Jesus Angulo

Lipschitz Regularization of Images supported on Surfaces using Riemannian Morphological Operators

Jesús Angulo

MINES ParisTech, PSL-Research University,
CMM-Centre de Morphologie Mathématique, France
jesus.angulo@mines-paristech.fr

Abstract. Different imaging modalities produce nowadays images on smooth surfaces, represented by images painted on meshes or point clouds. These Riemannian images are often nonsmooth and their regularization can be needed in many applications. This paper deals with the approximation of a bounded nonsmooth image painted on a surface by a sequence of more regular functions, having in particular Lipschitz gradient, and without any hypothesis of differentiability. We adopt here a geometric framework known as Lasry–Lions regularization. The aim of the present contribution is to consider the extension of Lasry–Lions regularization to Riemannian manifolds. We show that the key ingredients for such regularization are Riemannian morphological operators.

1 Introduction

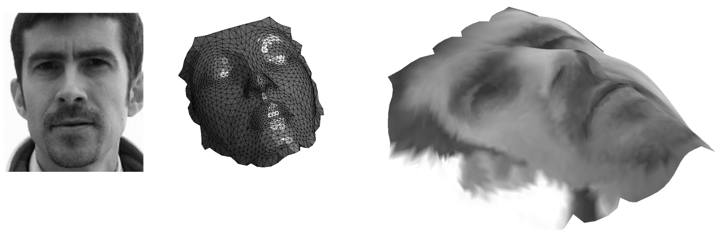
Mathematical morphology is a nonlinear image processing methodology based on two basic operators, dilation and erosion, which correspond respectively to the convolution in the $(\max, +)$ algebra and its dual. More precisely, in Euclidean (translation invariant) mathematical morphology the pair of adjoint and dual operators, dilation (sup-convolution) $(f \oplus b)(x)$ and erosion (inf-convolution) $(f \ominus b)(x)$ of an image $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, are given by [25,26]:

$$\begin{cases} \delta_b(f)(x) = (f \oplus b)(x) = \sup_{y \in E} \left\{ f(y) + b(y - x) \right\}, \\ \varepsilon_b(f)(x) = (f \ominus b)(x) = \inf_{y \in E} \left\{ f(y) - b(y + x) \right\}, \end{cases}$$

where $b : \mathbb{R}^n \to \overline{\mathbb{R}}$ is the structuring function which determines the effect of the operator. The structuring function plays a similar role to the kernel in classical convolution filtering. The structuring function is typically a parametric family $b_\lambda(x)$, where $\lambda > 0$ is the scale parameter. In particular, the canonic structuring function is the parabolic shape (i.e., square of the Euclidean distance):

$$b_\lambda(x) = q_\lambda(x) = -\frac{\|x\|^2}{2\lambda}.$$

such that the corresponding dilation and erosion are equivalent to the Lax-Oleinik operators or viscosity solution of the standard Hamilton-Jacobi PDE:



(a) (b)

Fig. 1. (a) Euclidean real-valued function + smooth surface (as a discrete mesh); (b) real-valued function on smooth surface.

$u_t(t,x) \mp \|u_x(t,x)\|^2 = 0$, $(t,x) \in (0,+\infty) \times E$; $u(0,x) = f(x)$, $x \in E$. Theory of morphological filtering is based on opening and closing operators, obtained respectively by product composition of erosion-dilation and dilation-erosion. Opening (resp. closing) is increasing, idempotent and anti-extensive (resp. extensive). Evolved filters are obtained by composition of openings and closings [25,26].

Morphological operators are classically defined for images supported on Euclidean spaces. However, different imaging modalities produce nowadays images on smooth surfaces represented by meshes or point clouds. Let us consider for instance the image depicted in Fig.1(c), which corresponds to an grey-scale image painted on a smooth surface. Such support space can be modeled as a Riemannian manifold $\mathcal{M}$. These Riemannian images are often nonsmooth and their regularization can be needed in different applications. A recent work has introduced mathematical morphology for real valued images whose support space is a Riemannian manifold [1].

The problem we consider here concerns the approximation of a bounded nonsmooth image painted on a surface by a sequence of more regular functions, having in particular Lipschitz gradient, and without any hypothesis of differentiability. Current state-of-the-art techniques for regularizing such images is mainly based on heat-kernel and diffusion-like PDE, see for instance [27].

We adopt here a framework known as Lasry–Lions regularization [18,3]. Working on this geometric framework, the aim of the present contribution is to consider the interest of some recent theoretical results on the extension of Lasry–Lions regularization to Riemannian manifolds [8,9] in order to obtain Lipschitz regularized Riemannian images from smooth surfaces. We show that key ingredients for such regularization are Riemannian morphological operators.

Paper organization. The rest of the paper is organized as follows.

- The recently proposed canonic framework of morphological operators for images on Riemannian manifolds [1] is reminded in Section 2. It involves the use of a structuring function based on the scaled square of geodesic distance.

These morphological operators are the basic ingredients for the regularization discussed in the paper. More general Riemannian morphological operators are also formulated in [1].
- Section 3 reviewed the main results of Lasry–Lions theory for regularization in Hilbert spaces. It is discussed also the relationship with the more classical Moreau–Yosida regularization of convex analysis as well as some extensions of Lasry–Lions for rather general families of functions.
- We discuss in Section 4 our extension of Lasry–Lions regularization for Riemannian manifolds. In particular, we focus on the case of bounded functions on bounded domains defined on (compact and finite dimensional) Cartan–Hadamard manifolds.
- In Section 5 it is illustrated the application of this theory to Lipschitz regularization of images supported on surfaces.
- Some conclusions and perspectives in Section 6 close the paper.

Notation on smooth functions in Hilbert spaces. Let $f \in BUC(\mathbb{R}^n)$ be space of bounded uniformly continuous scalar functions in $\mathbb{R}^n$, i.e., assume there exists m continuous, nondecreasing on $[0,+\infty[$ such that $m(0)=0$, $m(t+s) \leq m(t)+m(s)$, for $t,s \geq 0$ and

$$|f(x)-f(y)| \leq m\left(\|x-y\|\right), \quad \forall x,y \in \mathbb{R}^n$$

The setting of this paper concerns the approximation of f by a sequence f_λ of functions in $C_b^{1,1}(\mathbb{R}^n)$ such that f_λ converges uniformly to f in $\mathbb{R}^n$, where

$$C_b^{1,1}(\mathbb{R}^n) = \left\{ f \in C_b^1(\mathbb{R}^n) \ : \ \nabla f \text{ is Lipschitz on } \mathbb{R}^n \right\},$$

with $C_b^1(\mathbb{R}^n) = \left\{ f \in C^1(\mathbb{R}^n) \ : \ f, \nabla f \text{ are bounded on } \mathbb{R}^n \right\}$. Therefore, $C_b^{1,1}$ represents the class of bounded continuously differentiable with a Lipschitz continuous gradient function.

2 Riemannian morphological operators

We consider here that $\mathcal{M}$ is a finite dimensional compact and complete manifold. Let $d_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$, $(x,y) \mapsto d_{\mathcal{M}}(x,y)$ be the geodesic distance on $\mathcal{M}$.

