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Stabilization of photon-number states via single-photon corrections: a first convergence analysis under an ideal set-up

H. B. Silveira  P. S. Pereira da Silva  P. Rouchon

Abstract—This paper presents a first mathematical convergence analysis of a Fock states feedback stabilization scheme via single-photon corrections. This measurement-based feedback has been developed and experimentally tested in 2012 by the cavity quantum electrodynamics group of Serge Haroche and Jean-Michel Raimond. Here, we consider the infinite-dimensional Markov model corresponding to the ideal set-up where detection errors and feedback delays have been disregarded. In this ideal context, we show that any goal Fock state can be stabilized by a Lyapunov-based feedback for any initial quantum state belonging to the dense subset of finite rank density operators with support in a finite photon-number sub-space. Closed-loop simulations illustrate the performance of the feedback law.

I. INTRODUCTION

In [8], a photon-number states (Fock state) feedback stabilization scheme via single-photon corrections was described and experimentally tested. Such control problem is relevant for quantum information applications [6], [4]. The quantum state $\rho$ corresponds to the density operator of a microwave field stored inside a superconducting cavity and described as a quantum harmonic oscillator. At each sample step $k \in \mathbb{N}$, a probe atom is launched inside the cavity. The measurement outcome $y_k$ detected by a sensor is the energy-state of this probe atom after its interaction with the microwave field. Each probe atom is considered as a two-level system: either it is detected in the lowest energy state $\left| g \right\rangle$, or the highest energy state $\left| e \right\rangle$. Consequently, the measurement outcomes correspond to a discrete-valued output $y_k$ with only two distinct possibilities: $g$ or $e$. Similarlly, the control inputs $u_k$ are also discrete-valued with 3 distinct possibilities: $-1, 0, +1$. The open-loop value $u_k = 0$ corresponds to a dispersive atom/field interaction: it achieves in fact a Quantum Non-Demolition measurement of Fock states [2]. The two other values $u_k = \pm 1$ correspond to resonant atom/field interactions where the probe atom and the field exchange energy quanta: these values achieve single-photon corrections.

Although the feedback law proposed and implemented in [8] considered imperfect detections on $y_k$ and delays in the control, here we focus on an ideal-set up, that is, detection errors and control delays have been disregarded. Theorem 2 shows that, by adding an arbitrarily small term to the Lyapunov function used in [8], one ensures almost sure stabilization of any goal Fock state for the closed-loop ideal set-up. This is achieved by relying on an infinite-dimensional Markov model of the ideal set-up that takes into account the back-action of the measurement outcome $y_k$ on the quantum state $\rho_{k+1}$.

Loosely speaking, in [8], the control value $u_k$ at each sampling step $k$ was chosen so as to minimize the conditional expectation of the Lyapunov function $V(\rho_k) = \text{Tr}(d(N)\rho_k)$, where $N$ is the photon-number operator, $d(n) = (n - \overline{n})^2$ and $\overline{n} = \langle \overline{n}\rangle$ is the goal Fock state. However, in closed-loop, the difference between such $V$ and its conditional expectation is not strictly positive: such $V$ does not become a strict Lyapunov function in closed-loop and additional arguments have to be considered to prove convergence. These additional arguments are related to Lasalle invariance. They are well established in a smooth context where the control $u$ is a smooth function of the state $\rho$. This cannot be the case here since $u$ is a discrete-valued control. In order to overcome such technical difficulties, we propose, similarly to [1], to add the arbitrarily small term $-\epsilon \sum_{n=0}^{\infty} \langle n | \rho_k | n \rangle^2$ to $V(\rho_k)$, where $\epsilon > 0$. This slightly modified control-Lyapunov function becomes then a strict Lyapunov function in closed-loop that simplifies notably the convergence analysis. Moreover, the developed convergence analysis is done in the infinite-dimensional setting in the following sense: we show that, for any initial density operator $\rho_0$ with a finite photon-number support ($\rho_0 | n \rangle = 0$ for $n$ large enough), the closed-loop trajectory $k \mapsto \rho_k$ remains also with a finite photon-number support with a uniform bound on the maximum photon-number. This almost finite-dimensional behavior simplifies the convergence analysis despite the fact that such condition on $\rho_0$ is met on a dense subset of density operators (Hilbert-Schmidt topology on the Banach space of Hilbert-Schmidt self-adjoint operators).

The paper is organized as follows. Section II presents the ideal Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [8] and precisely formulates the Fock state stabilization problem here treated (see Definition 1). Section III establishes the proposed solution to the control problem in two distinct parts. Firstly, Section III-A considers the case where the initial condition $\rho_0$ is a diagonal density operator (see Theorem 1).
with a resonant interaction with the cavity field used in [8] is given by:

\[ H \]

Remark: The set of trace-class, self-adjoint, non-negative operators on \( H \) is the separable complex Hilbert space \( L_2(\mathbb{C}) \) with orthonormal basis \( \{ \phi_n \}_{n \in \mathbb{N}} \) of Fock states (photon-number). Hence, \( H = \{ \sum_{n \in \mathbb{N}} \psi_n |n\} \), \( (\psi_0, \psi_1, \ldots) \in \mathbb{C} \). Let \( \mathbb{B} \) be the set of all density operators on \( H \), that is, the set of trace-class, self-adjoint, non-negative operators on \( H \) with unit trace. The sample step, corresponding to a sampling period around 100\( \mu s \), is indexed by \( k \in \mathbb{N} = \{ 0, 1, 2, \ldots \} \). Let \( \rho_k \in \mathbb{B} \) the quantum state and \( y_k \in \{ g, e \} \) the measurement outcome.

