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Stabilization of photon-number states via single-photon corrections: a first convergence analysis under an ideal set-up

H. B. Silveira  P. S. Pereira da Silva  P. Rouchon

Abstract—This paper presents a first mathematical convergence analysis of a Fock states feedback stabilization scheme via single-photon corrections. This measurement-based feedback has been developed and experimentally tested in 2012 by the cavity quantum electrodynamics group of Serge Haroche and Jean-Michel Raimond. Here, we consider the infinite-dimensional Markov model corresponding to the ideal set-up where detection errors and feedback delays have been disregarded. In this ideal context, we show that any goal Fock state can be stabilized by a Lyapunov-based feedback for any initial quantum state belonging to the dense subset of finite rank density operators with support in a finite photon-number sub-space. Closed-loop simulations illustrate the performance of the feedback law.

I. INTRODUCTION

In [8], a photon-number states (Fock state) feedback stabilization scheme via single-photon corrections was described and experimentally tested. Such control problem is relevant for quantum information applications [6], [4]. The quantum state \( \rho \) corresponds to the density operator of a microwave field stored inside a superconducting cavity and described as a quantum harmonic oscillator. At each sample step \( k \in \mathbb{N} \), a probe atom is launched inside the cavity. The measurement outcome \( y_k \) detected by a sensor is the energy-state of this probe atom after its interaction with the microwave field. Each probe atom is considered as a two-level system: either it is detected in the lowest energy state \(|g\rangle\), or the highest energy state \(|e\rangle\). Consequently, the measurement outcomes corresponds to a discrete-valued output \( y_k \) with only two distinct possibilities: \( g \) or \( e \). Similarly, the control inputs \( u_k \) are also discrete-valued with 3 distinct possibilities: \(-1, 0, +1\). The open-loop value \( u_k = 0 \) corresponds to a dispersive atom/field interaction: it achieves in fact a quantum Non-Demolition measurement of Fock states [2]. The two other values \( u_k = \pm 1 \) correspond to resonant atom/field interactions where the probe atom and the field exchange energy quanta: these values achieve single-photon corrections.

Although the feedback law proposed and implemented in [8] considered imperfect detections on \( y_k \) and delays in the control, here we focus on an ideal-set up, that is, detection errors and control delays have been disregarded. Theorem 2 shows that, by adding an arbitrarily small term to the Lyapunov function used in [8], one ensures almost sure global stabilization of any goal Fock state for the closed-loop ideal set-up. This is achieved by relying on an infinite-dimensional Markov model of the ideal set-up that takes into account the back-action of the measurement outcome \( y_k \) on the quantum state \( \rho_{k+1} \).

Loosely speaking, in [8], the control value \( u_k \) at each sampling step \( k \) was chosen so as to minimize the conditional expectation of the Lyapunov function \( V(\rho) = \text{Tr}(d(N)\rho_k) \), where \( N \) is the photon-number operator, \( d(n) = (n - \overline{n})^2 \) and \( \overline{n} = \langle \overline{n} \rangle \) is the goal Fock state. However, in closed-loop, the difference between such \( V \) and its conditional expectation is not strictly positive: such \( V \) does not become a strict Lyapunov function in closed-loop and additional arguments have to be considered to prove convergence. These additional arguments are related to Lasalle invariance. They are well established in a smooth context where the control \( u \) is a smooth function of the state \( \rho \). This cannot be the case here since \( u \) is a discrete-valued control. In order to overcome such technical difficulties, we propose, similarly to [1], to add the arbitrarily small term \(-\epsilon \sum_{n=0}^{\infty} \langle \rho | n \rangle^2 \) to \( V(\rho_k) \), where \( \epsilon > 0 \). This slightly modified control-Lyapunov function becomes then a strict-Lyapunov function in closed-loop that simplifies notably the convergence analysis. Moreover, the developed convergence analysis is done in the infinite-dimensional setting in the following sense: we show that, for any initial density operator \( \rho_0 \) with a finite photon-number support \( \langle \rho_0 | n \rangle = 0 \) for \( n \) large enough), the closed-loop trajectory \( k \mapsto \rho_k \) remains also with a finite photon-number support with a uniform bound on the maximum photon-number. This almost finite-dimensional behavior simplifies the convergence analysis despite the fact that such condition on \( \rho_0 \) is met on a dense subset of density operators (Hilbert-Schmidt topology on the Banach space of Hilbert-Schmidt self-adjoint operators).

The paper is organized as follows. Section II presents the ideal Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [8] and precisely formulates the Fock state stabilization problem here treated (see Definition 1). Section III establishes the proposed solution to the control problem in two distinct parts. Firstly, Section III-A considers the case where the initial condition \( \rho_0 \) is a diagonal density operator (see Theorem 1).
Only the main ideas of the convergence proof are outlined. The technical details are given in Section [V]. Afterwards, in Section [III-B] the main result of the paper is presented: the general solution is obtained from Theorem [I] for \( \rho_0 \) belonging to a dense subset (see Theorem [2]). The simulation results are exhibited in Section [V]. The proof of some intermediate results and computations required in Sections [III] and [V] are presented in Appendices [A][C]. Finally, the concluding remarks are given in Section [VI].

II. IDEAL MARKOV MODEL

Denote by \( \mathcal{H} \) the separable complex Hilbert space \( L_2(\mathbb{C}) \) with orthonormal basis \( \{ |n\rangle, n \in \mathbb{N} \} \) of Fock states (photons-number). Hence, \( \mathcal{H} = \{ \sum_{n \in \mathbb{N}} \psi_n |n\rangle, \langle \psi_0, \psi_1, \ldots \} \subset L_2(\mathbb{C}) \). Let \( \mathcal{D} \) be the set of all density operators on \( \mathcal{H} \), that is, the set of trace-class, self-adjoint, non-negative operators on \( \mathcal{H} \) with unit trace. The sample step, corresponding to a sampling period around 100\( \mu \text{s} \), is indexed by \( k \in \mathbb{N} = \{ 0, 1, 2, \ldots \} \), where \( u_{k} \in \{ -1, 0, 1 \} \) is the control, \( \rho_k \in \mathcal{D} \) the quantum state and \( y_k \in \{ g, e \} \) the measurement outcome. The ideal Markov model of the controlled microwave superconducting cavity used in [8] is given by:

\[
\rho_{k+1}^g = \frac{M_g(u_k)\rho_k M_g(u_k)^\dagger}{\text{Tr}(M_g(u_k)\rho_k M_g(u_k))} \quad \text{when } y_k = g,
\]
\[
\rho_{k+1}^e = \frac{M_e(u_k)\rho_k M_e(u_k)^\dagger}{\text{Tr}(M_e(u_k)\rho_k M_e(u_k))} \quad \text{when } y_k = e,
\]

where the measurements outcomes \( y_k = g \) and \( y_k = e \) occur with probability

\[ p_{g,k} = \text{Tr}(M_g(u_k)\rho_k M_g(u_k)^\dagger) \]
\[ p_{e,k} = \text{Tr}(M_e(u_k)\rho_k M_e(u_k)^\dagger) = 1 - p_{g,k}. \]

respectively, \( u_k \) is the quantum state and \( y_k \) is the measurement outcome. (Quantum Non-Demolition measurement of photons)

\[ M_g(0) = \cos \left( \frac{\theta_0 N + \phi_0}{2} \right), \quad M_e(0) = \sin \left( \frac{\theta_0 N + \phi_0}{2} \right), \]