In this framework, canonic morphological operators are defined as follows [1].

Definition 1. *Given a Riemannian image $f : \mathcal{M} \to \overline{\mathbb{R}}$, for any $\lambda > 0$, we define for every $x \in \mathcal{M}$ the canonical Riemannian dilation of f of scale parameter λ as*

$$\delta_\lambda(f)(x) = \sup_{y \in \mathcal{M}} \left\{ f(y) - \frac{1}{2\lambda} d_{\mathcal{M}}(x,y)^2 \right\} \tag{1}$$

and the canonical Riemannian erosion of f of parameter λ as

$$\varepsilon_\lambda(f)(x) = \inf_{y \in \mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(x,y)^2 \right\} \tag{2}$$

We note that they are just the supremal convolution $(f \oplus \mathfrak{q}_\lambda)$ and infimal convolution $(f \ominus \mathfrak{q}_\lambda)$ of image f by the "quadratic geodesic structuring function"

$$\mathfrak{q}_\lambda(x; y) = -\frac{1}{2\lambda} d_{\mathcal{M}}(x, y)^2. \tag{3}$$

An obvious property of the canonical Riemannian dilation and erosion is the *duality by the involution* $f(x) \mapsto \complement f(x) = -f(x)$, i.e., $\delta_\lambda(f) = \complement \varepsilon_\lambda(\complement f)$. As in classical Euclidean morphology, the adjunction relationship is fundamental for the construction of the rest of morphological operators.

Proposition 1. *For any two real-valued images defined on the same Riemannian manifold $\mathcal{M}$, i.e., $f, g : \mathcal{M} \to \overline{\mathbb{R}}$, the pair $(\varepsilon_\lambda, \delta_\lambda)$ is called the canonical Riemannian adjunction*

$$\delta_\lambda(f)(x) \le g(x) \Leftrightarrow f(x) \le \varepsilon_\lambda(g)(x) \tag{4}$$

This result implies in particular that the canonical Riemannian dilation commutes with the supremum and the dual erosion with the infimum, i.e., for a given collection of images $f_i \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$, $i \in I$, we have

$$\delta_\lambda\left(\bigvee_{i\in I} f_i\right) = \bigvee_{i\in I} \delta_\lambda(f_i); \quad \varepsilon_\lambda\left(\bigwedge_{i\in I} f_i\right) = \bigwedge_{i\in I} \varepsilon_\lambda(f_i).$$

Classical properties of Euclidean dilation and erosion have the equivalent for Riemannian manifold $\mathcal{M}$, and they do not dependent on the geometry of $\mathcal{M}$.

Proposition 2. *Let $\mathcal{M}$ be a Riemannian manifold, and let $f, g \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$ two real-valued images $\mathcal{M}$. We have the following properties for the canonical Riemannian operators.*

1. *(Increaseness) If $f(x) \le g(x)$, $\forall x \in \mathcal{M}$ then $\delta_\lambda(f)(x) \le \delta_\lambda(g)(x)$ and $\varepsilon_\lambda(f)(x) \le \varepsilon_\lambda(g)(x)$, $\forall x \in \mathcal{M}$ and $\forall \lambda > 0$.*
2. *(Extensivity and anti-extensivity) $\delta_\lambda(f)(x) \ge f(x)$ and $\varepsilon_\lambda(f)(x) \le f(x)$, $\forall x \in \mathcal{M}$ and $\forall \lambda > 0$.*
3. *(Ordering property) If $0 < \lambda_1 < \lambda_2$ then $\delta_{\lambda_2}(f)(x) \ge \delta_{\lambda_1}(f)(x)$ and $\varepsilon_{\lambda_2}(f)(x) \le \varepsilon_{\lambda_1}(f)(x)$.*
4. *(Invariance under isometry) If $T : \mathcal{M} \to \mathcal{M}$ is an isometry of $\mathcal{M}$ and if f is invariant under T, i.e., $f(Tz) = f(z)$ for all $z \in \mathcal{M}$, then the Riemannian dilation and erosion are also invariant under T, i.e., $\delta_\lambda(f)(Tz) = \delta_\lambda(f)(z)$ and $\varepsilon_\lambda(f)(Tz) = \varepsilon_\lambda(f)(z)$, $\forall z \in \mathcal{M}$ and $\forall \lambda > 0$.*
5. *(Extrema preservation) We have $\sup \delta_\lambda(f) = \sup f$ and $\inf \varepsilon_\lambda(f) = \inf f$, moreover if f is lower (resp. upper) semicontinuous then every minimizer (resp. maximizer) of $\varepsilon_\lambda(f)$ (resp. $\delta_\lambda(f)$) is a minimizer (resp. maximizer) of f, and conversely.*

In addition, using the classical result on adjunctions in complete lattices [26,15], we state that the composition products of the pair $(\varepsilon_\lambda, \delta_\lambda)$ lead to the adjoint opening and adjoint closing as follows.

Definition 2. *Given an image* $f\mathcal{M} \to \overline{\mathbb{R}}$*, the canonical Riemannian opening and canonical Riemannian closing of scale parameter* λ *are respectively given by*

$$\gamma_\lambda(f)(x) = \sup_{z\in\mathcal{M}} \inf_{y\in\mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(z,y)^2 - \frac{1}{2\lambda} d_{\mathcal{M}}(z,x)^2 \right\}, \tag{5}$$

and

$$\varphi_\lambda(f)(x) = \inf_{z\in\mathcal{M}} \sup_{y\in\mathcal{M}} \left\{ f(y) - \frac{1}{2\lambda} d_{\mathcal{M}}(z,y)^2 + \frac{1}{2\lambda} d_{\mathcal{M}}(z,x)^2 \right\}. \tag{6}$$

Having the canonical Riemannian opening and closing, all the other morphological filters defined by composition of them are easily obtained.

In Fig. 2 are depicted some examples of morphological operators applied on a real-valued function on a smooth surface. The geodesic distances $d_{\mathcal{S}}(x,y)$ are calculated using the function `all_shortest_paths()` from the toolbox MatlabBGL [19]. This function uses either the Floyd–Warshall algorithm or the Johnson's algorithm for finding shortest path in the weighted graph of faces of the mesh.

