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Stabilization of photon-number states via single-photon corrections: 
a first convergence analysis under an ideal set-up

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Abstract—This paper presents a first mathematical convergence analysis of a Fock states feedback stabilization scheme via single-photon corrections. This measurement-based feedback has been developed and experimentally tested in 2012 by the cavity quantum electrodynamics group of Serge Haroche and Jean-Michel Raimond. Here, we consider the infinite-dimensional Markov model corresponding to the ideal set-up where detection errors and feedback delays have been disregarded. In this ideal context, we show that any goal Fock state can be stabilized by a Lyapunov-based feedback for any initial quantum state belonging to the dense subset of finite rank density operators with support in a finite photon-number sub-space. Closed-loop simulations illustrate the performance of the feedback law.

I. INTRODUCTION

In [8], a photon-number states (Fock state) feedback stabilization scheme via single-photon corrections was described and experimentally tested. Such control problem is relevant for quantum information applications [6], [4]. The quantum state $\rho$ corresponds to the density operator of a microwave field stored inside a super-conducting cavity and described as a quantum harmonic oscillator. At each sample step $k \in \mathbb{N}$, a probe atom is launched inside the cavity. The measurement outcome $y_k$ detected by the sensor is the energy-state of this probe atom after its interaction with the microwave field. Each probe atom is considered as a two-level system: either it is detected in the lowest energy state $|g\rangle$, or the highest energy state $|e\rangle$. Consequenly, the measurement outcomes corresponds to a discrete-valued output $y_k$ with only two distinct possibilities: $g$ or $e$. Similarly, the control inputs $u_k$ are also discrete-valued with 3 distinct possibilities: $-1, 0, +1$. The open-loop value $u_k = 0$ corresponds to a dispersive atom/field interaction: it achieves in fact a Quantum Non-Demolition measurement of Fock states [2]. The two other values $u_k = \pm 1$ correspond to resonant atom/field interactions where the probe atom and the field exchange energy quanta: these values achieve single-photon corrections.

Although the feedback law proposed and implemented in [8] considered imperfect detections on $y_k$ and delays in the control, here we focus on an ideal-set up, that is, detection errors and control delays have been disregarded. Theorem 2 shows that, by adding an arbitrarily small term to the Lyapunov function used in [8], one ensures almost sure global stabilization of any goal Fock state for the closed-loop ideal set-up. This is achieved by relying on an infinite-dimensional Markov model of the ideal set-up that takes into account the back-action of the measurement outcome $y_k$ on the quantum state $\rho_{k+1}$.

Loosely speaking, in [8], the control value $u_k$ at each sampling step $k$ was chosen so as to minimize the conditional expectation of the Lyapunov function $V(\rho_k) = \text{Tr}(d(N)\rho_k)$, where $N$ is the photon-number operator, $d(n) = (n - \bar{\gamma})^2$ and $\bar{\gamma} = \|\gamma\|$ is the goal Fock state. However, in closed-loop, the difference between such $V$ and its conditional expectation is not strictly positive: such $V$ does not become a strict Lyapunov function in closed-loop and additional arguments have to be considered to prove convergence. These additional arguments are related to Lasalle invariance. They are well established in a smooth context where the control $u$ is a smooth function of the state $\rho$. This cannot be the case here since $u$ is a discrete-valued control. In order to overcome such technical difficulties, we propose, similarly to [1], to add the arbitrarily small term $-\epsilon \sum_{n=0}^{\infty} \langle n | \rho_k | n \rangle^2$ to $V(\rho_k)$, where $\epsilon > 0$. This slightly modified control-Lyapunov function becomes then a strict-Lyapunov function in closed-loop that simplifies notably the convergence analysis. Moreover, the developed convergence analysis is done in the infinite-dimensional setting in the following sense: we show that, for any initial density operator $\rho_0$ with a finite photon-number support ($\rho_0 | n \rangle = 0$ for $n$ large enough), the closed-loop trajectory $k \mapsto \rho_k$ remains also with a finite photon-number support with a uniform bound on the maximum photon-number. This almost finite-dimensional behavior simplifies the convergence analysis despite the fact that such condition on $\rho_0$ is met on a dense subset of density operators (Hilbert-Schmidt topology on the Banach space of Hilbert-Schmidt self-adjoint operators).