(2)

when \( u_k = +1 \) the atom enters the cavity in the state \( |e\rangle \) with a resonant interaction with the cavity field

\[ M_g(+1) = \frac{\sin \left( \frac{\theta_0 \sqrt{N}}{\sqrt{N}} \right)}{\sqrt{N}} a^\dagger, \quad M_e(+1) = \cos \left( \frac{\theta_0 \sqrt{N}}{\sqrt{N}} + 1 \right), \]

(3)

when \( u_k = -1 \) it enters in \( |g\rangle \) with a resonant interaction

\[ M_g(-1) = \cos \left( \frac{\theta_0 \sqrt{N}}{\sqrt{N}} \right), \quad M_e(-1) = a \frac{\sin \left( \frac{\theta_0 \sqrt{N}}{\sqrt{N}} \right)}{\sqrt{N}}, \]

(4)

and \( \theta_0, \phi_0 \in \mathbb{R} \) are adjustable control parameters. For each \( u_k \in \{ -1, 0, 1 \} \), \( M_g(u_k) \) and \( M_e(u_k) \) are (linear) operators on \( \mathcal{H} \) defined in the obvious way according to the definitions in Appendix [A]. They are indeed well-defined operators on \( \mathcal{H} \), despite the fact that \( a \) and \( a^\dagger \) are unbounded.

1As usual in quantum physics, it is here assumed that the measurement outcome \( y_k = g \) cannot occur when \( \text{Tr}(M_g(u_k)\rho_k M_g(u_k)) = 0 \), for \( y = g, e \).

2For instance, \( M_g(+1)|n\rangle = \left( \sin \left( \frac{\theta_0 \sqrt{N}}{\sqrt{N}} \right) \right) \sqrt{n+1} |n+1\rangle \). In order for the definition of \( M_e(-1) \) to be consistent, it is assumed \( \sin(0)/0 = 1 \).

Operators. It is clear that \( M_g(u_k) M_e(u_k) \) are bounded operators on \( \mathcal{H} \) with \( \mathcal{M}_g(u_k) M_g(u_k) + M_e(u_k) M_e(u_k) = I \) (identity operator), \( M_e(-1) = M_e^\dagger(1+1) = a \sin \left( \frac{\theta_0 \sqrt{N}}{\sqrt{N}} \right) / \sqrt{N} \), and \( M_g(-1), M_g(0), M_e(0), M_e(2) \) are self-adjoint. It is easy to see that if the initial condition \( \rho_0 \) is a density operator then, for all realizations of the ideal Markov process \((1) - (4)\), \( \rho_k \) is a density operator for \( k \in \mathbb{N} \).

Notice that \( \pi = |\pi\rangle \langle \pi| \) is a steady state of the Markov process \((1) - (4)\) with \( u_k = 0 \), where \( \pi \in \mathcal{N} \) is arbitrary. The control problem here treated is given as follows:

Definition 1: For the ideal Markov process \((1) - (4)\), the control problem is to find a feedback law \( u_k = f(\rho_k) \) such that, given an initial condition \( \rho_0 \) and \( \pi \in \mathcal{N} \), the closed-loop trajectory \( \rho_k \) converges almost surely towards the goal Fock state \( \pi = |\pi\rangle \langle \pi| \) as \( k \to \infty \).

The almost sure convergence above is with respect to the probabilities amplitudes \( P_n(\rho) = \text{Tr}(\langle n|\rho|n\rangle) = \langle n|\rho|n\rangle \) of \( \rho \), that is, \( \lim_{k \to \infty} P_n(\rho_k) = P_n(\pi) \) for each \( n \in \mathbb{N} \). In other words, \( \lim_{k \to \infty} P_n(u_k) = 1 \text{ and } \lim_{k \to \infty} P_n(u_k) = 0 \) when \( n \neq \pi \). The solution proposed in this paper for the control problem above is developed in Section [III].

III. STABILIZATION OF FOCK STATES

Given any operator \( A : \mathcal{H} \to \mathcal{H} \), let \( A_{mn} = \langle m|A|n\rangle \) for \( m, n \in \mathbb{N} \). Hence, \( A_{nm} \) is the \( n \)-th diagonal element of \( A \), while \( A_{mn} \) with \( m \neq n \) correspond to its “off-diagonal” elements. One says that the operator \( A \) is diagonal when \( A_{mn} = 0 \) for all \( m, n \in \mathbb{N} \) with \( m \neq n \). One shall begin by solving the control problem given in Definition [1] in the particular case where the initial condition \( \rho_0 \) is diagonal (see Theorem [1] in Section [III-A]). Afterwards, in Section [III-B] the solution to the general non-commutative case is presented (see Theorem [2]); its solution relies essentially on the diagonal case.

A. Diagonal case

For each \( n^* \in \mathbb{N} \), define

\[ D_{n^*} = \{ \rho \in \mathcal{D} \mid \rho \text{ is diagonal and } \rho |n^*\rangle = 0, \forall n > n^* \}. \]

Consider the set \( D_{n^*} = \bigcup_{n^* \in \mathbb{N}} D_{n^*} \subset \mathcal{D} \). Note that \( D_{n^*} \subset D_{n^*+1} \), and that each element \( \rho \) of \( D_{n^*} \) is “finite dimensional” in the following sense: \( \rho \in \mathcal{D} \) is in \( D_{n^*} \), if and only if \( \rho = \sum_{n=0}^{n^*} \rho_{nn} |n\rangle \langle n| \), and \( \rho \in D_{n^*} \) may be considered as an operator from \( \mathcal{H} \) to the finite-dimensional space \( \mathcal{H}_{n^*} = \text{span}\{0, \ldots, |n^*\rangle\} \). As a density matrix on \( \mathcal{H}_{n^*} \). One defines the functions \( n_{\text{min}}, D_{n^*} \to \mathbb{N} \), \( n_{\text{max}}, D_{n^*} \to \mathbb{N} \) and \( n_{\text{length}}, D_{n^*} \to \mathbb{N} \), respectively by:

- \( n_{\text{min}}(\rho) \) is the smallest \( n \in \mathbb{N} \) such that \( \rho |n\rangle \neq 0 \);
- \( n_{\text{max}}(\rho) \) is the greatest \( n \in \mathbb{N} \) such that \( \rho |n\rangle \neq 0 \);
- \( n_{\text{length}}(\rho) = n_{\text{max}}(\rho) - n_{\text{min}}(\rho) \).

It is clear that, given \( \rho \in D_{n^*} \), one has \( \rho \in D_{n^*} \), if and only if \( n_{\text{max}}(\rho) \leq n^* \). The next result exhibits the properties of the state \( \rho_k \) of \((1) - (4)\) with respect to these functions.

3Note that if \( \rho = |n\rangle \langle n| \) for some \( n \in \mathbb{N} \), then \( \rho \in D_n \).
**Proposition 1:** For every realization of the ideal Markov process (1–4) with initial condition $ρ_0 \in D_*$, one has that $ρ_k \in D_*$ for all $k \in \mathbb{N}$ with:

- If $u_k = 0$ or $u_k = -1$, then $n_{\max}(ρ_{k+1}) \leq n_{\max}(ρ_k)$ and $n_{\length}(ρ_{k+1}) \leq n_{\length}(ρ_k)$;
- If $u_k = +1$, then $n_{\max}(ρ_{k+1}) \leq n_{\max}(ρ_k) + 1$ and $n_{\length}(ρ_{k+1}) \leq n_{\length}(ρ_k)$.