3 From Moreau–Yosida regularization to Lasry-Lions regularization

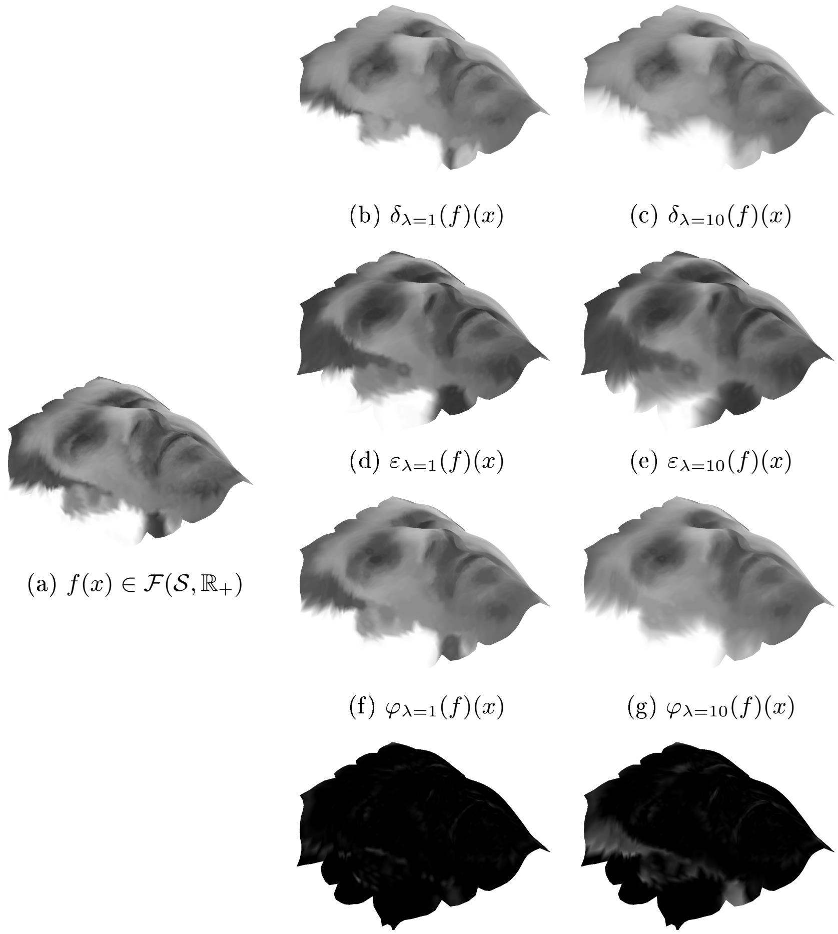
3.1 Moreau–Yosida regularization

In the field of convex analysis [21,23,4,16] and variational analysis [24], Moreau–Yosida regularization consists in computing a regularized version of a scalar function defined on a vector space (Euclidean or Hilbert space), by means of a Euclidean erosion using quadratic structuring functions. Its origin goes back to the work of Yosida [30] on maximal monotone operators. More precisely, we have the following definition.

Definition 3. *Given a lower semicontinuous function* $f : E \to \mathbb{R} \cup \{+\infty\}$*, the one-parameter Moreau–Yosida regularization of* f*, also called Moreau envelope, is defined as*

$$f_\lambda(x) = \inf_{y\in E} \left\{ f(y) + \frac{1}{2\lambda} \|x-y\|^2 \right\} = (f \ominus q_\lambda)(x). \tag{7}$$

Obviously, the Moreau–Yosida regularizers has all the properties of the quadratic Euclidean erosion. Associated to the anti-extensivity and ordering with λ, one has that f_λ increases pointwise to f as λ decreases to 0; the convergence is uniform on bounded sets when f is uniformly continuous on bounded sets In addition, if $\lambda \in (0,L)$ then f_λ is everywhere finite and Lipschitz continuous on bounded sets. The following classical result due to Moreau [20] summarizes the additional properties of the Moreau–Yosida approximation in the convex setting.



(a) $f(x) \in \mathcal{F}(\mathcal{S}, \mathbb{R}_+)$

(b) $\delta_{\lambda=1}(f)(x)$ (c) $\delta_{\lambda=10}(f)(x)$

(d) $\varepsilon_{\lambda=1}(f)(x)$ (e) $\varepsilon_{\lambda=10}(f)(x)$

(f) $\varphi_{\lambda=1}(f)(x)$ (g) $\varphi_{\lambda=10}(f)(x)$

(h) $\varphi_{\lambda=1}(f)(x) - f(x)$ (i) $\varphi_{\lambda=10}(f)(x) - f(x)$

Fig. 2. Morphological operators applied on a real-valued function on a smooth surface: (a) original image $f : S \to \mathbb{R}$, (b) and (c) canonic Riemannian dilation $\delta_\lambda(f)$ with $\lambda = 1$ and $\lambda = 10$, (d) and (e) canonic Riemannian erosion $\varepsilon_\lambda(f)$ with $\lambda = 1$ and $\lambda = 10$, (f) and (g) canonic Riemannian closing $\varphi_\lambda(f)$ with $\lambda = 1$ and $\lambda = 10$, (h) and (i) residues between the original image f and the closings $\varphi_\lambda(f)$.

Theorem 1. *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous proper function. Then, for any $\lambda > 0$, the Moreau–Yosida approximate f_λ of f is a convex $C^{1,1}$ function. Moreover, the gradient of f_λ converges (in the set convergence sense [2]) to the subdifferential of f.*

A dual result is naturally obtained for upper semicontinuous concave functions, by replacing the quadratic erosion $(f \ominus q_\lambda)$ by a quadratic dilation $(f \oplus q_\lambda)$.

In general, when f is not convex, f_λ is not smooth, even in the one-dimensional case $E = \mathbb{R}$. This is a strong limitation in order to regularize images which are rarely convex functions.

3.2 Lasry–Lions regularization: original formulation

As a generalization of the use of Moreau–Yosida approach, Lasry–Lions regularization [18] is a theory of nonsmooth approximation for functions in Hilbert spaces using combinations of Euclidean dilation and erosion with quadratic structuring functions, which leads to the approximation of bounded lower or upper-semicontinuous functions with Lipschitz continuous derivatives which approximate f, without assuming convexity of f. The main results of this approach are summarized in the following theorem [18].

Theorem 2 (Lasry and Lions, 1986). *Let $f \in BUC(\mathbb{R}^n)$. For all $0 < \mu < \lambda$, let us define the Lasry–Lions regularizers, according to our morphological framework, based on dilation and erosion by quadratic structuring function:*

$$(f_\lambda)^\mu(x) = \left((f \ominus q_\lambda) \oplus q_\mu\right)(x), \tag{8}$$
$$(f^\lambda)_\mu(x) = \left((f \oplus q_\lambda) \ominus q_\mu\right)(x). \tag{9}$$

Then the functions $(f_\lambda)^\mu$ and $(f^\lambda)_\mu$ belong to $C_b^{1,1}(\mathbb{R}^n)$. Lasry–Lions regularizers converges uniformly to f as λ goes to 0. In addition,

$$|(f_\lambda)^\mu(x) - (f_\lambda)^\mu(y)| \leq m(\|x-y\|) \; ; \; |(f_\lambda)^\mu(x) - f(x)| \leq m(t_\lambda + t_\mu) + \frac{t_\lambda^2}{2\lambda};$$
$$\sup_{\mathbb{R}^n} |(f_\lambda)^\mu(x) - f(x)| \leq m(t_\lambda) \; ; \; \sup_{\mathbb{R}^n} |(f^\lambda)_\mu(x) - f(x)| \leq m(t_\lambda);$$
$$|\nabla(f_\lambda)^\mu(x) - \nabla(f_\lambda)^\mu(y)| \leq M_{\lambda,\mu}\|x-y\| \; ; \; |\nabla(f^\lambda)_\mu(x) - \nabla(f^\lambda)_\mu(y)| \leq M_{\lambda,\mu}\|x-y\|$$
$$\sup_{\mathbb{R}^n} |\nabla(f_\lambda)^\mu| \leq \frac{t_\lambda}{\lambda} \; ; \; \sup_{\mathbb{R}^n} |\nabla(f^\lambda)_\mu| \leq \frac{t_\lambda}{\lambda};$$

where t_λ is the maximum positive root of $t_\lambda^2 = 2\lambda m(\lambda)$ and $M_{\lambda,\mu} = \max(\mu^{-1}, (\lambda-\mu)^{-1})$.