The paper is organized as follows. Section III presents the ideal Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [8] and precisely formulates the Fock state stabilization problem here treated (see Definition 1). Section III establishes the proposed solution to the control problem in two distinct parts. Firstly, Section III-A considers the case where the initial condition $\rho_0$ is a diagonal density operator (see Theorem 1).
Only the main ideas of the convergence proof are outlined. The technical details are given in Section V. Afterwards, in Section III-B, the main result of the paper is presented: the general solution is obtained from Theorem 1 for the control problem here treated is given as follows:

Definition 1: For the ideal Markov process (1–4), the control problem is to find a feedback law \( u_k = f(p_k) \) such that, given an initial condition \( \rho_0 \) in \( \mathbb{N} \), the closed-loop trajectory \( \rho_k \) converges almost surely towards the goal Fock state \( \bar{\pi} = |\pi|/|\pi| \) as \( k \to \infty \).

The almost sure convergence above is with respect to the probabilities amplitudes \( P_n(\rho) = \text{Tr}(\langle n|\rho\rangle = \langle n|\rho|n\rangle \) of \( \rho \), that is, \( \lim_{k \to \infty} P_n(\rho_k) = P_n(\bar{\pi}) \) for each \( n \in \mathbb{N} \). In other words, \( \lim_{k \to \infty} P_n(\rho_k) = 1 \) and \( \lim_{k \to \infty} P_n(\rho_k) = 0 \) when \( n \neq \bar{\pi} \). The solution proposed in this paper for the control problem above is developed in the next section.

III. STABILIZATION OF FOCK STATES

Given any operator \( A: \mathcal{H} \to \mathcal{H} \), let \( A_{mn} = \langle m|A|n\rangle \) for \( m, n \in \mathbb{N} \). Hence, \( A_{mn} \) is the \( n \)-th diagonal element of \( A \), while \( A_{mn} \) with \( m \neq n \) correspond to its “off-diagonal” elements. One says that the operator \( A \) is diagonal when \( A_{mn} = 0 \) for all \( m, n \in \mathbb{N} \) with \( m \neq n \). One shall begin by solving the control problem given in Definition 1 in the particular case where the initial condition \( \rho_0 \) is diagonal (see Theorem 2 in Section III-A). Afterwards, in Section III-B the solution to the general non-commutative case is presented (see Theorem 2). Its solution relies essentially on the diagonal case.

A. Diagonal case

For each \( n^* \in \mathbb{N} \), define

\[ D_{n^*} = \{ \rho \in \mathbb{D} | \rho \text{ is diagonal and } \rho|n\rangle = 0, \forall n > n^* \} \]

Consider the set \( D_{n^*} = \bigcup_{n^* \in \mathbb{N}} D_{n^*} \subset \mathbb{D} \). Note that \( D_{n^*} \subset D_{n^*+1} \), and that each element \( \rho \) of \( D \) is “finite dimensional” in the following sense: \( \rho \in \mathbb{D} \) is in \( D_{n^*} \) if and only if \( \rho = \sum_{|n| = 0} \rho_{nn} |n\rangle \langle n| \), and \( \rho \in D_{n^*} \) may be considered as an operator from \( \mathcal{H} \) to the finite-dimensional space \( \mathcal{H}_{n^*} = \text{span}(\{|0\rangle, \ldots, |n^*\rangle\}) \), or a density matrix on \( \mathcal{H}_{n^*} \). One defines the functions \( n_{\min}: D_{n^*} \to \mathbb{N}, n_{\max}: D_{n^*} \to \mathbb{N}, n_{\text{length}}: D_{n^*} \to \mathbb{N} \) respectively by:

\[ n_{\min}(\rho) = \text{the smallest } n \in \mathbb{N} \text{ such that } \rho|n\rangle \neq 0; \]
\[ n_{\max}(\rho) = \text{the greatest } n \in \mathbb{N} \text{ such that } \rho|n\rangle \neq 0; \]
\[ n_{\text{length}}(\rho) = n_{\max}(\rho) - n_{\min}(\rho). \]

It is clear that, given \( \rho \in D_{n^*} \), one has \( \rho \in D_{n^*} \) if and only if \( n_{\max}(\rho) \leq n^* \). The next result exhibits the properties of the state \( \rho_k \) of (1–4) with respect to these functions.

1 As usual in quantum physics, it is here assumed that the measurement outcome \( y_k = y \) cannot occur when \( \text{Tr}(M_g(u_k)\rho_k M_g^*(u_k)) = 0 \), for \( y = g, e \).