**Proof:** See Appendix B.

Take a goal photon-number $π \in \mathbb{N}$. As in [1], consider the following Lyapunov function $V_ε: D_* \to \mathbb{R}$ defined as

$$V_ε(ρ) = \text{Tr} (d(N)ρ) - ε \sum_{n \in \mathbb{N}} ρ^2_{nn}, \quad \text{for } ρ \in D_*,$$

where $ε > 0$ is a real number and $d(n) = (n-π)^2$ as defined in [8]. The feedback law $u: D_* \to \{-1,0,1\}$ is given by

$$u = f(ρ) \triangleq \text{Argmin } V_ε(ρ_{k+1}) \quad \text{with } ρ_k = ρ, \quad u_k = u.$$

Note that for each $ρ \in D_*$ and $n^* \geq n_{\max}(ρ)$, $d(N)ρ$ in (5) is a well-defined self-adjoint, non-trace-class operator on $H$, by considering $d(N)$ as an operator on $H_\rho$, and $ρ$ as an operator from $H$ to $H_\rho$. Indeed, $d(N)ρ = \sum_{n=0}^{n^*} ρ_{nn}(n-π)^2/n|n\rangle \langle n|$. Thus, (5) is well-defined. Moreover, since $H_\rho$ is invariant under $ρ \in D_*$ for $n^* \geq n_{\max}(ρ)$, it is clear that $\text{Tr} (d(N)ρ) = \text{Tr}_{H_\rho}(d(N)ρ)$, where the right-hand side one considers $ρ$ as an operator on the finite-dimensional space $H_\rho$, and the trace is taken over $H_\rho$.

We have the following convergence result when $ρ_0 \in D_*:

**Theorem 1:** Let $π \in \mathbb{N}$ and $ε > 0$. In (2–4), assume that $ϕ_0/π$ and $(θ_0/π)^2$ are irrational numbers, and take $ϕ_R = π/2 - πϕ_0$. Consider the closed-loop Markov process (1–4) with $u_k = f(ρ_k)$, where the feedback law $f$ is as in (6). Then, given any initial condition $ρ_0 \in D_*$, one has that $ρ_k$ converges almost surely towards $π = |π⟩⟨π|$ as $k \to ∞$.

Its proof is decomposed into two steps:

**First Step.** Choose $π \in \mathbb{N}$ and $ε > 0$. Let $n_0 = n_{\length}(ρ_0)$, $r_0 = n_{\min}(ρ_0)$. Then, there exists an integer $n_0 > n_0 + r_0 + π + 1$ (depending on $n_0, r_0, π$ and $ε$) such that, for all closed-loop realizations $ρ_k$, one has $ρ_k \in D_{n_0}$ for $k \in \mathbb{N}$.

**Second Step.** Choose irrational numbers $ϕ_0/π$ and $(θ_0/π)^2$ in (2–4), and take $ϕ_R = π/2 - πϕ_0$. In $D_{n_0}$, $V_ε$ is a strict super-martingale: for all density operators $ρ \in D_{n_0}$, one has

$$E[V_ε(ρ_{k+1}) | ρ_k, u_k = f(ρ)] = \langle V_ε(ρ, f(ρ)) - Q_ε(ρ, f(ρ)) \rangle,$$

where $Q_ε(ρ, f(ρ)) \geq 0$, and $Q_ε(ρ, f(ρ)) = 0$ if and only if $ρ = π$. The almost sure convergence follows then from usual results on strict super-martingales for Markov processes with compact state spaces.

The complete proof of the two steps above is presented in Section [N]. The general case where the initial condition $ρ_0$ is not necessarily diagonal is treated in the next subsection.

**B. General case**

Consider, for each $n^* \in \mathbb{N}$,

$$D_{n^*} = \{ρ \in D : |ρ|n = 0, ∀n > n^* \} \subset D_{n^*+1},$$

and let $D_* = \bigcup_{n^* \in \mathbb{N}} D_{n^*}$. It is clear that $ρ \in D_*$ if and only if $ρ = \sum_{n,m=0}^∞ ρ_{mn}|m⟩⟨n|$. Consequently, $D_*$ is a dense subset of $D$ when $D$ is endowed with the subspace topology induced from the Hilbert-Schmidt norm.

Indeed, let $J_2$ be the complex Banach space of all Hilbert-Schmidt operators on $H$ with the Hilbert-Schmidt norm $||B||_2 = (\sum_{m,n=0}^∞ |B_{mn}|^2)^{1/2}$, for $B \in J_2$ [7], [3]. Since $D \subset J_2$ and $ρ \in D_{n^*}$ has the form $ρ = \sum_{n,m=0}^∞ ρ_{mn}|m⟩⟨n|$, the density property of $D_*$ in $D$ is clear.

One has that $ρ \in D_{n^*}$ may be considered as an operator from $H$ to the finite-dimensional space $H_{n^*}$, or as a density matrix on $H_{n^*}$. Hence, $d(N)ρ$ is a well-defined trace-class operator on $H$, by considering $d(N)$ as an operator on $H_{n^*}$ and $ρ \in D_{n^*}$ as an operator from $H$ to $H_{n^*}$. Indeed, $d(N)ρ = \sum_{n=0}^{∞} ρ_{mn}(m-π)^2|m⟩⟨n|$, and it is trace-class because its range is finite-dimensional [7], [3]. Consequently, the Lyapunov function $V_ε$ in (5), the feedback in (6) and $n_{\max}$ can be extended to $D_*$. Define the map $Δ : D_* \to D_* \subset D_*$ as $Δ = \sum_{n=0}^{∞} ρ_{mn}(m-π)^2|m⟩⟨n|$. Note that $Δ$ extracts the diagonal of $ρ \in D_*$. It is easy to see that $n_{\max}(Δρ) = n_{\max}(ρ)$ and $(Δρ)_{nn} = ρ_{nn}, ρ \in D_*$. Moreover, $Δρ = ρ$ when $ρ \in D_*$. Other properties of the map $Δ$ are given in the next result:

**Proposition 2:** Let $ρ \in D_*, u \in \{-1,0,1\}, y = y_e$. Take $α = \text{Tr} (M_y(u)ρM_y^†(u))$. Then:

- $\text{Tr}(Δρ) = \text{Tr}(Δ(Δρ))$, for every diagonal bounded operator $A: H \to H$;
- $V_ε(ρ) = \text{Tr}(Δρ)$, for $ε > 0$;
- $α^{-1}M_y(u)M_y^†(u)$ belongs to $D_*$ with $Δ(α^{-1}M_y(u)M_y^†(u)) = α^{-1}M_y(u)(Δρ)M_y^†(u)$;
- $[M_y(u)(Δρ)M_y^†(u)]_{nn} = [M_y(u)ρM_y^†(u)]_{nn}$, for all $n \in \mathbb{N}$. In particular, $α = \text{Tr} (M_y(u)ρM_y^†(u)).$

**Proof:** See Appendix C.

Now, let $ε > 0$ and $π = |π⟩⟨π|$, where $π \in \mathbb{N}$. Assume that $ρ_0 \in D_*$. Let $ρ_k, k \in \mathbb{N}$, be the corresponding closed-loop trajectory for a fixed realization of (1–4) with feedback $u_k = f(ρ_k)$, where $f$ is as in (6). It is immediate from the proposition above that:

- $ρ_k \in D_*$, for $k \in \mathbb{N}$;
- $Δρ_k \in D_*, k \in \mathbb{N}$, is the corresponding closed-loop trajectory of (1–4) for the initial condition $Δρ_0$, the same realization (and with the same transition probabilities $p_{e,k}$ and $p_{k,k}$), as well as the same feedback $u_k = f(ρ_k) = f(Δρ_k)$;
- $\text{Tr}(|n⟩⟨n|Δρ_k) = \text{Tr}(|n⟩⟨n|Δρ_0)$, for any $n \in \mathbb{N}$.