If f is uniformly continuous on balls (bounded domains), the regularizers converge uniformly on balls to f. If f is lower-semicontinuous and bounded below, the $(f_\lambda)^\mu \in C_b^{1,1}(\mathbb{R}^n)$ for $0 < \mu < \lambda$, and $(f_\lambda)^\mu$ converges pointwise to f when $\lambda \to 0$. Dually, if f is upper-semicontinuous and bounded above, the $(f^\lambda)_\mu \in C_b^{1,1}(\mathbb{R}^n)$ for $0 < \mu < \lambda$, and $(f^\lambda)_\mu$ converges pointwise to f when $\lambda \to 0$.

As a general rule, if $f \in BUC(\mathbb{R}^n)$ enjoys more regularity or symmetry, the functions $(f_\lambda)^\mu$ and $(f^\lambda)_\mu$ will also enjoy more regularity and symmetry preserving. For instance,

- If f is convex (resp. concave) then $(f_\lambda)^\mu$ is also convex (resp. $(f^\lambda)_\mu$ is also concave).
- If f is invariant to a group of isometries, so are $(f_\lambda)^\mu$ and $(f^\lambda)_\mu$. This fact is interesting for critical point theory.
- The set of minima (resp. maxima) of f is preserved by $(f_\lambda)^\mu$ (resp. $(f^\lambda)_\mu$).

For an analysis on the second-order differentiability of approximations f_λ and $(f_\lambda)^\mu$ (i.e., existence and expressions of Hessian) see [22].

3.3 Lasry–Lions regularizers from a mathematical morphology viewpoint

We note that Lasry–Lions regularizers can be seen in a qualitative sense as quadratic Euclidean pseudo-opening and pseudo-closing. We remind that, given a function $f : E \to \overline{\mathbb{R}}$, its opening $\gamma_\lambda(f)$ and closing $\varphi_\lambda(f)$ by the quadratic structuring function q_λ are given by the composition of the corresponding dilation and erosion, i.e.,

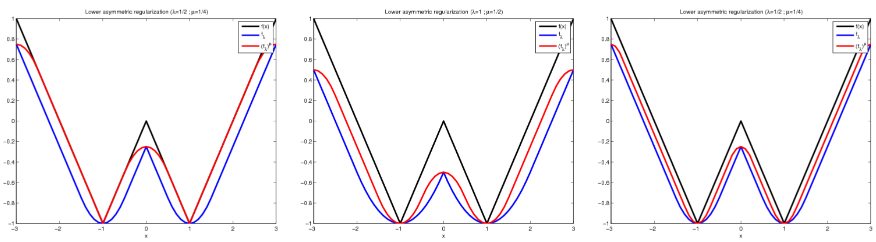
$$\gamma_\lambda(f)(x) = \left((f \ominus q_\lambda) \oplus q_\lambda\right)(x),$$
$$\varphi_\lambda(f)(x) = \left((f \oplus q_\lambda) \ominus q_\lambda\right)(x).$$

For the case of Riemannian images, those correspond just to canonic Riemannian opening (5) and closing (6). Openings and closings have the following properties [25]: they are increasing operators (ordering preserving); idempotent operators (stable at the iteration) $\gamma_\lambda \circ \gamma_\lambda(f) = \gamma_\lambda(f)$, $\varphi_\lambda \circ \varphi_\lambda(f) = \varphi_\lambda(f)$; and hold the following ordering: for $0 < \lambda_1 \leq \lambda_2$, we have $\gamma_{\lambda_2}(f)(x) \leq \gamma_{\lambda_1}(f)(x) \leq f(x) \leq \varphi_{\lambda_1}(f)(x) \leq \varphi_{\lambda_2}(f)(x)$. Lasry–Lions regularizers are also increasing and have the same ordering property with respect to λ, i.e.,

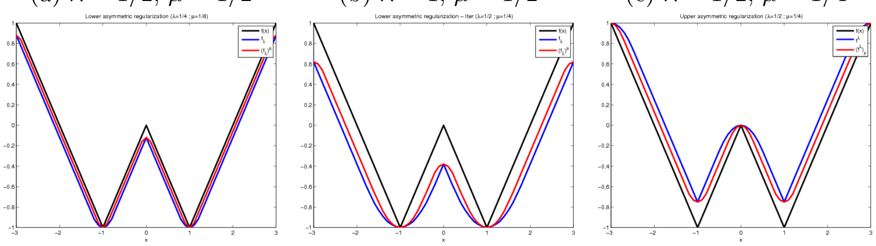
- (Increaseness) If $f \leq g$ then
$$(f_\lambda)^\mu \leq (g_\lambda)^\mu, \text{ and } (f^\lambda)_\mu \leq (g^\lambda)_\mu.$$
- (Ordering) If $\lambda_1 \geq \lambda_2 > \mu_2 \geq \mu_1 > 0$ then
$$(f_{\lambda_1})^{\mu_1} \leq (f_{\lambda_2})^{\mu_2} \leq f \leq (f^{\lambda_2})_{\mu_2} \leq (f^{\lambda_1})_{\mu_1}.$$

Thus, the fundamental difference with respect to openings and closing is the lack of idempotency in Lasry–Lions regularizers since the scale parameter of quadratic dilation and erosion are different, i.e., $0 < \mu < \lambda$. We note that the Lipschitz constant of the regularized gradient $M_{\lambda,\mu} = \max(\mu^{-1}, (\lambda - \mu)^{-1})$ becomes $+\infty$ when $\lambda = \mu$.

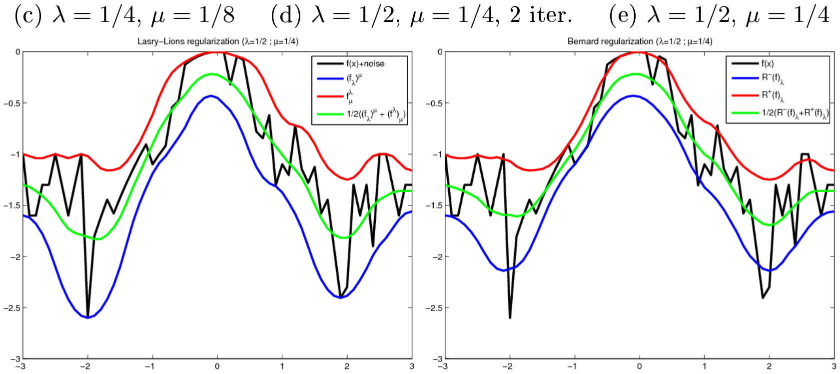
Fig. 3 depicts the behavior of Lasry–Lions regularizers on a 1D function. The case (a) corresponds to a quadratic opening ($\lambda = \mu$) which is not smooth.