The ideal Markov model of the controlled microwave superconducting cavity used in [8] is given by:

\[
\rho_{k+1} = \begin{cases} 
\rho_G^{k+1} = \frac{M_g(u_k) \rho_k M_g^\dagger(u_k)}{Tr(M_g(u_k) \rho_k M_g^\dagger(u_k))} & \text{when } y_k = g, \\
\rho_c^{k+1} = \frac{M_c(u_k) \rho_k M_c^\dagger(u_k)}{Tr(M_c(u_k) \rho_k M_c^\dagger(u_k))} & \text{when } y_k = e,
\end{cases}
\]

where the measurements outcomes \( y_k = g \) and \( y_k = e \) occur with probability \( p_{g,k} = Tr(M_g(u_k) \rho_k M_g^\dagger(u_k)) \) and \( p_{c,k} = Tr(M_c(u_k) \rho_k M_c^\dagger(u_k)) = 1 - p_{g,k} \), respectively, \( u_k = 0 \) corresponds to a dispersive interaction of the launched atom with the cavity field (Quantum Non-Demolition measurement of photons).

\[
M_g(0) = \cos \left( \frac{\phi_0 N + \phi_e}{2} \right), \quad M_c(0) = \sin \left( \frac{\phi_0 N + \phi_e}{2} \right),
\]

when \( u_k = +1 \) the atom enters the cavity in the state \( |e\rangle \) with a resonant interaction with the cavity field.

\[
M_g(+1) = \sin \left( \frac{\phi_0 \sqrt{N}}{\sqrt{N}} \right) a^\dagger, \quad M_c(+1) = \cos \left( \frac{\phi_0 \sqrt{N}}{\sqrt{N}} \right),
\]

when \( u_k = -1 \) it enters in \( |g\rangle \) with a non-resonant interaction.

\[
M_g(-1) = \cos \left( \frac{\phi_0 \sqrt{N}}{\sqrt{N}} \right), \quad M_c(-1) = \sin \left( \frac{\phi_0 \sqrt{N}}{\sqrt{N}} \right),
\]

and \( \phi_0, \phi_e, \theta_0 \in \mathbb{R} \) are adjustable control parameters. For each \( u_k \in \{-1, 0, 1\} \), \( M_g(u) \) and \( M_c(u) \) are linear operators on \( H \) defined in the obvious way according to the definitions in Appendix A. They are indeed well-defined operators on \( H \), despite the fact that \( a \) and \( a^\dagger \) are unbounded.

Consider the set \( D_{n^*} = \{ \rho \in \mathbb{B} | \rho \text{ is diagonal and } \rho |n\rangle = 0, \forall n > n^* \} \).

For each \( n^* \in \mathbb{N} \), define

\[
D_{n^*} = \{ \rho \in \mathbb{B} | \rho \text{ is diagonal and } \rho |n\rangle = 0, \forall n > n^* \}.
\]

Consider the set \( D_{n^*} = \bigcup_{n=n^*} \mathbb{N} D_{n^*} \subset \mathbb{C} \). Note that \( D_{n^*} \subset D_{n^*+1} \), and that each element \( \rho \) of \( D_{n^*} \) is “finite dimensional” in the following sense: \( \rho \in \mathbb{B} \) is in \( D_{n^*} \), if and only if \( \rho = \sum_{n=0}^{n^*} \rho_{nn} |n\rangle \langle n| \), and \( \rho \in D_{n^*} \) may be considered as an operator from \( H \) to the finite-dimensional space \( H_{n^*} = \text{span} \{ |0\rangle, \ldots, |n^*\rangle \} \), or as a density matrix on \( H_{n^*} \). One defines the functions \( n_{\text{min}}: D_{n^*} \rightarrow \mathbb{N}, n_{\text{max}}: D_{n^*} \rightarrow \mathbb{N}, n_{\text{length}}: D_{n^*} \rightarrow \mathbb{N} \) respectively by:

\[
\begin{align*}
& n_{\text{min}}(\rho) = \text{the smallest } n \in \mathbb{N} \text{ such that } \rho |n\rangle \neq 0; \\
& n_{\text{max}}(\rho) = \text{the greatest } n \in \mathbb{N} \text{ such that } \rho |n\rangle \neq 0; \\
& n_{\text{length}}(\rho) = n_{\text{max}}(\rho) - n_{\text{min}}(\rho).
\end{align*}
\]

It is clear that, given \( \rho \in D_{n^*} \), one has \( \rho \in D_{n^*} \), if and only if \( n_{\text{max}}(\rho) \leq n^* \). The next result exhibits the properties of the state \( \rho_k \) of (1)–(4) with respect to these functions.

\[\text{Note that if } \rho = |n\rangle \langle n| \text{ for some } n \in \mathbb{N}, \text{ then } \rho \in D_n.\]
**Proposition 1:** For every realization of the ideal Markov process (1)-(4) with initial condition $\rho_0 \in D_*$, one has that $\rho_k \in D_*$ for all $k \in \mathbb{N}$ with:

- If $u_k = 0$ or $u_k = -1$, then $n_{\text{max}}(\rho_{k+1}) \leq n_{\text{max}}(\rho_k)$ and $n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)$;
- If $u_k = +1$, then $n_{\text{max}}(\rho_{k+1}) \leq n_{\text{max}}(\rho_k) + 1$ and $n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)$.

**Proof:** See Appendix [B].

Take a goal photon-number $\pi \in \mathbb{N}$. As in [1], consider the following Lyapunov function $V_\pi: D_\pi \to \mathbb{R}$ defined as

$$V_\pi(\rho) = \text{Tr} (d(N)\rho) - e \sum_{n \in \mathbb{N}} \rho^2_{nn}, \quad \text{for } \rho \in D_\pi,$$

where $e > 0$ is a real number and $d(n) = (n - \pi)^2$ as defined in [8]. The feedback law $u: D_\pi \to \{-1, 0, 1\}$ is given by

$$u = f(\rho) \triangleq \text{Argmin}_{v \in \{-1, 0, 1\}} \mathbb{E}[V_\pi(\rho_{k+1}) \mid \rho_k = \rho, u_k = v].$$

Note that for each $\rho \in D_\pi$ and $n^* \geq n_{\text{max}}(\rho)$, $d(N)\rho$ in (5) is a well-defined self-adjoint, non-negative, trace-class operator on $\mathcal{H}$, by considering $d(N)$ as an operator on $\mathcal{H}_{\pi}$, and $\rho$ as an operator from $\mathcal{H}$ to $\mathcal{H}_{\pi}$. Indeed, $d(N)\rho = \sum_{n=0}^{n^*} \rho_{nn} (n - \pi)^2 |n\rangle \langle n|$.