2 For instance, \( M_g(+1) = \frac{\sin \left( \frac{\phi_u \sqrt{N}}{2} \right)}{\sqrt{N}} \alpha^\dagger \), \( M_g(-1) = \alpha \frac{\sin \left( \frac{\phi_u \sqrt{N}}{2} \right)}{\sqrt{N}} \).

3 Note that if \( \rho = |n\rangle\langle n| \) for some \( n \in \mathbb{N} \), then \( \rho \in D_{n} \).
Proposition 1: For every realization of the ideal Markov process (1–4) with initial condition $\rho_0 \in D_*$, one has that $\rho_k \in D_*$ for all $k \in \mathbb{N}$ with:

- If $u_k = 0$ or $u_k = -1$, then $n_{\text{max}}(\rho_{k+1}) \leq n_{\text{max}}(\rho_k)$ and $n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)$;
- If $u_k = +1$, then $n_{\text{max}}(\rho_{k+1}) \leq n_{\text{max}}(\rho_k) + 1$ and $n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)$.

Proof: See Appendix A.

Take a goal photon-number $\overline{\pi} \in \mathbb{N}$. As in [1], consider the following Lyapunov function $V_e: D_* \to \mathbb{R}$ defined as

$$V_e(\rho) = \text{Tr}(d(N)|\rho) - \epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^{\pi}, \quad \text{for } \rho \in D_*,$$  \hspace{1cm} (5)

where $\epsilon > 0$ is a real number and $d(n) = (n - \overline{\pi})^2$ as defined in [8]. The feedback law $u: D_* \to \{-1, 0, 1\}$ is given by

$$u = f(\rho) \triangleq \arg\min_{v \in \{-1, 0, 1\}} \mathbb{E}[V_e(\rho_{k+1}) | \rho_k = v, u_k = v].$$  \hspace{1cm} (6)

Note that for each $\rho \in D_*$ and $n^* \geq n_{\text{max}}(\rho)$, $d(N)|\rho$ in (5) is a well-defined self-adjoint, non-negative, trace-class operator on $\mathcal{H}$, by considering $d(N)$ as an operator on $\mathcal{H}_n$, and $\rho$ as an operator from $\mathcal{H}$ to $\mathcal{H}_n$. Indeed, $d(N)|\rho = \sum_{n=0}^{n^*} \rho_{nn}(n - \overline{\pi})^2|n\rangle\langle n|$. Thus, (5) is well-defined. Moreover, since $\mathcal{H}_n$ is invariant under $\rho \in D_*$ for $n^* \geq n_{\text{max}}(\rho)$, it is clear that $\text{Tr}(d(N)|\rho) = \text{Tr}_{\mathcal{H}_n}(d(N)|\rho)$, where the right-hand side one considers $\rho$ as an operator on the finite-dimensional space $\mathcal{H}_n$ and the trace is taken over $\mathcal{H}_n$.

We have the following convergence result when $\rho_0 \in D_*:

Theorem 1: Let $\overline{\pi} \in \mathbb{N}$ and $\epsilon > 0$. In (2)–(4), assume that $\phi_0/\pi$ and $(\theta_0/\pi)^2$ are irrational numbers, and take $\phi_R = \pi/2 - \pi\phi_0$. Consider the closed-loop Markov process (1–4) with $u_k = f(\rho_k)$, where the feedback law $f$ is as in (6).

Then, given any initial condition $\rho_0 \in D_*$, one has that $\rho_k$ converges almost surely towards $\overline{\pi} = \overline{\pi}/\overline{\pi}$ as $k \to \infty$. Its proof is decomposed into two steps:

First Step. Choose $\overline{\pi} \in \mathbb{N}$ and $\epsilon > 0$. Let $n_0 = n_{\text{length}}(\rho_0)$, $r_0 = n_{\text{min}}(\rho_0)$. Then, there exists an integer $n_1 > n_0 + r_0 + \overline{\pi} + 1$ (depending on $n_0, r_0, \overline{\pi}$ and $\epsilon$) such that, for all closed-loop realizations $\rho_k$, one has $\rho_k \in D_{m_0}$ for $k \in \mathbb{N}$.