(a) $\lambda = 1/2$, $\mu = 1/2$ (b) $\lambda = 1$, $\mu = 1/2$ (c) $\lambda = 1/2$, $\mu = 1/4$



(c) $\lambda = 1/4$, $\mu = 1/8$ (d) $\lambda = 1/2$, $\mu = 1/4$, 2 iter. (e) $\lambda = 1/2$, $\mu = 1/4$



(f) $\lambda = 1$, $\mu = 1/2$ (g) $\lambda = 1$

Fig. 3. Lasry-Lions regularization of a 1D signal (original signal in black, operator in λ in blue and composed operator in μ in red: (a) quadratic opening, (b) to (c) lower regularizer $(f_\lambda)^\mu$, (d) iterated lower regularizer, (e) upper regularizer $(f^\lambda)_\mu$. Bottom row, comparison between Lasry–Lions regularizers (f) and Bernard regularizers (g). Green curves correspond to the averages.

By comparing for instance the lower regularizer in (c) and the equivalent upper regularizer, we note a strong asymmetric result of the regularized function. Therefore, the choice of lower $(f_\lambda)^\mu$ or upper $(f^\lambda)_\mu$ as regularizer involves a certain asymmetric behavior. Bernard [8] has recently proposed a more symmetric pair of regularizers.

Theorem 3 (Bernard, 2010). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a (locally) bounded function. The Bernard regularizers correspond to the operators*

$$R^-_\lambda(f)(x) = (((f \ominus q_\lambda) \oplus q_{2\lambda}) \ominus q_\lambda)(x), \tag{10}$$
$$R^+_\lambda(f)(x) = (((f \oplus q_\lambda) \ominus q_{2\lambda}) \oplus q_\lambda)(x), \tag{11}$$

and have the following properties:

- *Regularization. For each function f and each $\lambda > 0$, functions $R^-_\lambda(f)$ and $R^+_\lambda(f)$ are $C^{1,1}_b(\mathbb{R}^n)$.*
- *Approximation. If f is uniformly continuous, then $R^-_\lambda(f)$ and $R^+_\lambda(f)$ are $C^{1,1}_b(\mathbb{R}^n)$ and converge uniformly to f as $\lambda \to 0$.*

Bernard regularizers are again compositions of quadratic erosion and dilation: $R^-_\lambda(f) = \varepsilon_\lambda \circ \delta_{2\lambda} \circ \varepsilon_\lambda(f)$ and $R^+_\lambda(f) = \delta_\lambda \circ \varepsilon_{2\lambda} \circ \delta_\lambda(f)$. Then, by the semi-group law of quadratic dilation and erosion, they can be rewritten as

$$R^-_\lambda(f) = \varphi_\lambda \circ \gamma_\lambda(f),$$
$$R^+_\lambda(f) = \gamma_\lambda \circ \varphi_\lambda(f).$$

In mathematical morphology filtering theory, Bernard regularizers are the well known $\gamma \circ \varphi$ and $\varphi \circ \gamma$ filters obtained just by composition of opening and closing using the same structuring function. These filters are known to be increasing and idempotent operators. However they are neither ordered between them nor ordered with respect to initial function.

Fig. 3(d) shows an example of Bernard lower $R^-_\lambda(f)$ and upper $R^+_\lambda(f)$ regularizers, which are compared with the corresponding Lasry–Lions regularizers $(f_\lambda)^\mu$ and $(f^\lambda)_\mu$ for the same λ. In particular, it is also given the average between the lower and upper regularizers. Even if Bernand regularizers are more symmetric than the original Lasry–Lions regularizers, in practice, they are not very different. Other more symmetric ones can be formulated inspired from the evolved morphological filters, see below the alternate and alternate sequential regularizers that we propose. It is not obvious if the idempotency brings any interest to the problem regularization/approximation.

3.4 Lasry–Lions regularization: extensions

In the seminal paper [18], it was conjectured that this regularization works in any arbitrary Banach space (or even in metric spaces) and under other properties of the function f to be regularized. This has been the motivation to extend Lasry–Lions regularization according to the following different directions.

- More general kernels than the quadratic one, including non-concave/non-convex [10].
- Generalization to semicontinuous, quadratically minorized/majorized functions defined on $\mathbb{R}^n$, using quadratic (concave and smooth) kernels [3].
- Extended to Banach spaces [28].
- Extension to Banach and metric spaces, with kernels adapted to the properties of the norm [12].
- Generalization to finite dimensional compact manifolds [8,9].

Let us focus on and summarize the approach introduced by Attouch and Aze in [3] to semicontinuous, non necessarily bounded, quadratically minorized/majorized functions defined on $\mathbb{R}^n$. This theory gives a clear insight of how Lasry–Lions regularization works and its relationship with Moreau–Yosida regularization. First of all, we need the recall the definition of weakly-convex/concave functions in $\mathbb{R}^n$.

Definition 4 (Weakly-convex/concave functions in $\mathbb{R}^n$). *A function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said weakly convex, or convex up to square, or paraconvex, if there exists some constant $c \geq 0$ such that $g(\cdot) + \frac{c}{2}\|\cdot\|^2$ is convex, i.e.,*

$$g\left(tx + (1-t)y\right) \leq tg(x) + (1-t)g(y) + t(1-t)\frac{c}{2}\|x-y\|^2$$

for all $x, y \in \mathbb{R}^n$ and all t in $[0,1]$.

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is said weakly concave, or concave up to square, or paraconcave, if there exists some constant $c \geq 0$ such that $f(\cdot) - \frac{c}{2}\|\cdot\|^2$ is concave, i.e.,

$$f\left(tx + (1-t)y\right) \geq tf(x) + (1-t)f(y) - t(1-t)\frac{c}{2}\|x-y\|^2$$

for all $x, y \in \mathbb{R}^n$ and all t in $[0,1]$.

This first result gives the link of quadratically minorized/majorized functions and weakly concave/convex functions using a first quadratic operator.

Proposition 3 (Attouch and Aze, 1993). *Let $f : E \subseteq \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $g : E \subseteq \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$.*

- *If $f(x) \geq -\frac{c}{2}(1 + \|x\|^2)$, $c \geq 0$ (quadratically minorized), then for any $0 < \lambda < \frac{1}{c}$ the quadratic erosion (Moreau–Yosida regularization) $(f \ominus q_\lambda)(x)$ is a λ^{-1}-weakly concave function; i.e., $f(\cdot) - \frac{\|\cdot\|^2}{2\lambda}$ is concave.*
- *If $g(x) \leq \frac{d}{2}(1 + \|x\|^2)$, $d \geq 0$ (quadratically majorized), then for any $0 < \mu < \frac{1}{d}$ the quadratic dilation (dual Moreau–Yosida regularization) $(g \oplus q_\lambda)(x)$ is a μ^{-1}-weakly convex function; i.e., $g(\cdot) + \frac{\|\cdot\|^2}{2\mu}$ is convex.*

The second result establishes how weakly convex or concave functions becomes $C^{1,1}$ by the second quadratic operator.