Moreover, the feedback law $u$ is invariant under $\rho \in D_\pi$ for $n^* \geq n_{\text{max}}(\rho)$, it is clear that $\text{Tr}(d(N)\rho) = \text{Tr}(d(N)f(\rho))$, where $f(\rho)$ is as in (6).

We have the following convergence result when $\rho_0 \in D_\pi$:

**Theorem 1:** Let $\pi \in \mathbb{N}$ and $e > 0$. In (2)-(4), assume that $\phi_0/\pi$ and $(\theta_0/\pi)^2$ are irrational numbers, and take $\phi_R = \pi/2 - \pi \phi_0$. Consider the closed-loop Markov process (1)-(4) with $u_k = f(\rho_k)$, where the feedback law $f$ is as in (6). Then, given any initial condition $\rho_0 \in D_\pi$, one has that $\rho_k$ converges almost surely towards $\overline{\pi} = |\pi\rangle \langle \pi|$ as $k \to \infty$.

Its proof is decomposed into two steps:

**First Step.** Choose $\pi \in \mathbb{N}$ and $e > 0$. Let $n_0 = n_{\text{length}}(\rho_0)$, $r_0 = n_{\text{max}}(\rho_0)$. Then, there exists an integer $n_0 > n_0 + r_0 + \pi + 1$ (depending on $n_0, r_0, \pi$ and $e$) such that, for all closed-loop realizations $\rho_k$, one has $\rho_k \in D_{n_0}$ for $k \in \mathbb{N}$.

**Second Step.** Choose irrational numbers $\phi_0/\pi$ and $(\theta_0/\pi)^2$ in (2)-(4), and take $\phi_R = \pi/2 - \pi \phi_0$. In $D_{n_0}$, $V_\pi$ is a strict super-martingale: for all density operators $\rho$ in $D_{n_0}$, one has

$$\mathbb{E}[V_\pi(\rho_{k+1}) \mid \rho_k = \rho, u_k = f(\rho)] - V_\pi(\rho) = -QV_\pi(\rho, f(\rho)),$$

where $QV_\pi(\rho, f(\rho)) \geq 0$, and $QV_\pi(\rho, f(\rho)) = 0$ if and only if $\rho = \overline{\pi}$. The almost sure convergence follows then from usual results on strict super-martingales for Markov processes with compact state spaces.

The complete proof of the two steps above is presented in Section [V]. The general case where the initial condition $\rho_0$ is not necessarily diagonal is treated in the next subsection.

**B. General case**

Consider, for each $n^* \in \mathbb{N}$,

$$D_{n^*} = \{ \rho \in \mathbb{D} \mid \rho|n\rangle = 0, \forall n > n^* \} \subset D_{n^*+1}.$$
IV. SIMULATION RESULTS

This section presents the closed-loop simulation results concerning the application of Theorem 2 above to the ideal Markov process (1)-(4). The quantum experimental results exhibited in [8] used the following control parameter values in (2)-(4): $\phi_0/\pi = 0.252$ and $\theta_0/\pi \approx 2/\sqrt{\pi + 1}$. However, according to the assumptions in Theorem 2, $\phi_0/\pi$ and $(\theta_0/\pi)^2$ should be irrational numbers. Hence, here one chooses $\phi_0/3.14 = 0.252$ and $\theta_0/3.14 = 2/\sqrt{\pi + 1}$. One takes $\rho_0 = \sum_{n=0}^{15} \rho_{n}^2/n^2/16 \in D_3$ as the initial condition, $\pi = 10$ for the goal Fock state $\overline{\eta} = |\overline{\eta}\rangle|\overline{\eta}\rangle$, and $\epsilon = 10^3$ as the gain for the feedback $u_k = f(\rho_k)$ in (5)-(6). Figure 1 exhibits the simulation results for one closed-loop realization with such choices and a final sample step of 120. It shows: the dynamics of the populations of $\rho_k$ (top), the controls $u_k$ (middle) and the simulated outcomes $y_k$ (bottom). The populations of $\rho_k$ correspond to the following observables:

$$A_1 = \sum_{n=0}^{15} |n\rangle\langle n| (n < \pi), A_2 = |\overline{\eta}\rangle\langle \overline{\eta}| (n = \pi), A_3 = \sum_{n>\pi} |n\rangle\langle n| (n > \pi).$$

Therefore, one sees from the dynamics of the populations that $\rho_k$ converges to $\overline{\eta}$ as $k \to \infty$, which is in accordance with Theorem 2. Note that $\langle |\overline{\eta}\rangle \rho_k |\overline{\eta}\rangle \approx 1$ and $u_k = 0$ for all $k > 45$.

Recall that Theorem 2 assumes that $\epsilon > 0$. In order to further analyze the performance of the Lyapunov-based feedback law here proposed, we now make a comparison with the one used experimentally in [8], which corresponds to take $\epsilon = 0$ in (5), i.e. to disregard the term $-\epsilon \sum_{n \in \mathbb{N}} \rho_{n}^2$. Figure 2 presents the simulation results for one closed-loop realization of such case. The control parameters, $\rho_0$ and $\pi = 10$ are the same as above. Note that $\langle |\overline{\eta}\rangle \rho_k |\overline{\eta}\rangle \approx 1$ and $u_k = 0$ for all $k > 78$. In order to make a comparison in terms of the speed of convergence, define the settling time $k_s$ to be the smallest $k \in \mathbb{N}$ such that $\langle |\overline{\eta}\rangle \rho_k |\overline{\eta}\rangle > 0.9$ for all $k \geq k_s$. One has $k_s = 45$ for the case $\epsilon = 10^3$ above, and $k_s = 78$ for $\epsilon = 0$. Therefore, in the two realizations here simulated, the choice of $\epsilon = 10^3$ reduced the settling time $k_s$ by nearly 42% with respect to $\epsilon = 0$. This behavior is typical on an average basis, thereby justifying the term $-\epsilon \sum_{n \in \mathbb{N}} \rho_{n}^2$ in (5). Table I shows the average value $\overline{k}_s$ and the standard deviation $\sigma$ of $k_s$ for $\epsilon \in \{0, 0.1, 1, 10, 10^2, 10^3, 10^4, 10^5\}$, where a total of 5000 realizations were simulated for each $\epsilon$. Notice that when $\epsilon$ is relatively large or relatively small in comparison to $\epsilon = 10^3$, the average settling time $\overline{k}_s$ deteriorated. Furthermore, although for $\epsilon = 10^5$ one has that $\overline{k}_s$ increased by nearly 22% in comparison to $\epsilon = 10^3$, the standard deviation $\sigma$ decreased by nearly 62%. Computer simulations have suggested that a choice of $\epsilon > 0$ which may perhaps significantly improve $\overline{k}_s$ generally depends on the initial condition $\rho_0$ and on the goal Fock state $|\overline{\eta}\rangle\langle \overline{\eta}|$, and it has to be determined heuristically.