From these arguments, Theorem [N] and the fact that $Δπ = π$, one immediately obtains the following generic solution to the control problem, that is, when the initial condition $ρ_0$ belongs to the dense subset $D_*$ of $D$.

**Theorem 2:** Let $π \in \mathbb{N}$ and $ε > 0$. In (2–4), assume that $ϕ_0/π$ and $(θ_0/π)^2$ are irrational numbers, and take $ϕ_R = π/2 - πϕ_0$. Consider the closed-loop Markov process (1–4) with $u_k = f(ρ_k)$, where the feedback law $f$ is as in (6). Then, given any initial condition $ρ_0 \in D_*$, one has that $ρ_k$ converges almost surely towards $π = |π⟩⟨π|$ as $k \to ∞$.
IV. SIMULATION RESULTS

This section presents the closed-loop simulation results concerning the application of Theorem 2 above to the ideal Markov process (1–4). The quantum experimental results exhibited in [8] used the following control parameter values in (2–4): $\rho_0/\pi = 0.252$ and $\theta_0/\pi \approx 2/\sqrt{\pi} + 1$. However, according to the assumptions in Theorem 2, $\rho_0/\pi$ and $(\theta_0/\pi)^2$ should be irrational numbers. Hence, here one chooses $\rho_0/3.14 = 0.252$ and $\theta_0/3.14 = 2/\sqrt{\pi} + 1$. One takes $\rho_0 = \sum_{n=0}^{15} |n\rangle\langle n|/16 \in \mathbb{D}$, as the initial condition, $\pi = 10$ for the goal Fock state $\rho = |\pi\rangle\langle \pi|$, and $\epsilon = 10^3$ as the gain for the feedback $u_k = f(\rho_k)$ in (5)–(6). Figure 1 exhibits the simulation results for one closed-loop realization with such choices and a final sample step of 120. It shows: the dynamics of the populations of $\rho_k$ (top), the controls $u_k$ (middle) and the simulated outcomes $y_k$ (bottom). The populations of $\rho_k$ correspond to the following observables: 

$A_1 = \sum_{n=1}^{9} |n\rangle\langle n| \quad (n < \pi)$,

$A_2 = |\pi\rangle\langle \pi| \quad (n = \pi)$,

$A_3 = \sum_{n=14}^{15} |n\rangle\langle n| \quad (n > \pi)$.

Therefore, one sees from the dynamics of the populations that $\rho_k$ converges to $|\pi\rangle\langle \pi|$ as $k \rightarrow \infty$, which is in accordance with Theorem 2. Note that $\langle |\pi\rangle |\pi\rangle \approx 1$ and $u_k = 0$ for all $k > 45$.

Recall that Theorem 2 assumes that $\epsilon > 0$. In order to further analyze the performance of the Lyapunov-based feedback law here proposed, we now make a comparison with the one used experimentally in [8], which corresponds to take $\epsilon = 0$ in (5), i.e. to disregard the term $-\epsilon \sum_{n\in\mathbb{N}} p_{nn}^2$. Figure 2 presents the simulation results for one closed-loop realization of such case. The control parameters, $\rho_0$ and $\pi = 10$ are the same as above. Note that $\langle |\pi\rangle |\pi\rangle \approx 1$ and $u_k = 0$ for all $k > 78$. In order to make a comparison in terms of the speed of convergence, define the settling time $k_s$ to be the smallest $k \in \mathbb{N}$ such that $\langle |\pi\rangle |\pi\rangle > 0.9$ for all $k \geq k_s$. One has $k_s = 45$ for the case $\epsilon = 10^3$ above, and $k_s = 78$ for $\epsilon = 0$. Therefore, in the two realizations here simulated, the choice of $\epsilon = 10^3$ reduced the settling time $k_s$ by nearly 42% with respect to $\epsilon = 0$. This behavior is typical on an average basis, thereby justifying the term $-\epsilon \sum_{n\in\mathbb{N}} p_{nn}^2$ in (5). Table I shows the average value $\bar{k}_s$ and the standard deviation $\sigma$ of $k_s$ for $\epsilon \in \{0, 0.1, 1, 10, 10^2, 10^3, 10^4, 10^5\}$, where a total of 5000 realizations were simulated for each $\epsilon$. Notice that when $\epsilon$ is relatively large or relatively small in comparison to $\epsilon = 10^3$, the average settling time $\bar{k}_s$ deteriorated. Furthermore, although for $\epsilon = 10^5$ one has that $\bar{k}_s$ increased by nearly 22% in comparison to $\epsilon = 10^3$, the standard deviation $\sigma$ decreased by nearly 62%. Computer simulations have suggested that a choice of $\epsilon > 0$ which may perhaps significantly improve $\bar{k}_s$ generally depends on the initial condition $\rho_0$ and on the goal Fock state $\rho = |\pi\rangle\langle \pi|$, and it has to be determined heuristically.

V. PROOF OF THEOREM 1 (DIAGONAL CASE)

Proof of the First Step:
Let $\epsilon > 0$. Define $V: D_s \rightarrow \mathbb{R}$ and $W: D_s \rightarrow \mathbb{R}$ as

$V(\rho) = \text{Tr} (d(N)\rho)$,

$W(\rho) = -\sum_{n\in\mathbb{N}} p_{nn}^2$.

respectively. Note that $V_1 = V + eW$. Define:

- $Q_W(\rho, u) = W(\rho) - \mathbb{E} [W(\rho_{k+1}) | \rho_k = \rho, u_k = u]$, 
- $Q_V(\rho, u) = V(\rho) - \mathbb{E} [V(\rho_{k+1}) | \rho_k = \rho, u_k = u]$, 
- $Q_{V_1}(\rho, u) = V_1(\rho) - \mathbb{E} [V_1(\rho_{k+1}) | \rho_k = \rho, u_k = u]$,
for $\rho \in D_*$ and $u \in \{-1, 0, 1\}$. The proof of Theorem 1 is a straightforward consequence of the next proposition:

**Proposition 1.** Let $\epsilon > 0$ and $n_0, r_0, \pi \in \mathbb{N}$. There exists an integer $m_0 > n_0 + r_0 + \pi + 1$ (depending on $\epsilon, n_0, r_0, \pi$) such that, for each $\rho \in D_*$ with $\text{length}(\rho) \leq n_0$, if $\text{max}(\rho) = m_0$, then

$$Q_V(\rho, -1) > \max \{Q_V(\rho, 0), Q_V(\rho, +1)\}.$$

In fact, given $\rho_0 \in D_*$, let $n_0 = \text{length}(\rho_0)$ and $r_0 = \text{min}(\rho_0)$. Note that $\text{max}(\rho_0) = m_0$ for $\rho_0 \in D_*$ with $\text{length}(\rho_0) \leq n_0$. By Proposition 1, $\rho_0 \in D_*$ with $\text{length}(\rho_0) \leq n_0$ for all $k \in \mathbb{N}$. Since $u = f(\rho)$ maximizes $Q_V(\rho, u)$, Proposition 1 implies that when $\text{max}(\rho_k) = m_0$ for some $k \in \mathbb{N}$, then the input $u_k$ will always be equal to $-1$, and hence Proposition 1 ensures that $\text{max}(\rho_k+1) \leq \text{max}(\rho_k) = m_0$. Therefore, $\text{max}(\rho_k) \leq m_0$, $k \in \mathbb{N}$, showing the First Step.