Theorem 4 (Attouch and Aze, 1993). *Let f be a c-weakly convex function in $E \subseteq \mathbb{R}^n$. Let us introduce the corresponding convex function $\phi(\cdot) = f(\cdot) + \frac{c}{2}\|\cdot\|^2$. Then for any $0 < \lambda < \frac{1}{c}$ we have,*

- *Quadratic erosion f_λ belongs to the class of $C^{1,1}(\mathbb{R}^n)$ functions.*
- *$\forall x \in E$, f_λ is λ^{-1}-weakly concave.*
- *$\forall x \in E$, f_λ is $\big((1-\lambda c)^{-1}c\big)$-weakly convex.*
- *Gradient of quadratic erosion Df_λ is $\max(\lambda^{-1}, (1-\lambda c)^{-1}c)$-Lipschitz continuous.*

A dual result is obtained for a c-weakly concave function by considering the quadratic dilation.

Thus, we can draw the following conclusions on the composition of the couple of operators underlaying Lasry–Lions regularizers.

- Given a quadratically majorized function in $\mathbb{R}^n$ of parameter c, the quadratic dilation with $\lambda < c^{-1}$ produces a λ-weakly convex function.
- Then for any $\mu < \lambda$ (strictly smaller than the dilation scale), the corresponding quadratic erosion produces a function which belongs to the class $C^{1,1}(\mathbb{R}^n)$.
- By the way, this function is now both weakly convex and weakly concave.

Quadratically minorized and majorized are rather general conditions. However, images are typically functions obtained by combination of such functions; i.e., bright (dark) areas or intensity peaks can be modeled as quadratically minorized (majorized) areas. Nevertheless, images can be considered in most of situations as bounded functions.

4 Lasry–Lions regularization on Riemannian manifolds

4.1 From Hilbert spaces to Riemannian manifolds

Some recent works provide us the elements for an extension of these regularization tools to images on Riemannian manifolds. On the one hand, as widely discussed in [5,6], the results of Moreau–Yosida regularization can be extended to functions on a Cartan–Hadamard manifold. On the other hand, Lasry–Lions regularization itself has been recently generalized to finite dimensional compact manifolds [8,9], in the framework of recent progresses on sub-solutions of Hamilton-Jacobi equations [14].

The convexity being a crucial notion of this theory, it is replaced in Riemannian manifolds by the notion of geodesic convexity. We remind that a subset C of a Riemannian manifold $\mathcal{M}$ is said to be a geodesically convex set if, given any two points in C, there is a minimizing geodesic contained within C that joins those two points. Now, let C be a geodesically convex subset of $\mathcal{M}$. A function $f : C \to \mathbb{R}$ is said to be a geodesically convex function if the composition $f \circ \gamma : [0,T] \to \mathbb{R}$ is a convex function in the usual sense for every unit speed geodesic arc $\gamma : [0,T] \to \mathcal{M}$ contained within C. The notions of weakly convex and weakly convex function (named also semi-convex and semi-concave function), which are intimately related to $C_b^{1,1}$ functions, are also generalized to the case of manifolds, see for instance [14]. Finally, the notions of quadratically geodesically minorized $f(x) \geq -\frac{c}{2}(1 + d(x,x_0)^2)$ and majorized $g(x) \leq \frac{c'}{2}(1 + d(x,x_0)^2)$ assumptions appears also naturally in the Riemannian framework, where $d(\cdot,\cdot)$ is the geodesic distance of the manifold. The extension of the definition of Lipschitz gradient to Riemannian manifolds is not obvious since the gradient at two different points belong to different fibres. Thus a possible definition involves a notion of local (pointwise) Lipschitz constant using the metric of the tangent space. Then the corresponding global Lipschitz constant is given by the supremum of local Lipschitz constants for all points [14].

In order to transpe Lasry–Lions regularization to the case of a Riemannian manifold $\mathcal{M}$, it seems intuitive that the canonic Riemannian structuring function $-\frac{d_{\mathcal{M}}(x,y)^2}{2\lambda}$ should be a concave and smooth function: in order to obtain a weakly convex (concave) function from a geodesically quadratic majorized (minorized) function, the square of geodesic distance function should be a convex function. In addition, the smoothness of the gradient is obtained locally and therefore the corresponding function will be locally $C_b^{1,1}$.

4.2 Lasry–Lions regularization on Cartan–Hadamard manifolds

More precisely, let us focus on the case where $\mathcal{M}$ is a finite dimensional compact Cartan–Hadamard manifold (thus also geodesically complete): every two points can be connected by a minimizing geodesic and the curvature is bounded in bounded sets. We remind that a Cartan–Hadamard manifold is a simply connected Riemannian manifold $\mathcal{M}$ with sectional curvature $K \leq 0$ [17]. Let A be a closed convex subset of $\mathcal{M}$. Then the distance function to A, $x \mapsto d_{\mathcal{M}}(x,A)$, where $d_{\mathcal{M}}(x,A) = \inf\{d_{\mathcal{M}}(x,y) \;:\; y \in A\}$ is C^1 smooth on $\mathcal{M}\backslash A$ and, moreover, the square of the distance function $x \mapsto d_{\mathcal{M}}(x,A)^2$ is C^1 smooth and convex on all of M [6].

An assumption of non-positive curvature of $\mathcal{M}$ is a sufficient condition in order that $d_{\mathcal{M}}$ be uniformly locally convex around the diagonal $\mathcal{M} \times \mathcal{M}$. Consequently, if $\mathcal{M}$ is a Cartan–Hadamard manifold, the structuring function $x \mapsto \mathfrak{q}(x,y)$, $\forall y \in \mathcal{M}$, is always a concave function; or equivalently, $-\mathfrak{q}(x,y)$ is a convex function.

We can now formulate the result which extends the theory discussed in previous section for Hilbert spaces to Cartan–Hadamard manifolds.

Theorem 5. *Let $\mathcal{M}$ be a finite-dimensional compact Cartan–Hadamard manifold. Let $\Omega \subset \mathcal{M}$ be a bounded set of $\mathcal{M}$. Given a function $f : \Omega \to \mathbb{R}$, for all $0 < \mu < \lambda$, let us define the Riemannian Lasry–Lions regularizers:*

$$\Gamma^{-}_{\lambda,\mu}(f)(x) = (f_\lambda)^\mu(x) = \sup_{z\in\mathcal{M}} \inf_{y\in\mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(z,y)^2 - \frac{1}{2\mu} d_{\mathcal{M}}(z,x)^2 \right\}$$

$$\Gamma^{+}_{\lambda,\mu}(f)(x) = (f^\lambda)_\mu(x) = \inf_{z\in\mathcal{M}} \sup_{y\in\mathcal{M}} \left\{ f(y) - \frac{1}{2\lambda} d_{\mathcal{M}}(z,y)^2 + \frac{1}{2\mu} d_{\mathcal{M}}(z,x)^2 \right\}$$

We have $(f_\lambda)^\mu \leq f$ *and* $(f^\lambda)_\mu \geq f$.