V. PROOF OF THEOREM 1 (DIAGONAL CASE)

Proof of the First Step:

Let $\epsilon > 0$. Define $V : D_+ \to \mathbb{R}$ and $W : D_+ \to \mathbb{R}$ as

$$V(\rho) = \text{Tr} (d(N)\rho), \quad W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{n}^2, \quad (7)$$

respectively. Note that $V = V + eW$. Define:

$$Q_W(\rho, u) = W(\rho) - \mathbb{E} [W(\rho_{k+1}) | \rho_k = \rho, u_k = u],$$

$$Q_V(\rho, u) = V(\rho) - \mathbb{E} [V(\rho_{k+1}) | \rho_k = \rho, u_k = u],$$

$$Q_{V_\epsilon}(\rho, u) = V_\epsilon(\rho) - \mathbb{E} [V_\epsilon(\rho_{k+1}) | \rho_k = \rho, u_k = u],$$

where $V_\epsilon(\rho) = V(\rho) - \epsilon \sum_{n \in \mathbb{N}} \rho_{n}^2$.
for \( \rho \in D_* \) and \( u \in \{-1, 0, 1\} \). The proof of Theorem 1 is a straightforward consequence of the next proposition:

**Proposition 3:** Let \( \epsilon > 0 \) and \( q_0, r_0, \pi \in \mathbb{N} \). There exists an integer \( m_0 > n_0 + r_0 + \pi + 1 \) (depending on \( \epsilon, q_0, r_0, \pi \)) such that, for each \( \rho \in D_* \) with \( n_{\text{length}}(\rho) \leq n_0 \), if \( n_{\text{max}}(\rho) = m_0 \), then

\[
Q_V(\rho, -1) > \max \{ Q_V(\rho, 0), Q_V(\rho, +1) \}.
\]

In fact, given \( \rho_0 \in D_* \), let \( n_0 = n_{\text{length}}(\rho_0) \) and \( r_0 = n_{\text{min}}(\rho_0) \). Note that \( n_{\text{max}}(\rho_0) \leq n_0 \). By Proposition 1, \( \rho_k \in D_* \) with \( n_{\text{length}}(\rho_k) \leq n_0 \) for all \( k \in \mathbb{N} \). Since \( u = f(\rho) \) maximizes \( Q_V(\rho, f(\rho)) \), Proposition 3 implies that when \( n_{\text{max}}(\rho_k) = m_0 \) for some \( k \in \mathbb{N} \), then the input \( u_k \) will be always be equal to \(-1\), and hence Proposition 1 ensures that \( n_{\text{max}}(\rho_k+1) \leq n_{\text{max}}(\rho_k) = m_0 \). Therefore, \( n_{\text{max}}(\rho_k) \leq m_0, \) \( k \in \mathbb{N} \), showing the First Step.

The following two lemmas are instrumental for showing Proposition 3. Their proofs are given in Appendix D and Appendix E respectively.

**Lemma 1:** Given an arbitrary nonzero \( \theta_0 \in \mathbb{R} \), fix any \( a \in \mathbb{R} \) such that \( 0 < a < 1/2 \). For all nonzero \( N_0, N \in \mathbb{N} \), there exists an integer \( N > N \) big enough such that,

\[
0 < 1/2 - a \leq \sin^2 \left( \frac{\theta_0}{2\sqrt{N}} \right) \leq 1/2 + a,
\]

for \( n = N, N+1, \ldots, N + N_0 - 1 \).

**Lemma 2:** Let \( \rho \in D_* \): Then,

- \( |Q_W(\rho, u)| \leq 1 \), for each \( u \in \{-1, 0, 1\} \);
- \( Q_V(\rho, 0) = 0 \);
- \( Q_V(\rho, +1) = -\sum_{n \in \mathbb{N}} \rho_{nn}[2(n - \pi) + 1] \sin^2 \left( \frac{\theta_0}{2\sqrt{N}} \right) \);
- \( Q_V(\rho, -1) = \sum_{n \in \mathbb{N}} \rho_{nn}[2(n - \pi) - 1] \sin^2 \left( \frac{\theta_0}{2\sqrt{N}} \right) \).

The proof of Proposition 3 is shown in the sequel. Proof: Let \( \epsilon > 0 \) and \( n_0, r_0, \pi \in \mathbb{N} \). One has to show that there exists \( m_0 > n_0 + r_0 + \pi + 1 \) such that, if \( \rho \in D_* \) with \( n_{\text{length}}(\rho) \leq n_0 \), then \( u = -1 \) always maximizes \( Q_V(\rho, u) \) whenever \( n_{\text{max}}(\rho) = m_0 \). From Lemma 2 and the fact that \( Q_V = Q_V + \epsilon Q_W \), to complete the proof it suffices to show that:

- If \( \rho \in D_* \) is such that \( n_{\text{length}}(\rho) \leq n_0 \) and \( n_{\text{max}}(\rho) \geq n_0 + \pi + 1 \), then \( Q_V(\rho, +1) \leq 0 \);
- There exists \( m_0 > n_0 + r_0 + \pi + 1 \) such that \( Q_V(\rho, -1) > 2 \epsilon \), whenever \( \rho \in D_* \) is such that \( n_{\text{length}}(\rho) \leq n_0 \) and \( n_{\text{max}}(\rho) = m_0 \).

Note that

\[
Q_V(\rho, +1) = -\sum_{n = n_{\text{min}}(\rho)}^{n_{\text{max}}(\rho)} \rho_{nn}[2(n - \pi) + 1] \sin^2 \left( \frac{\theta_0}{2\sqrt{N}} \right),
\]

for any \( \rho \in D_* \). Thus, if \( n_{\text{length}}(\rho) \leq n_0 \) and \( n_{\text{max}}(\rho) \geq \pi + n_0 \), then \( n_{\text{min}}(\rho) \geq \pi + 1 \), and hence the first claim is shown.