The following two lemmas are instrumental for proving Proposition 2. Their proofs are given in Appendix D and Appendix E respectively.

**Lemma 1.** Given an arbitrary nonzero $\theta_0 \in \mathbb{R}$, fix any $a \in \mathbb{R}$ such that $0 < a < 1/2$. For all nonzero $n_0, N \in \mathbb{N}$, there exists an integer $\bar{N} > N$ big enough such that,

$$0 < 1/2 - a \leq \sin^2 \left(\frac{\theta_0}{2\sqrt{n}}\right) \leq 1/2 + a,$$

for $n = \bar{N}, \bar{N} + 1, \ldots, \bar{N} + n_0 - 1$.

**Lemma 2.** Let $\rho \in D_*$. Then:

- $|Q_V(\rho, u)| \leq 1$, for each $u \in \{-1, 0, 1\}$;
- $Q_V(\rho, 0) = 0$;
- $Q_V(\rho, +1) = -\sum_{n \in \mathbb{N}} \rho_{nn}[2(n - \pi) + 1] \sin^2 \left(\frac{\theta_0}{2\sqrt{n+1}}\right)$;
- $Q_V(\rho, -1) = \sum_{n \in \mathbb{N}} \rho_{nn}[2(n - \pi) - 1] \sin^2 \left(\frac{\theta_0}{2\sqrt{n}}\right)$.

The proof of Proposition 2 is shown in the sequel. 

**Proof:** Let $\epsilon > 0$ and $n_0, r_0, \pi \in \mathbb{N}$. One has to show that there exists $m_0 > n_0 + r_0 + \pi + 1$ such that, if $\rho \in D_*$ with $\text{length}(\rho) \leq n_0$, then $u = -1$ always maximizes $Q_V(\rho, u)$ whenever $\text{max}(\rho) = m_0$. From Lemma 2 and the fact that $Q_V = Q_V + cQ_W$, to complete the proof it suffices to show that:

- If $\rho \in D_*$ is such that $\text{length}(\rho) \leq n_0$ and $\text{max}(\rho) \geq n_0 + \pi$, then $Q_V(\rho, +1) \leq 0$;
- There exists $m_0 > n_0 + r_0 + \pi + 1$ such that $Q_V(\rho, -1) > 2\epsilon$, whenever $\rho \in D_*$ is such that $\text{length}(\rho) \leq n_0$ and $\text{max}(\rho) = m_0$. 

Note that

$$Q_V(\rho, +1) = -\sum_{n = \text{min}(\rho)}^{\text{max}(\rho)} \rho_{nn}[2(n - \pi) + 1] \sin^2 \left(\frac{\theta_0}{2\sqrt{n+1}}\right),$$

for any $\rho \in D_*$. Thus, if $\text{length}(\rho) \leq n_0$ and $\text{max}(\rho) \geq \pi + n_0$, then $\text{min}(\rho) \geq \pi$, and hence the first claim is shown.

Now, fix $0 < a < 1/2$ and let $N \geq \frac{1}{\sqrt{2a}} \left[\frac{2}{1/2 - a} + 2\pi + 1\right]$. Applying Lemma 1 for $N_0 = n_0 + r_0 + 1$ and such choice of $N$, one gets $\bar{N} > N$ in which $0 < 1/2 - a \leq \sin^2 \left(\frac{\theta_0}{2\sqrt{n+1}}\right)$.

As $N$ is an integer, it follows that $N \geq \pi + 1$.

For $n = N, N + 1, \ldots, N + n_0 + r_0$. Take $m_0 = N + n_0 + r_0$. Let $\rho \in D_*$ with $\text{length}(\rho) \leq n_0$ and $\text{max}(\rho) = m_0$. Note that $m_0 > n_0 + r_0 + \pi + 1$ and $\text{min}(\rho) \geq \pi + n_0 + r_0$. From Lemma 1 and the inequality above for $1/2 - a$, one obtains

$$Q_V(\rho, -1) = \sum_{n = \text{min}(\rho)}^{m_0} \rho_{nn}[2(n - \pi) - 1] \sin^2 \left(\frac{\theta_0}{2\sqrt{n}}\right),$$

$$\geq \sum_{n = \text{min}(\rho)}^{m_0} \rho_{nn}[2(n - \pi) - 1](1/2 - a),$$

$$= \left[2(\pi - n) - 1\right](1/2 - a) \sum_{n = \text{min}(\rho)}^{m_0} \rho_{nn}.$$
Theorem 3: [5, Theorem 1, p. 195] Let \( \Omega \) be a probability space and let \( W \) be a measurable space. Consider that \( X_k: \Omega \to W, k \in \mathbb{N}, \) is a Markov chain with respect to the natural filtration. Let \( Q: W \to \mathbb{R} \) and \( V: W \to \mathbb{R} \) be measurable non-negative functions with \( V(X_k) \) integrable for all \( k \in \mathbb{N} \). If \( \mathbb{E} [V(X_{k+1}) | X_k] - V(X_k) = -Q(X_k) \), for \( k \in \mathbb{N} \), then \( \lim_{k \to \infty} Q(X_k) = 0 \) almost surely.

Indeed, let \( \mathcal{F}_t \) be the complex Banach space of all trace-class operators on \( \mathcal{H} \) with the trace norm \( | \cdot |_1 \), that is, \( |B|_1 = \text{Tr}(|B|) \), where \( |B| \triangleq \sqrt{B^*B} \), for \( B \in \mathcal{F}_t \). Recall that \( |B| \leq |A|_1 \) and \( |B| \leq \|B\|_1 \) for every \( B \in \mathcal{F}_t \) and each bounded operator \( A: \mathcal{H} \to \mathcal{H}, \) where \( \| \cdot \|_1 \) is the usual operator norm (sup norm of bounded operators) [7], [3]. Consider the subspace topology on \( D_{m_0} \) with respect to \( \mathcal{F}_t \). One has that the closed-loop trajectory \( \rho_k, k \in \mathbb{N} \), is a Markov chain with phase space \( D_{m_0} \) (with respect to the natural filtration and the Borel algebra on \( D_{m_0} \)). It is clear that \( D_{m_0} \) is compact, and that \( Q_e \) and \( V_e \) are non-negative and continuous on \( D_{m_0} \), for all \( e > 0 \), where \( \alpha_e \triangleq \min_{\rho \in D_{m_0}} V_e(\rho) \). The theorem above implies that \( \rho_k \) converges almost surely towards \( \bar{\rho} \) as \( k \to \infty \) (with respect to the trace norm). This completes the proof of Theorem [1]

VI. CONCLUDING REMARKS

This paper provided a convergence analysis of Fock states stabilization via single-photon corrections under an ideal set-up, that is, assuming perfect measurement detection and no control delays. In terms of convergence speed, the simulation results here presented have justified the inclusion of the term \( -e \sum_{n \in \mathbb{N}} \ell_{mn}^2 \) in the Lyapunov-based feedback law [3]–[5]. It is straightforward to verify that the convergence analysis developed in this paper remains valid for: (i) any other function \( d(n) \) in [3] satisfying \( d(\bar{\rho}) = 0, d(n) \) is increasing for \( n > \bar{\rho} \) and \( d(n) \) is decreasing for \( n < \bar{\rho} \); and (ii) \( e > 0 \) dependent on \( n \), that is, to take the term \( -\sum_{n \in \mathbb{N}} \epsilon_n \rho_{mn}^2 \). However, it is an open problem how to choose the function \( d(n) \) and the gains \( \epsilon_n > 0 \) so as to achieve the best convergence speed.