- *Let* f *be a bounded uniformly continuous function in* Ω. *Then, for all* $0 < \mu < \lambda$, *the functions* $(f_\lambda)^\mu$ *and* $(f^\lambda)_\mu$ *are locally of class* $C_b^{1,1}(\Omega)$ *and converge uniformly to* f *on* Ω.
- *Assume that there exists* $c, c' > 0$, *such that we have the following growing conditions for bounded semicontinuous functions:*

$$f(x) \geq -\frac{c}{2}(1 + d(x, x_0)^2), \quad g(x) \leq \frac{c'}{2}(1 + d(x, x_0)^2), \;\; x_0 \in \mathcal{M}.$$

  Then, for all $0 < \mu < \lambda < \Lambda$, *the function*$(f_\lambda)^\mu$ *and for all* $0 < \mu < \lambda < \Lambda$ *the function* $(g^\lambda)_\mu$ *are locally* $C_b^{1,1}(\Omega)$ *and they converge point-wise respectively to* f *and* g.
- *In addition, if* f *is a geodesically convex function (resp. concave) the* $(f_\lambda)^\mu$ *is also convex (resp.* $(f^\lambda)_\mu$ *is concave).*

Proof. The result considered here is a particular case of those given by Bernard [8] (Theorem 4) for the case of functions on finite dimensional compact manifolds. The proofs in [8,9] are based on partition of unity. That involves the use of local charts in the partition, such that the regularizers do not have an explicit expression as the one provided here. In other terms, in [8] (Theorem 4) the regularizer is obtained as the sum of a regularization on the local chart of partition of the function.

If we consider the following result on localization of Riemannian canonic quadratic erosion from [6] (Proposition 2.1):

Proposition 4 (Azagra and Ferrera, 2006). *Let* $\mathcal{M}$ *be a Riemannian manifold, and let* $f : \mathcal{M} \to \overline{\mathbb{R}}$ *a function satisfying that* $f(x) \geq -\frac{c}{2}(1 + d(x, x_0)^2)$ *(quadratically minorized) for some* $c > 0$, $x_0 \in \mathcal{M}$. *Then, for all* $\lambda \in (0, \frac{1}{2c})$ *and for all* $\rho > \overline{\rho}$, *we have that*

$$\varepsilon_\lambda(f)(x) = \inf_{y \in B_\rho(x)} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(x, y)^2 \right\}, \tag{12}$$

where

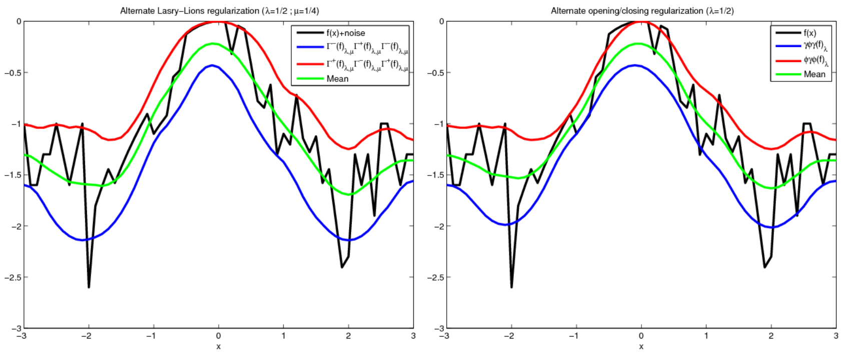
$$\overline{\rho} = \overline{\rho}(x, \lambda, c) = \sqrt{\lambda \frac{2f(x) + c(2d(x, x_0)^2 + 1)}{1 - 2\lambda c}},$$

and where the geodesic ball of center x *and radius* ρ *is defined by* $B_\rho(x) = \{y : d_{\mathcal{M}}(x, y) \leq \rho\}$.

Then it becomes clear that Riemannian canonic quadratic erosion and dilation can be computed locally on bounded sets and that the property of local $C_b^{1,1}$ of our explicit Riemannian Lasry–Lions regularizers is equivalent to that of [8] (Theorem 4) by using in particular the exponential chart.

The result on convergence of Lasry–Lions operators is obtained by the convergence of Moreau–Yosida regularizer shown in [6] (Proposition 2.3).

Finally, the result of convexity is a consequence of the one obtained in [6] (Corollary 4.4) for the Moreau–Yosida regularization in Cartan–Hadamard manifolds.



(a) $\lambda = 1$, $\mu = 1/2$ (b) $\lambda = 1$

Fig. 4. Comparison between alternate Lasry–Lions regularizers (a) and classical alternate morphological filters (b).

4.3 Composed regularizers

By composition of the lower $\Gamma^{-}_{\lambda,\mu}(f) = (f_\lambda)^\mu(x)$ and upper $\Gamma^{+}_{\lambda,\mu}(f) = (f^\lambda)_\mu(x)$ regularizers, it is possible to formulate other evolved Riemannian morphological regularizers inspired on classical morphological filters [25,26] and which preserves the properties of approximation and regularization, but producing more symmetric filtering effects.

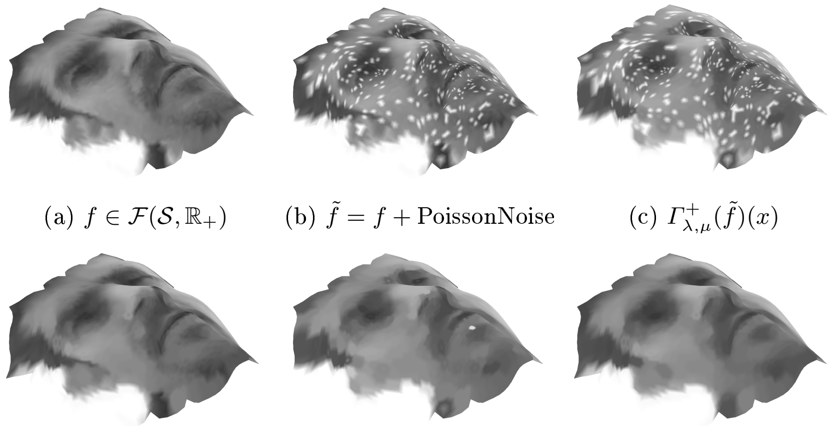
- Alternate regularizers ($0 < \mu < \lambda < \Lambda$):

$$f \mapsto \Gamma^{-}_{\lambda,\mu}\Gamma^{+}_{\lambda,\mu}\Gamma^{-}_{\lambda,\mu}(f),$$
$$f \mapsto \Gamma^{+}_{\lambda,\mu}\Gamma^{-}_{\lambda,\mu}\Gamma^{+}_{\lambda,\mu}(f).$$

- Alternate sequential regularizers ($0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \cdots < \mu_n < \lambda_n < \Lambda$):

$$f \mapsto \Gamma^{+}_{\lambda_n,\mu_n}\Gamma^{-}_{\lambda_n,\mu_n}\cdots\Gamma^{+}_{\lambda_2,\mu_2}\Gamma^{-}_{\lambda_2,\mu_2}\Gamma^{+}_{\lambda_1,\mu_1}\Gamma^{-}_{\lambda_1,\mu_1}(f),$$
$$f \mapsto \Gamma^{-}_{\lambda_n,\mu_n}\Gamma^{+}_{\lambda_n,\mu_n}\cdots\Gamma^{-}_{\lambda_2,\mu_2}\Gamma^{+}_{\lambda_2,\mu_2}\Gamma^{-}_{\lambda_1,\mu_1}\Gamma^{+}_{\lambda_1,\mu_1}(f).$$

In addition, we can also consider the average between $\Gamma^{-}_{\lambda,\mu}(f)$ and $\Gamma^{+}_{\lambda,\mu}(f)$, or the average between the alternate ones, as appropriate regularized version of f. Fig.4 gives an example of alternate Lasry–Lions regularizers on the 1D signal and they are compared to the classical alternate morphological filters (composition of openings and closings).