Second Step. Choose irrational numbers $\phi_0/\pi$ and $(\theta_0/\pi)^2$ in (2)–(4), and take $\phi_R = \pi/2 - \pi\phi_0$. In $D_{m_0}$, $V_e$ is a strict super-martingale: for all density operators $\rho$ in $D_{m_0}$, one has

$$\mathbb{E}[V_e(\rho_{k+1}) | \rho_k = \rho, u_k = f(\rho)] - V_e(\rho) = -Q_{V_e}(\rho, f(\rho)),$$

where $Q_{V_e}(\rho, f(\rho)) \geq 0$, and $Q_{V_e}(\rho, f(\rho)) = 0$ if and only if $\rho = \overline{\pi}$. The almost sure convergence follows then from usual results on strict super-martingales for Markov processes with compact state spaces.

The complete proof of the two steps above is presented in Section of the paper. The general case where the initial condition $\rho_0$ is not necessarily diagonal is treated in the next subsection.

B. General case

Consider, for each $n^* \in \mathbb{N}$,

$$\mathbb{D}_{n^*} = \{\rho \in \mathbb{D} | \rho|n\rangle = 0, \forall n > n^*\} \subset \mathbb{D}_{n^*+1},$$

and let $\mathbb{D}_* = \bigcup_{n^* \in \mathbb{N}} \mathbb{D}_{n^*} \supset D_*$. It is clear that $\rho \in \mathbb{D}$ in $\mathbb{D}_{n^*}$ if and only if $\rho = \sum_{n=n^*}^{\infty} \rho_{nn}|n\rangle\langle n|$. Consequently, $\mathbb{D}_*$ is a dense subset of $\mathbb{D}$ when $\mathbb{D}$ is endowed with the subspace topology induced from the Hilbert-Schmidt norm. Indeed, let $\mathcal{J}_2$ be the complex Banach space of all Hilbert-Schmidt operators on $\mathcal{H}$ with the Hilbert-Schmidt norm $\|B\|_2 = (\sum_{n,n} |B_{nn}|^2)^{1/2}$, for $B \in \mathcal{J}_2$ [7], [3]. Since $\mathbb{D} \subset \mathcal{J}_2$ and $\rho \in \mathbb{D}_*$ has the form $\rho = \sum_{n=n^*}^{\infty} \rho_{nn}|n\rangle\langle n|$, the density property of $\mathbb{D}_*$ in $\mathbb{D}$ is clear.

One has that $\rho \in \mathbb{D}_*$ may be considered as an operator from $\mathcal{H}$ to the finite-dimensional space $\mathcal{H}_n$, or as a density matrix on $\mathcal{H}_{n^*}$. Hence, $d(N)|\rho$ is a well-defined trace-class operator on $\mathcal{H}$, by considering $d(N)$ as an operator on $\mathcal{H}_n$, and $\rho$ as an operator from $\mathcal{H}$ to $\mathcal{H}_{n^*}$. Indeed, $d(N)|\rho = \sum_{n=n^*}^{\infty} \rho_{nn}(n - \overline{\pi})^2|n\rangle\langle n|$. Thus, (5) is well-defined. Moreover, since $\mathcal{H}_{n^*}$ is invariant under $\rho \in D_*$ for $n^* \geq n_{\text{max}}(\rho)$, it is clear that $\text{Tr}(d(N)|\rho) = \text{Tr}_{\mathcal{H}_{n^*}}(d(N)|\rho)$, where the right-hand side one considers $\rho$ as an operator on the finite-dimensional space $\mathcal{H}_{n^*}$ and the trace is taken over $\mathcal{H}_{n^*}$.

We have the following convergence result when $\rho_0 \in D_*:

Proposition 2: Let $\rho_0 \in D_*$. Let $\rho_k, k \in \mathbb{N}$, be the corresponding closed-loop trajectory for a fixed realization of (1–4) with feedback $u_k = f(\rho_k)$, where $f$ is as in (6). It is immediate from the proposition above that:

- $\rho_k \in \mathbb{D}_*$, for $k \in \mathbb{N}$;
- $\Delta \rho_k \in D_*$, $k \in \mathbb{N}$, is the corresponding closed-loop trajectory of (1–4) for the initial condition $\Delta \rho_0$, the same realization (and with the same transition probabilities $p_{e,k}$ and $p_{g,k}$), as well as the same feedback $u_k = f(\rho_k)$;
- $\text{Tr}(|n\rangle\langle n|\rho_k) = \text{Tr}(|n\rangle\langle n|\Delta \rho_k)$, for any $n \in \mathbb{N}$.