Finally, the feedback law used in [8], which corresponds to \( \epsilon = 0 \), was tailored for an experimental set-up with measurement imperfections and control delays. The convergence analysis of such realistic situation will be investigated in the future.

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APPENDIX

A. Basic properties of the operators \( \mathcal{N} \), \( \alpha \) and \( \alpha^\dagger \)

Fix \( n^* \in \mathbb{N} \) and let \( \mathcal{H}_{n^*} = \text{span}\{0, \ldots, [n^*]\} \). Consider the (linear) operators \( \mathcal{N}: \mathcal{H}_{n^*} \to \mathcal{H}_{n^*}, \alpha: \mathcal{H}_{n^*} \to \mathcal{H}_{n^*+1}, \alpha^\dagger: \mathcal{H}_{n^*+1} \to \mathcal{H}_{n^*} \), defined respectively as \( \mathcal{N}(n) = n |n\rangle \), \( \alpha(0) = 0, \alpha(n) = \sqrt{n} |n-1\rangle \) for \( n \geq 1 \).

\(^5\)One also recalls that if \( A \) is a bounded operator on \( \mathcal{H} \) and \( B \in \mathcal{F}_t \), then \( AB, BA \in \mathcal{F}_t \) with \( \text{Tr}(AB) = \text{Tr}(BA) \).

\[ a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \] Note that these operators cannot be extended to \( \mathcal{H} \). Let \( f: \mathbb{N} \to \mathbb{R} \) be a function. Define the operator \( f(N): \mathcal{H}_{n^*} \to \mathcal{H}_{n^*} \) by \( f(N)(n) = f(n)|n\rangle \), for each \( n = 0, \ldots, n^* \). It is clear that \( f(N) \) can be extended to \( \mathcal{H} \) whenever \( f \) is a bounded function. Given \( f: \mathbb{N} \to \mathbb{R} \) and an integer \( m \), one defines \( g: \mathbb{N} \to \mathbb{R} \) as: \( g(n) = f(n+m) \), when \( n+m \geq 0 \); and \( g(n) = 0 \), when \( n+m < 0 \). One abuses notation letting \( f(N+m) \) stand for \( f(N) \). Given two functions \( f, g: \mathbb{N} \to \mathbb{R} \), it is clear that \( f(N)g(N) = g(N)f(N) = (fg)(N) \) and \( f(g)(N) = f(N) + g(N) \). For example: \( aa^\dagger = \mathcal{N} + 1 \). 

B. Proof of Proposition [2]

Fix any \( \rho \in D_* \) and let \( n \in \mathbb{N} \). In particular, \( \rho(n) = \rho_{nn} |n\rangle \). It then follows from (2)–(4) that:

\[ M_g(0)\rho M_g^\dagger(0)|n\rangle = \rho_{nn} \cos^2 \left( \frac{\phi_n + \phi_{n+1}}{2} \right) |n\rangle, \]

\[ M_e(0)\rho M_e^\dagger(0)|n\rangle = \rho_{nn} \sin^2 \left( \frac{\phi_n + \phi_{n+1}}{2} \right) |n\rangle, \]

\[ M_g(1)\rho M_g^\dagger(1)|n\rangle = \left\{ \begin{array}{ll} 0, & \text{for } n = 0, \\ \rho_{n-1,n-1} \sin^2 \left( \frac{\phi_{n-1} + \phi_{n}}{2} \right), & \text{for } n \geq 1. \end{array} \right. \]

\[ M_g(1)\rho M_g^\dagger(-1)|n\rangle = \rho_{n+1,n+1} \sin^2 \left( \frac{\phi_{n+1} + \phi_{n}}{2} \right), \]

\[ M_e(1)\rho M_e^\dagger(-1)|n\rangle = \rho_{n+1,n+1} \sin^2 \left( \frac{\phi_{n+1} + \phi_{n}}{2} \right). \]

Therefore:

\[ M_g(0)\rho M_g^\dagger(0) = \sum_{n \in \mathbb{N}} \rho_{mm} \cos^2 \left( \frac{\phi_n + \phi_{n+1}}{2} \right) |n\rangle \langle n|, \]

\[ M_e(0)\rho M_e^\dagger(0) = \sum_{n \in \mathbb{N}} \rho_{mm} \sin^2 \left( \frac{\phi_n + \phi_{n+1}}{2} \right) |n\rangle \langle n|, \]

\[ M_g(1)\rho M_g^\dagger(1) = \sum_{n \in \mathbb{N}} \rho_{n-1,n-1} \sin^2 \left( \frac{\phi_{n-1} + \phi_{n}}{2} \right) |n\rangle \langle n|, \]

\[ M_g(-1)\rho M_g^\dagger(-1) = \sum_{n \in \mathbb{N}} \rho_{n+1,n+1} \sin^2 \left( \frac{\phi_{n+1} + \phi_{n}}{2} \right) |n\rangle \langle n|. \]

By assumption, \( \rho_0 \in D_* \). Then, (1), (8)–(13) above and induction on \( k \) show the assertions in Proposition [1]

C. Computation of \( Q_V(\rho, u) \)

Fix any \( \rho \in D_* \) and \( \bar{\rho} \in \mathbb{N} \). Recall that \( V(\rho) = \text{Tr}(d(N)|\rho) \), where \( d: \mathbb{N} \to \mathbb{R} \) be given by \( d(n) = (n-\bar{\rho})^2 \). Note that (1) implies that, for each \( u \in \{-1,0,1\}, \)

\[ \mathbb{E}[V(\rho_{k+1}) | u_k = u] = \text{Tr} \left( d(N)M_g(u)\rho M^\dagger_g(u) + d(N)M_e(u)\rho M^\dagger_e(u) \right). \]
Take $u = 0$. From (33)–(35) in Appendix B, one has

\[ \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = 0] \]

\[ = \text{Tr} \left( d(N) M_g(0) \rho M_g^T(0) \right) + \text{Tr} \left( d(N) M_c(0) \rho M_c^T(0) \right) \]

\[ = \text{Tr} \left( d(N) \left[ M_g(0) \rho M_g^T(0) + M_c(0) \rho M_c^T(0) \right] \right) \]

\[ = \text{Tr}(d(N)\rho) = V(\rho). \]

In particular,

\[ Q_V(\rho, 0) = 0. \]  \hspace{1cm} (15)

Now, take $u = +1$. Then, (14) above and (10)–(11) in Appendix B provide that

\[ \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = +1] \]

\[ = \text{Tr}(d(N)\rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N+1} \right) d(N + 1) \right) \]

\[ = V(\rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N+1} \right) [d(N + 1) - d(N)] \right). \]