(a) $f \in \mathcal{F}(\mathcal{S}, \mathbb{R}_+)$ (b) $\tilde{f} = f + \text{PoissonNoise}$ (c) $\Gamma^+_{\lambda,\mu}(\tilde{f})(x)$

(d) $\Gamma^-_{\lambda,\mu}(\tilde{f})(x)$ (e) $\Gamma^+_{\lambda,\mu}\Gamma^-_{\lambda,\mu}\Gamma^+_{\lambda,\mu}(\tilde{f})(x)$ (f) $\Gamma^-_{\lambda,\mu}\Gamma^+_{\lambda,\mu}\Gamma^-_{\lambda,\mu}(\tilde{f})(x)$

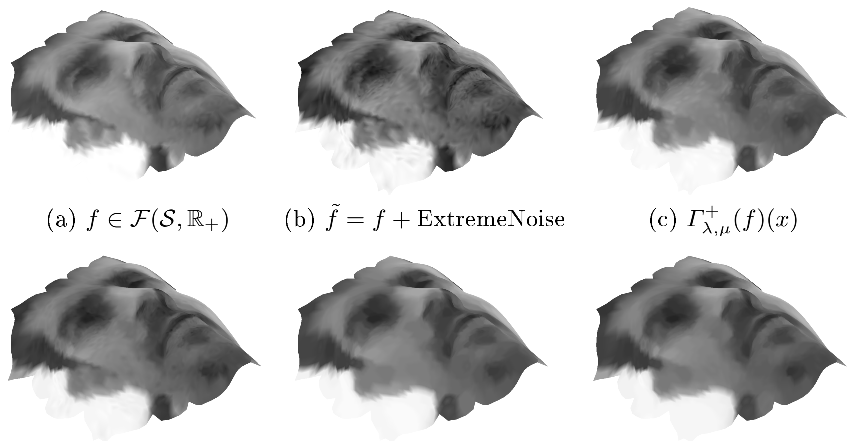
Fig. 5. Restoring image missing parts: Original image (a) has been corrupted with Poisson noise in (b). From (c) to (f), filtered image by composition of different Lasry-Lions regularizers (for all the cases $\lambda = 1/2$; $\mu = 1/4$).

5 Application to Lipschitz Regularization of Images supported on Surfaces

In order to illustrate the relevance of Lasry–Lions approach for Lipschitz regularization of images supported on surfaces, we consider two examples given in Fig. 5 and Fig. 6. In both cases, the original face image $f : \mathcal{S} \to \mathbb{R}$ has been corrupted, $\tilde{f} = f + \text{noise}$ and the aim is to restore as well as possible f from a regularization of $\tilde{f}$. We note the image is bounded in $[0, M]$ and support $\mathcal{S}$ is also a bounded set.

The case considered in Fig. 5 corresponds to suppress some parts of the image. The effect is simulated by a Poisson noise which is then thresholded. Then, the corresponding parts are set to the maximum value M. In this case, it is obvious that the upper regularizer $\Gamma^+_{\lambda,\mu}$ is without interest. On the contrary, the lower regularizer $\Gamma^-_{\lambda,\mu}$ produces a nice restoration. Again, as expected the alternate regularizer starting form $\Gamma^-_{\lambda,\mu}$ yields much better results than the one starting from $\Gamma^+_{\lambda,\mu}$.

Fig. 6 involves another problem where the image has been corrupted with a non-gaussian noise. By the nature of the noise, the average upper and lower regularizers performs now better than $\Gamma^+_{\lambda,\mu}$ or $\Gamma^-_{\lambda,\mu}$ separately. A similar behavior is observed for the averaged alternate regularizers. Obviously, by changing the values of λ and μ the Lipschitz regularization effect can be tuned; but from the experiments we have observed stable effect and a nice preservation of the contours of the main image structures.



(a) $f \in \mathcal{F}(\mathcal{S}, \mathbb{R}_+)$ (b) $\tilde{f} = f + \text{ExtremeNoise}$ (c) $\mathit{\Gamma}^+_{\lambda,\mu}(f)(x)$

(d) $1/2\left(\mathit{\Gamma}^+_{\lambda,\mu}(\tilde{f})(x) + \mathit{\Gamma}^-_{\lambda,\mu}(\tilde{f})(x)\right)$ (e) $\mathit{\Gamma}^+_{\lambda,\mu}\mathit{\Gamma}^-_{\lambda,\mu}\mathit{\Gamma}^+_{\lambda,\mu}(\tilde{f})(x)$ (f) $1/2\left(\mathit{\Gamma}^+_{\lambda,\mu}\mathit{\Gamma}^-_{\lambda,\mu}\mathit{\Gamma}^+_{\lambda,\mu}(\tilde{f})(x) + \mathit{\Gamma}^+_{\lambda,\mu}\mathit{\Gamma}^-_{\lambda,\mu}\mathit{\Gamma}^+_{\lambda,\mu}(\tilde{f})(x)\right)$

Fig. 6. Restoring image acquisition noise: Original image (a) has been corrupted with non-Gaussian noise in (b). From (c) to (f), filtered image by composition of different Lasry-Lions regularizers (for all the cases $\lambda = 1/2$; $\mu = 1/4$).

6 Conclusions and perspectives

The goal of this study was to bring a powerful approximation/regularization theory well-known in convex analysis to the morphological image processing domain. In particular we discussed its interest for filtering images painted on curved supports, such as meshes.

We have shown the remarkable regularization properties using two basic morphological operators: dilation and erosion with quadratic structuring functions. Namely, the regularity properties of this structuring function are transferred to the image approximations, without computing (discrete) derivatives. In addition, no new maxima/minima are created in the regularized image. In the context of morphological image processing, the latter property makes the regularized images as an appropriate preprocessing for watershed segmentation as well as markers for levelings.

Using canonic Riemannian dilation and erosion, we have considered the theory of generalization from the Euclidean framework to the case of bounded images on Cartan–Hadamard manifolds (nonpositive sectional curvature). However, in practice we observe that it works for bounded images on bounded surfaces of positive and negative curvature. As discussed in [8] and [9], more general versions of Lasry–Lions regularization can be obtained in compact manifolds. In particular the case of compact nonnegative curvature manifolds is relevant for optimal transport problems [29]. Nevertheless, some recent parallel work [7] has provided a complete analysis of the generalization of Lasry–Lions regularization for bounded functions in manifolds of bounded sectional curvature. This study provides also a precise estimate of the Lipschitz constants.

Therefore, Lasry–Lions regularization is also appropriate for $CAT(0)$ spaces [11]. In ongoing research, we will consider in particular the case of regularization of functions on trees (dendrograms), as an example of $CAT(0)$ space. More generally, we will study the extension to other metric spaces appearing in non-Euclidean image processing as well as the case of weighed graph regularization.

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