Now, fix \( 0 < a < 1/2 \) and let \( N \geq \frac{2}{2 - a} + 2\pi + 1 \). Applying Lemma 1 for \( N_0 = n_0 + r_0 + 1 \) and such choice of \( N \), one gets \( N > N \) in which \( 0 < 1/2 - a \leq \sin^2 \left( \frac{\theta_0}{2\sqrt{N}} \right) \).

As \( N \) is an integer, it follows that \( N \geq \pi + 1 \).
**Theorem 3:** [5, Theorem 1, p. 195] Let $\Omega$ be a probability space and let $W$ be a measurable space. Consider that $X_k: \Omega \to W$, $k \in \mathbb{N}$, is a Markov chain with respect to the natural filtration. Let $Q: W \to \mathbb{R}$ and $V: W \to \mathbb{R}$ be measurable non-negative functions with $V(X_k)$ integrable for all $k \in \mathbb{N}$. If $\mathbb{E}[V(X_{k+1}) | X_k] - V(X_k) = -Q(X_k)$, for $k \in \mathbb{N}$, then $\lim_{k \to \infty} Q(X_k) = 0$ almost surely.

Indeed, let $\mathcal{J}_1$ be the complex Banach space of all trace-class operators on $\mathcal{H}$ with the trace norm $\| \cdot \|_1$, that is, $\| B \|_1 = \text{Tr}(|B|)$, where $|B| \triangleq \sqrt{B^*B}$, for $B \in \mathcal{J}_1$. Recall that $\| B \| \leq \| B \|_1$ and $\text{Tr}(AB) = \text{Tr}(BA) \|$, for every $B \in \mathcal{J}_1$ and each bounded operator $A: \mathcal{H} \to \mathcal{H}$, where $\| \cdot \|$ is the usual operator norm (sup norm of bounded operators) [7], [3]. Consider the subspace topology on $D_{mo}$ with respect to $\mathcal{J}_1$. One has that the closed-loop trajectory $\rho_k$, $k \in \mathbb{N}$, is a Markov chain with phase space $D_{mo}$ (with respect to the natural filtration and the Borel algebra on $D_{mo}$). It is clear that $D_{mo}$ is compact, and that $Q_\epsilon$ and $V_\epsilon$ are non-negative and continuous on $D_{mo}$, for all $\epsilon > 0$, where $\alpha_\epsilon \triangleq \min_{\rho \in D_{mo}} V_\epsilon(\rho)$. The theorem above implies that $\rho_k$ converges almost surely towards $\overline{\rho}$ as $k \to \infty$ (with respect to the trace norm). This completes the proof of Theorem 1.

**VI. CONCLUDING REMARKS**

This paper provided a convergence analysis of Fock states stabilization via single-photon corrections under an ideal set-up, that is, assuming perfect measurement detection and no control delays. In terms of convergence speed, the simulation results here presented have justified the inclusion of the term $-\epsilon \sum_{n \in \mathbb{N}} P_{nn}^2$ in the Lyapunov-based feedback law (5). It is straightforward to verify that the convergence analysis developed in this paper remains valid for: (i) any other function $d(n)$ in (5) satisfying $d(\n) = 0$, $d(n)$ is increasing for $n > \n$ and $d(n)$ is decreasing for $n < \n$; and (ii) $\epsilon > 0$ dependent on $n$, that is, to take the term $-\epsilon \sum_{n \in \mathbb{N}} \epsilon_n \rho_{nn}^2$. However, it is an open problem how to choose the function $d(n)$ and the gains $\epsilon_n > 0$ so as to achieve the best convergence speed.

Finally, the feedback law used in [8], which corresponds to $\epsilon = 0$, was tailored for an experimental set-up with measurement imperfections and control delays. The convergence analysis of such realistic situation will be investigated in the future.

**VII. ACKNOWLEDGMENTS**

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**APPENDIX**

A. Basic properties of the operators $N$, $a$ and $a^\dagger$

Fix $n^* \in \mathbb{N}$ and let $\mathcal{H}_{n^*} = \text{span}\{0, \ldots, [n^*]\}$. Consider the (linear) operators $N$: $\mathcal{H}_{n^*} \to \mathcal{H}_{n^*}$, $a$: $\mathcal{H}_{n^*} \to \mathcal{H}_{n^*+1}$ defined respectively as $N(n) = n |n\rangle$, $a|0\rangle = 0$, $a|n\rangle = \sqrt{n} |n-1\rangle$ for $n \geq 1$.

One also recalls that if $A$ is a bounded operator on $\mathcal{H}$ and $B \in \mathcal{J}_1$, then $AB, BA \in \mathcal{J}_1$ with $\text{Tr}(AB) = \text{Tr}(BA)$.

$\alpha^\dagger|n\rangle = \sqrt{n + 1} |n + 1\rangle$. Note that these operators cannot be extended to $\mathcal{H}$. Let $f: \mathbb{N} \to \mathbb{R}$ be a function. Define the operator $f(N)$: $\mathcal{H}_{n^*} \to \mathcal{H}_{n^*}$ by $f(N)|n\rangle = f(n)|n\rangle$, for each $n = 0, \ldots, [n^*]$. It is clear that $f(N)$ can be extended to $\mathcal{H}$ whenever $f$ is a bounded function. Given $f: \mathbb{N} \to \mathbb{R}$ and an integer $m$, one defines $g: \mathbb{N} \to \mathbb{R}$ as: $g(n) = f(n + m)$, when $n + m \geq 0$, and $g(n) = 0$, when $n + m < 0$. One abuses notation letting $f(N + m)$ stand for $g(N)$. Given two functions $f, g: \mathbb{N} \to \mathbb{R}$, it is clear that $f(N)g(N) = g(N)f(N) = (fg)(N)$ and $(f + g)(N) = f(N) + g(N)$. Furthermore: $aa^\dagger = N + 1$, $a^\dagger a = N$, $a f(N) = f(N + 1)a$, $a^\dagger f(N) = f(N - 1)a^\dagger$.