By summing and subtracting $\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N+1} \right) d(N) \right)$,

\[ \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = +1] \]

\[ = \text{Tr}(d(N)\rho) + \text{Tr} \left[ \sin^2 \left( \frac{\theta_0}{2} \sqrt{N+1} \right) [d(N + 1) - d(N)] \right]. \]

In particular,

\[ Q_V(\rho, +1) = -\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N+1} \right) [d(N + 1) - d(N)] \right), \]

\[ = -\sum_{n \in \mathbb{N}} 2(n - \pi + 1) \sin^2 \left( \frac{\theta_0}{2} \sqrt{n+1} \right). \]  \hspace{1cm} (16)

Finally, take $u = -1$. Using (14) above and (12)–(13) in Appendix B, one has

\[ \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = -1] \]

\[ = \text{Tr}(d(N)\rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N - 1) \right) \]

\[ = V(\rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) [d(N - 1) - d(N)] \right). \]

By summing and subtracting $\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N) \right)$,

\[ \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = -1] \]

\[ = \text{Tr}(d(N)\rho) + \text{Tr} \left[ \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) [d(N - 1) - d(N)] \right]. \]

In particular,

\[ Q_V(\rho, -1) = -\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) [d(N - 1) - d(N)] \right), \]

\[ = \sum_{n \in \mathbb{N}} 2(n - \pi - 1) \sin^2 \left( \frac{\theta_0}{2} \sqrt{n} \right). \]  \hspace{1cm} (17)

D. Proof of Lemma 7

Assume that $N_0$ is even (otherwise one may take $N_0 + 1$ instead of $N_0$ in this proof). Define the function $\eta$: $\mathbb{N} \to \mathbb{R}$ by

\[ \eta(\ell) = \left[ \frac{2}{\theta_0} \left( \ell \frac{\pi}{2} + \frac{\pi}{4} \right) \right]^2. \]  \hspace{1cm} (18)

By definition, one has $\frac{\theta_0}{2} \sqrt{\eta(\ell)} = \ell \frac{\pi}{2} + \frac{\pi}{4}$ for all $\ell \in \mathbb{N}$. Let $h = \pi/4 - \arcsin \left( \sqrt{1/2 - a} \right)$. Using the definition of $h$ and the symmetric of the function $\sin^2(\cdot)$, it is easy to show that

\[ 1/2 - a \leq \sin^2(x + \pi/4) \leq a + 1/2, \quad \forall x \in [-h, h]. \]  \hspace{1cm} (19)

Let $\ell \in \mathbb{N}$ be even and big enough such that the following two conditions are simultaneously met:

\[ \eta(\ell) > N_0/2 + N, \quad \frac{1}{8} \theta_0 \rho_0 / \sqrt{\eta(\ell)} - N_0/2 \leq h. \]  \hspace{1cm} (20)

Now, take $N = \lfloor |\eta(\ell)| - N_0/2 + 1 \rfloor$. From (1)–(4), one has $|\eta(\ell)| - N_0/2 < N$. Hence, when $\theta_0 \rho_0 > 8N_0/\sqrt{\eta(\ell)}$, one obtains

\[ \left| \frac{1}{2} \theta_0 \rho_0 (\sqrt{\eta(\ell)} - N_0/2) \right| < h. \]

Then, the proof follows easily from (13), (19) and the fact that $\sin^2(x - \ell \pi/2) = \sin^2(x)$, for every even $x \in \mathbb{N}$.

E. Proof of Lemma 2

Proof of the first claim: Let $u \in \{-1, 0, 1\}, \rho \in D_x$. Recall that $W(\rho) = -\sum_{n \in \mathbb{N}} \rho_n^2$. Since $\text{Tr}(\rho) = \sum_{n \in \mathbb{N}} \rho_n = 1$, then

\[ -1 = -\sum_{n \in \mathbb{N}} \rho_n \leq W(\rho) \leq 0. \]

Note, by (1),

\[ \mathbb{E}[W(\rho_{k+1}) \mid \rho_k = \rho, u_k = u] = \rho_{g,k} W(\rho_{g,k}^T) + \rho_{c,k} W(\rho_{c,k}^T), \]

where $\rho_{g,k}, \rho_{c,k} \geq 0$ with $\rho_{g,k} + \rho_{c,k} = 1$. Thus

\[ -1 \leq \mathbb{E}[W(\rho_{k+1}) \mid \rho_k = \rho, u_k = u]. \]

Since $W(\rho, u)$ is the difference of two numbers that are between $-1$ and $0$, one concludes that $|W(\rho, u)| \leq 1$.

The second, third and fourth claims, are immediate from (15), (16) and (17) in Appendix C respectively.

F. Proof of Lemma 3

Proof of the first claim: Let $\rho \in D_{m_0}$. By (38)–(40) in Appendix B, one has

\[ M_g(0) \rho M_g^T(0) + M_c(0) \rho M_c^T(0) = \rho. \]

Taking $\rho_k = \rho$ in $u_k = 0$ in (1), define

\[ \rho' \triangleq \rho_{k+1} = \frac{M_g(0) \rho M_g^T(0)}{\text{Tr} \left( M_g(0) \rho M_g^T(0) \right)}, \quad \text{for } g = g, e. \]

Hence, $\alpha \rho' + (1 - \alpha) \rho' = \rho$, where $\alpha \triangleq \rho_{g,k} = \text{Tr} \left( M_g(0) \rho M_g^T(0) \right)$. In particular, $\alpha \rho_{n,n} + (1 - \alpha) \rho_{n,n} = \rho_{n,n}$, for $n \in \mathbb{N}$. Note that, if $\alpha = 0$, then $M_g(0) \rho M_g^T(0) = 0$, and so $\rho' = \rho$. Similarly, $\alpha = 1$ implies $\rho' = \rho$. Thus, the identity $\alpha \rho_{n,n} + (1 - \alpha) \rho_{n,n} = \rho_{n,n}$, for $n \in \mathbb{N}$, still holds when $\alpha = 0$ or $\alpha = 1$. From (1), (7) and $\alpha = \rho_{g,k}$, one has

\[ Q_W(\rho, 0) = W(\rho) - \left[ \rho_{g,k} W(\rho_{g,k}^T) + \rho_{c,k} W(\rho_{c,k}^T) \right] \]

\[ = \sum_{n \in \mathbb{N}} (\alpha \rho_{n,n}^2 + (1 - \alpha) \rho_{n,n}^2) \]

\[ = \alpha (1 - \alpha) \sum_{n \in \mathbb{N}} [\rho_{n,n}^2 - \rho_{n,n}^2] \geq 0. \]  \hspace{1cm} (21)

6More precisely, $\sin^2(\pi/2 - x) = 1 - \cos^2(\pi/2 - x) = 1 - \sin^2(x)$.\]
thereby showing the first part of the first claim. If $\rho = |m\rangle\langle m|$ for some $m \in \mathbb{N}$ with $0 < \alpha < 1$, then (8)-(10) in Appendix B imply that $\rho^g = \rho^r = \rho$, and so $Q_W(\rho, 0) = 0$. Now, one shows that $\rho = |m\rangle\langle m|$ for some $m \in \mathbb{N}$ whenever $Q_W(\rho, 0) = 0$. Suppose $Q_W(\rho, 0) = 0$. Then, (21) implies that $\alpha = 0$, or $\alpha = 1$, or $\rho^g_n = \rho^r_n$ for all $n \in \mathbb{N}$ with $0 < \alpha < 1$. Assume that $\alpha = 0$. Hence,

$$M_g(0)\rho M_g^\dagger(0) = \sum_{m,n \in \mathbb{N}} \rho_{mn} \cos^2(\frac{\phi_m + \phi_n}{2}) |m\rangle\langle n| = 0$$

by (8) in Appendix B. Suppose that $\rho \not= |m\rangle\langle m|$ for every $m \in \mathbb{N}$. Thus, there exists $n_1, n_2 \in \mathbb{N}$ with $n_1 \neq n_2$, $\rho_{n_1,n_1} > 0$, $\rho_{n_2,n_2} > 0$. Recall that $|\sin(x_1)| = \pm \sin(x_2)$ if and only if $x_1 + x_2 = \ell \pi$ or $x_2 - x_1 = \ell \pi$, where $\ell$ is an integer. Therefore, $\sin(\frac{\phi_{n_1} + \phi_{n_2}}{2}) = \pm \sin(\frac{\phi_{n_1} + \phi_{n_2}}{2})$, which contradicts the assumption that $\phi_n \pi$ is an irrational number and $\phi_R = \pi/2 - \tilde{n}_0 \phi_0$. One has shown that $\rho = |m\rangle\langle m|$ for some $m \in \mathbb{N}$ whenever $\alpha = 0$. If $\alpha = 1$, or $\rho^g_n = \rho^r_n$ for all $n \in \mathbb{N}$ with $0 < \alpha < 1$, then from similar arguments and computations one also concludes that $\rho = |m\rangle\langle m|$ for some $m \in \mathbb{N}$.

**Proof of the second claim:** Let $m \in \mathbb{N}$ and take $\rho = |m\rangle\langle m| \in D_*$. It is clear that $W(\rho) = -\sum_{n \in \mathbb{N}} \rho^g_n = -1$. From (10)-(13) in Appendix B one has that:

$$(M_g(\pm 1)\rho M_g^\dagger(\pm 1))_{nn} = \delta(n, m + 1) \sin^2\left(\frac{\theta_0}{2} \sqrt{m + 1}\right),$$

$$(M_c(\pm 1)\rho M_c^\dagger(\pm 1))_{nn} = \delta(n, m) \cos^2\left(\frac{\theta_0}{2} \sqrt{m + 1}\right),$$

$$(M_g(-1)\rho M_g^\dagger(-1))_{nn} = \delta(n, m) \cos^2\left(\frac{\theta_0}{2} \sqrt{m}\right),$$

$$(M_c(-1)\rho M_c^\dagger(-1))_{nn} = \delta(n + 1, m) \sin^2\left(\frac{\theta_0}{2} \sqrt{m}\right),$$

where $\delta(n, m)$ is the usual Kronecker delta: $\delta(n, m) = 0$ if $n \neq m$, and $\delta(n, m) = 1$ if $n = m$. In particular:

$$\text{Tr} \left( M_g(\pm 1)\rho M_g^\dagger(\pm 1) \right) = \sin^2\left(\frac{\theta_0}{2} \sqrt{m + 1}\right),$$

$$\text{Tr} \left( M_c(\pm 1)\rho M_c^\dagger(\pm 1) \right) = \cos^2\left(\frac{\theta_0}{2} \sqrt{m + 1}\right),$$

$$\text{Tr} \left( M_g(-1)\rho M_g^\dagger(-1) \right) = \cos^2\left(\frac{\theta_0}{2} \sqrt{m}\right),$$

$$\text{Tr} \left( M_c(-1)\rho M_c^\dagger(-1) \right) = \sin^2\left(\frac{\theta_0}{2} \sqrt{m}\right),$$

$$\sum_{n \in \mathbb{N}} \left( \frac{M_g(u)\rho M_g^\dagger(u)}{\text{Tr} M_g(u)\rho M_g^\dagger(u)} \right)^2_{nn} = 1,$$

assuming no division by 0. Now, using (11) and the above computations, one gets

$$\mathbb{E} [W(p_{k+1}) | p_k = \rho, u_k = \pm 1] = p_{\pm 1} W(p_{k+1}) + p_{\mp 1} W(p_{k+1}) = -\sum_{y = g,c} \left[ \text{Tr} \left( M_g(\pm 1)\rho M_g^\dagger(\pm 1) \right) \times \right. \left. \sum_{n \in \mathbb{N}} \left( \frac{M_g(\pm 1)\rho M_g^\dagger(\pm 1)}{\text{Tr} M_g(\pm 1)\rho M_g^\dagger(\pm 1)} \right)^2_{nn} \right]_{nn}$$

$$= -1 = W(\rho).$$

Therefore, $Q_W(|m\rangle\langle m|, \pm 1) = 0$.

**G. Proof of Proposition 2**

Fix $\rho \in D_*$. Since $\text{Tr} (d(N)\rho) = \text{Tr} (d(N)\Delta \rho)$ and $\rho_{mn} = (\Delta \rho)_{mn}$ for $n \in \mathbb{N}$, the first two assertions are immediate from the definitions. As for the third and fourth assertions, let $|\psi\rangle = \sum_{n \in \mathbb{N}} (m\langle n| \rho_{mn}) |n\rangle \in \mathcal{H}$. Note that $|\rho_{m}\rangle = \sum_{n = 0}^{n_{\max}(\rho)} \rho_{mn} |n\rangle$, for $m \in \mathbb{N}$. Using (2)-(4):

$$M_g(0)\rho M_g^\dagger(0) |\psi\rangle = \sum_{m,n = 0} \rho_{mn} \cos\left(\frac{\theta_0}{2} \sqrt{m + 1}\right) \cos\left(\frac{\theta_0}{2} \sqrt{n + 1}\right) |m\rangle\langle n| |\psi\rangle,$$

$$M_c(0)\rho M_c^\dagger(0) |\psi\rangle = \sum_{m,n = 0} \rho_{mn} \sin\left(\frac{\theta_0}{2} \sqrt{m + 1}\right) \sin\left(\frac{\theta_0}{2} \sqrt{n + 1}\right) |m\rangle\langle n| |\psi\rangle,$$

$$M_g(+1)\rho M_g^\dagger(+1) |\psi\rangle = \sum_{m = 0} \rho_{m-1,n} \sin\left(\frac{\theta_0}{2} \sqrt{m + 1}\right) \sin\left(\frac{\theta_0}{2} \sqrt{n + 1}\right) |m\rangle\langle n| |\psi\rangle + \rho_{m,n} |\psi\rangle,$$

$$M_c(+1)\rho M_c^\dagger(+1) |\psi\rangle = \sum_{m = 0} \rho_{m-1,n} \sin\left(\frac{\theta_0}{2} \sqrt{m + 1}\right) \sin\left(\frac{\theta_0}{2} \sqrt{n + 1}\right) |m\rangle\langle n| |\psi\rangle + \rho_{m,n} |\psi\rangle - \rho_{mn} |\psi\rangle.$$

Since $\Delta \rho \in D_* \subset D_*$, $n_{\max}(\Delta \rho) = n_{\max}(\rho)$ and $(\Delta \rho)_{mn} = \rho_{mn}$, the proof is straightforward from (8)-(13) in Appendix B.

**REFERENCES**