B. Proof of Proposition 2

Fix any $\rho \in D_\ast$ and let $n \in \mathbb{N}$. In particular, $\rho|n\rangle = \rho_{nn}|n\rangle$. It then follows from (2)–(4) that:

$M_g(0)\rho M_g^\dagger(0)|n\rangle = \rho_{nn} \cos^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle$, $M_\epsilon(0)\rho M_\epsilon^\dagger(0)|n\rangle = \rho_{nn} \sin^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle$, $M_{g}(+1)\rho M_g^\dagger(+1)|n\rangle = \left\{ \begin{array}{ll} 0, & \text{for } n = 0, \\ \rho_{n-1,n-1} \sin^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle, & \text{for } n \geq 1. \end{array} \right.$

$M_{\epsilon}(+1)\rho M_{\epsilon}^\dagger(+1)|n\rangle = \rho_{nn} \cos^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle$, $M_{g}(-1)\rho M_g^\dagger(-1)|n\rangle = \rho_{nn} \cos^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle$, $M_{\epsilon}(-1)\rho M_{\epsilon}^\dagger(-1)|n\rangle = \rho_{n+1,n+1} \sin^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle$. Therefore:

$M_g(0)\rho M_g^\dagger(0) = \sum_{n=0}^{\infty} \rho_{nn} \cos^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle\langle n|$, $M_\epsilon(0)\rho M_\epsilon^\dagger(0) = \sum_{n=0}^{\infty} \rho_{nn} \sin^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle\langle n|$, $M_{g}(+1)\rho M_g^\dagger(+1) = \sum_{n=0}^{\infty} \rho_{n-1,n-1} \sin^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle\langle n|$, $M_{\epsilon}(+1)\rho M_{\epsilon}^\dagger(+1) = \sum_{n=0}^{\infty} \rho_{nn} \cos^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle\langle n|$, $M_{g}(-1)\rho M_g^\dagger(-1) = \sum_{n=0}^{\infty} \rho_{n+1,n+1} \sin^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle\langle n|$, $M_{\epsilon}(-1)\rho M_{\epsilon}^\dagger(-1) = \sum_{n=0}^{\infty} \rho_{n+1,n+1} \sin^2 \left(\frac{\delta_0 + \phi a}{2}\right) |n\rangle\langle n|$. (13)

By assumption, $\rho_0 \in D_\ast$. Then, (1), (8)–(13) above and induction on $k$ show the assertions in Proposition 1.

C. Computation of $Q_V(\rho, u)$

Fix any $\rho \in D_\ast$ and $\mathbf{u} \in \mathbb{N}$. Recall that $V(\rho) = \text{Tr}(d(N)\rho)$, where $d: \mathbb{N} \to \mathbb{R}$ be given by $d(n) = (n - \mathbf{u})^2$. Note that (14) implies that, for each $u \in \{-1, 0, 1\}$,

$\mathbb{E}[\mathcal{V}[(\rho_{k+1}) | \rho_k = \rho, u_k = u]] = \text{Tr} \left( d(N) M_g(u) \rho M_g^\dagger(u) + d(N) M_\epsilon(u) \rho M_\epsilon^\dagger(u) \right)$. (14)
Take $u = 0$. From (8)–(9) in Appendix B one has
\[
\mathbb{E} [ V(\rho_{k+1}) \mid \rho_k = \rho, u_k = 0 ] = \text{Tr} \left( d(N) M_y(0) \rho M_y(0) + d(N) M_e(0) \rho M_e(0) \right) = \text{Tr} \left( d(N) \left[ M_y(0) \rho M_y(0) + M_e(0) \rho M_e(0) \right] \right) = \text{Tr} (d(N) \rho) = V(\rho).
\]
In particular,
\[
Q_V(\rho, 0) = 0. \tag{15}
\]
Now, take $u = +1$. Then, (14) above and (10)–(11) in Appendix B provide that
\[
\mathbb{E} [ V(\rho_{k+1}) \mid \rho_k = \rho, u_k = +1 ] = \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N + 1) \rho \right) + \text{Tr} \left( \cos^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N) \rho \right).
\]
By summing and subtracting $\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N) \rho \right)$,
\[
\mathbb{E} [ V(\rho_{k+1}) \mid \rho_k = \rho, u_k = +1 ] = \text{Tr} (d(N) \rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N + 1) \rho \right), \tag{16}
\]
In particular,
\[
Q_V(\rho, +1) = -\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N + 1) \rho \right) = -\sum_{n \in \mathbb{N}} [2(n - \eta) + 1] \sin^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) \tag{17}
\]
Finally, take $u = -1$. Using (14) above and (12)–(13) in Appendix B
\[
\mathbb{E} [ V(\rho_{k+1}) \mid \rho_k = \rho, u_k = -1 ] = \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N - 1) \rho \right) + \text{Tr} \left( \cos^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N) \rho \right).
\]
By summing and subtracting $\text{Tr} \left( \cos^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N) \rho \right)$,
\[
\mathbb{E} [ V(\rho_{k+1}) \mid \rho_k = \rho, u_k = -1 ] = \text{Tr} (d(N) \rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N - 1) \rho \right), \tag{18}
\]
In particular,
\[
Q_V(\rho, -1) = \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N - 1) \rho \right), \tag{19}
\]
Assume that $N_0$ is even (otherwise one may take $N_0 + 1$ instead of $N_0$ in this proof). Define the function $\eta: \mathbb{N} \to \mathbb{R}$ by
\[
\eta(\ell) = \left[ \frac{2}{\theta_0} \left( \ell \frac{\pi}{2} + \frac{\pi}{4} \right) \right]^2. \tag{20}
\]
By definition, one has $\frac{\theta_0}{2} \sqrt{\eta(\ell)} = \ell \frac{\pi}{2} + \frac{\pi}{4}$ for all $\ell \in \mathbb{N}$. Let $h = \pi/4 - \arcsin \left( \sqrt{1/2 - a} \right)$. Using the definition of $h$ and the symmetric\(^6\) of the function $\sin^2(\cdot)$, it is easy to show that
\[
1/2 - a \leq \sin^2(x + \pi/4) \leq a + 1/2, \quad \forall x \in [-h, h]. \tag{21}
\]
Let $\ell \in \mathbb{N}$ be even and big enough such that the following two conditions are simultaneously met:
\[
\eta(\ell) > N_0/2 + N, \quad \frac{1}{8} \theta_0 N_0 / \sqrt{\eta(\ell) - N_0/2} \leq h. \tag{22}
\]
Now, take $\eta = \lceil \eta(\ell) \rceil - N_0/2 + 1 > N$, where $\lceil \cdot \rceil$ denotes the greatest integer which is less or equal to $\eta$. By construction, $\eta(\ell)$ is in-between the points $N + N_0/2 - 1$ and $N + N_0/2$, and hence it is in the interval $[N, N + N_0 - 1]$. Then, for $n = N, \ldots, N + N_0 - 1$, one has that $|n - \eta(\ell)| < N_0/2$. Consider the function $\phi(x) = \frac{\theta_0}{2} \sqrt{x}$. From the fact that $\phi'(x) = \frac{\theta_0}{\sqrt{x}}$, by the mean value theorem applied to the function $\phi$ and the second inequality in (22), one obtains
\[
\left| \frac{\theta_0}{2} \sqrt{n} - \frac{\theta_0}{2} \sqrt{\eta(\ell)} \right| < h, \quad \text{for } n = N, \ldots, N + N_0 - 1.
\]
Then, the proof follows easily from (139), (19) and the fact that $\sin^2(x - \ell \pi/2) = \sin^2(x)$, for every even $\ell \in \mathbb{N}$.

E. Proof of Lemma \[\] \[\]
Proof of the first claim: Let $u \in \{-1, 0, 1\}, \rho \in D_\tau$. Recall that $W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{n n}$. Since $\text{Tr} (\rho) = \sum_{n \in \mathbb{N}} \rho_{n n} = 1$, then $-1 = -\sum_{n \in \mathbb{N}} \rho_{n n} \leq W(\rho) \leq 0$. Now, by (1), $\mathbb{E} [ W(\rho_{k+1}) \mid \rho_k = \rho, u_k = u ] = \rho_{g,k} W(\rho_{k+1}) + \rho_{e,k} W(\rho_{k+1})$, where $\rho_{g,k}, \rho_{e,k} \geq 0$ with $\rho_{g,k} + \rho_{e,k} = 1$. Thus $-1 \leq \mathbb{E} [ W(\rho_{k+1}) \mid \rho_k = \rho, u_k = u ] \leq 0$. Since $Q_W(\rho, u)$ is the difference of two numbers that are in-between $-1$ and $0$, one concludes that $|Q_W(\rho, u)| \leq 1$.

The second, third and fourth claims, are immediate from (15), (16) and (17) in Appendix C respectively.

F. Proof of Lemma \[\] \[\]
Proof of the first claim: Let $\rho \in D_{\text{m}}$. By (8)–(9) in Appendix B
\[
M_y(0) \rho M_y(0) + M_e(0) \rho M_e(0) = \rho, \tag{23}
\]
define $\rho^g \triangleq \rho_{k+1} = \frac{M_y(0) \rho M_y(0)}{\text{Tr} (M_y(0) \rho M_y(0))}$, for $g = g, e$.

Hence, $\alpha \rho^g + (1 - \alpha) \rho^e = \rho$, where $\alpha \triangleq \rho_{g,k} = \text{Tr} (M_y(0) \rho M_y(0))$. In particular, $\alpha \rho_{n n} + (1 - \alpha) \rho_{e n} = \rho_{n n}$, for $n \in \mathbb{N}$. Note that, if $\alpha = 0$, then $M_y(0) \rho M_y(0) = 0$, and so $\rho^e = \rho$. Similarly, $\alpha = 1$ implies $\rho^g = \rho$. Thus, the identity $\alpha \rho_{n n}^g + (1 - \alpha) \rho_{n n}^e = \rho_{n n}$, for $n \in \mathbb{N}$, still holds when $\alpha = 0$ or $\alpha = 1$. From (11) and $\alpha = \rho_{g,k}$, one has
\[
Q_W(\rho, 0) = W(\rho) - \left[ p_{g,k} W(\rho_{k+1}) + p_{e,k} W(\rho_{k+1}) \right]
= \sum_{n \in \mathbb{N}} \alpha (\rho_{n n}^g)^2 + (1 - \alpha) (\rho_{n n}^e)^2 - [\alpha \rho_{n n}^g + (1 - \alpha) \rho_{n n}^e]^2
= \alpha (1 - \alpha) \sum_{n \in \mathbb{N}} [\rho_{n n}^g - \rho_{n n}^e]^2 \geq 0. \tag{24}
\]

More precisely, $\sin^2(\pi/2 - x) = 1 - \cos^2(\pi/2 - x) = 1 - \sin^2(x)$.\]
thereby showing the first part of the first claim.

If \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) with \( 0 < \alpha < 1 \), then (8)–(2) in Appendix B imply that \( \rho^g = \rho^r = \rho \), and so \( Q_W(\rho, 0) = 0 \). Note, one shows that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) whenever \( Q_W(\rho, 0) = 0 \). Suppose \( Q_W(\rho, 0) = 0 \). Then, (21) implies that \( \alpha = 0 \), or \( \alpha = 1 \), or \( \rho^g_{nn} = \rho^r_{nn} \) for all \( n \in \mathbb{N} \) with \( 0 < \alpha < 1 \). Assume that \( \alpha = 0 \). Hence, \( M_g(0)M_g^j(0) = \sum_{n \in \mathbb{N}} \rho_{nn} \cos^2(\frac{\phi_n + \phi_d}{2})(n|n) = 0 \) by (8) in Appendix B. Suppose that \( \rho \neq |m\rangle\langle m| \) for every \( m \in \mathbb{N} \). Thus, there exists \( n_1, n_2 \in \mathbb{N} \) with \( n_1 \neq n_2 \), \( \rho_{n_1,n_1} > 0, \rho_{n_2,n_2} > 0 \). Recall that \( \sigma_{n_1} = \pm \sin({x_2}) \) if and only if \( x_1 + x_2 = \pi \) or \( x_2 - x_1 = \pi \), where \( \pi \) is an integer. Therefore, \( \sin(\frac{\phi_n + \phi_d}{2}) = \pm \sin(\frac{\phi_n + \phi_d}{2}) \), which contradicts the assumptions that \( \phi_0/\pi \) is an irrational number and \( \phi_R = \pi/2 - \pi\phi_0 \). One has shown that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) whenever \( \alpha = 0 \). If \( \alpha = 1 \), or \( \rho^g_{nn} = \rho^r_{nn} \) for all \( n \in \mathbb{N} \) with \( 0 < \alpha < 1 \), then from similar arguments and computations one also concludes that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \).

Proof of the second claim: Let \( m \in \mathbb{N} \) and take \( \rho = |m\rangle\langle m| \in D_* \). It is clear that \( W(\rho) = - \sum_{n \in \mathbb{N}} \rho^g_{nn} = -1 \). From (10)–(13) in Appendix B one has that:

\[
(\rho^g(\pm 1) \rho^g_{M^j}(\pm 1))_{nn} = \delta(n,m) \sin^2 \left( \frac{\theta_n}{2} \sqrt{m+1} \right), \\
(\rho^c(\pm 1) \rho^c_{M^j}(\pm 1))_{nn} = \delta(n,m) \cos^2 \left( \frac{\theta_n}{2} \sqrt{m+1} \right), \\
(\rho^g(-1) \rho^g_{M^j}(-1))_{nn} = \delta(n,m) \cos \left( \frac{\theta_n}{2} \sqrt{m} \right), \\
(\rho^c(-1) \rho^c_{M^j}(-1))_{nn} = \delta(n+1,m) \sin \left( \frac{\theta_n}{2} \sqrt{m} \right),
\]

where \( \delta(n,m) \) is the usual Kronecker delta: \( \delta(n,m) = 0 \) if \( n \neq m \), and \( \delta(n,m) = 1 \) if \( n = m \). In particular:

\[
\text{Tr} \left( (\rho^g(\pm 1) \rho^g_{M^j}(\pm 1))_{nn} \right) = \sin^2 \left( \frac{\theta_n}{2} \sqrt{m+1} \right), \\
\text{Tr} \left( (\rho^c(\pm 1) \rho^c_{M^j}(\pm 1))_{nn} \right) = \cos^2 \left( \frac{\theta_n}{2} \sqrt{m+1} \right), \\
\text{Tr} \left( (\rho^g(-1) \rho^g_{M^j}(-1))_{nn} \right) = \cos \left( \frac{\theta_n}{2} \sqrt{m} \right), \\
\text{Tr} \left( (\rho^c(-1) \rho^c_{M^j}(-1))_{nn} \right) = \sin \left( \frac{\theta_n}{2} \sqrt{m} \right),
\]

so:

\[
\sum_{n \in \mathbb{N}} \left( \frac{M_y(u) \rho M_y^j(u)}{\text{Tr} \left( M_y(u) \rho M_y^j(u) \right)} \right)_{nn}^2 = 1, \text{ for } u = \pm 1, y = g, c
\]

(assuming no division by 0). Now, using (11) and the above computations, one gets:

\[
\mathbb{E} \left[ W_0(p_{k+1}) | p_k = \rho, u_k = \pm 1 \right] = p_{g,k} W(p_{k+1}) + p_{c,k} W(p_{k+1}) \]

\[
= - \sum_{y=g,c} \left[ \text{Tr} \left( M_y(\pm 1) \rho M_y^j(\pm 1) \right) \times \sum_{n \in \mathbb{N}} \left( \frac{M_y(\pm 1) \rho M_y^j(\pm 1)}{\text{Tr} \left( M_y(\pm 1) \rho M_y^j(\pm 1) \right)} \right)_{nn}^2 \right] \]

\[
= -1 = W(\rho).
\]

Therefore, \( Q_W(|m\rangle\langle m|, \pm 1) = 0 \).

G. Proof of Proposition [2]

Fix \( \rho \in D_* \). Since \( \text{Tr} \left( d(N) \rho \right) = \text{Tr} \left( d(N) \Delta \rho \right) \) and \( \rho_{nn} = (\Delta \rho)_{nn} \) for \( n \in \mathbb{N} \), the first two assertions are immediate from the definitions. As for the third and fourth assertions, let \( |\psi\rangle = \sum_{m=0}^{\infty} (m|\psi\rangle) |m\rangle \in H \). Note that \( \rho = \sum_{m=0}^{n_{\max}(\rho)} \rho_{mn} |m\rangle \langle n| \), for \( m \in \mathbb{N} \). Using (2)–(4):

\[
M_y(0) M_y^j(0) |\psi\rangle = \sum_{m=0}^{n_{\max}(\rho)} \rho_{mn} \sin \left( \frac{\theta_n}{2} \sqrt{m+1} \right) \sin \left( \frac{\theta_n}{2} \sqrt{n+1} \right) \langle m|\psi\rangle |n\rangle,
\]

\[
M_y(\pm 1) M_y^j(\pm 1) |\psi\rangle = \sum_{m=0}^{n_{\max}(\rho)+1} \rho_{mn} \cos \left( \frac{\theta_n}{2} \sqrt{m+1} \right) \cos \left( \frac{\theta_n}{2} \sqrt{n+1} \right) \langle m|\psi\rangle |n\rangle,
\]

\[
M_y(-1) M_y^j(-1) |\psi\rangle = \sum_{m=0}^{n_{\max}(\rho)} \rho_{mn} \cos \left( \frac{\theta_n}{2} \sqrt{m} \right) \cos \left( \frac{\theta_n}{2} \sqrt{n} \right) \langle m|\psi\rangle |n\rangle,
\]

\[
M_y(1) M_y^j(1) |\psi\rangle = \sum_{m=0}^{n_{\max}(\rho)+1} \rho_{mn} \cos \left( \frac{\theta_n}{2} \sqrt{m+1} \right) \sin \left( \frac{\theta_n}{2} \sqrt{n+1} \right) \langle m|\psi\rangle |n\rangle.
\]

Since \( \Delta \rho \in D_* \subset D_* \), \( n_{\max}(\Delta \rho) = n_{\max}(\rho) \) and \( (\Delta \rho)_{nn} = \rho_{nn} \), the proof is straightforward from (8)–(13) in Appendix B.

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