From these arguments, Theorem 1 and the fact that $\Delta \overline{\pi} = \overline{\pi}$ one immediately obtains the following generic solution to the control problem, that is, when the initial condition $\rho_0$ belongs to the dense subset $D_*$ of $\mathbb{D}$:

Theorem 2: Let $\overline{\pi} \in \mathbb{N}$ and $\epsilon > 0$. In (2)–(4), assume that $\phi_0/\pi$ and $(\theta_0/\pi)^2$ are irrational numbers, and take $\phi_R = \pi/2 - \pi\phi_0$. Consider the closed-loop Markov process (1–4) with $u_k = f(\rho_k)$, where the feedback law $f$ is as in (6). Then, given any initial condition $\rho_0 \in D_*$, one has that $\rho_k$ converges almost surely towards $\overline{\pi} = \overline{\pi}/\overline{\pi}$ as $k \to \infty$. 


IV. SIMULATION RESULTS

This section presents the closed-loop simulation results concerning the application of Theorem 2 above to the ideal Markov process (1–4). The quantum experimental results exhibited in [8] used the following control parameter values in (2–4): \(\phi_0/\pi = 0.252\) and \(\theta_0/\pi \approx 2/\sqrt{\pi^2 + 1}\). However, according to the assumptions in Theorem 2, \(\phi_0/\pi\) and \((\theta_0/\pi)^2\) should be irrational numbers. Hence, here one chooses \(\phi_0/3.14 = 0.252\) and \(\theta_0/3.14 \approx 2/\sqrt{\pi^2 + 1}\). One takes \(\rho_0 = \sum_{n=0}^{15} n|n/n|n/16 \in \mathbb{D}\), as the initial condition, \(\pi = 10\) for the goal Fock state \(\pi = |\pi\rangle\langle\pi|\), and \(\epsilon = 10^3\) as the gain for the feedback \(u_k = f(\rho_k)\) in (5–6). Figure 1 exhibits the simulation results for one closed-loop realization with such choices and a final sample step of 120. It shows: the dynamics of the populations of \(\rho_k\) (top), the controls \(u_k\) (middle) and the simulated outcomes \(y_k\) (bottom). The populations of \(\rho_k\) correspond to the following observables: 
\[A_1 = \sum_{n=3}^{10} |n/n|n, \quad A_2 = |\pi\rangle\langle\pi| (n = \pi), \quad A_3 = \sum_{n=\pi}^{\infty} |n/n|n, \quad (n > \pi),\]
Therefore, one sees from the dynamics of the populations that \(\rho_k\) converges to \(\pi\) as \(k \to \infty\), which is in accordance with Theorem 2. Note that \(\langle|\pi\rangle\rho_k|\pi\rangle \approx 1\) and \(u_k = 0\) for all \(k > 45\).

Recall that Theorem 2 assumes that \(\epsilon > 0\). In order to further analyze the performance of the Lyapunov-based feedback law here proposed, we now make a comparison with the one used experimentally in [8], which corresponds to take \(\epsilon = 0\) in (5), i.e. to disregard the term \(-\epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2\). Figure 2 presents the simulation results for one closed-loop realization of such case. The control parameters, \(\rho_0\) and \(\pi = 10\) are the same as above. Note that \(\langle|\pi\rangle\rho_k|\pi\rangle \approx 1\) and \(u_k = 0\) for all \(k > 78\). In order to make a comparison in terms of the speed of convergence, define the settling time \(k_s\) to be the smallest \(k \in \mathbb{N}\) such that \(\langle|\pi\rangle\rho_k|\pi\rangle > 0.9\) for all \(k \geq k_s\). One has \(k_s = 45\) for the case \(\epsilon = 10^3\) above, and \(k_s = 78\) for \(\epsilon = 0\). Therefore, in the two realizations here simulated, the choice of \(\epsilon = 10^3\) reduced the settling time \(k_s\) by nearly 42% with respect to \(\epsilon = 0\). This behavior is typical on an average basis, thereby justifying the term \(-\epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2\) in (5). Table I shows the average value \(\bar{k}_s\) and the standard deviation \(\sigma\) of \(k_s\) for \(\epsilon \in \{0, 0.1, 1, 10, 10^2, 10^3, 10^4, 10^5\}\), of a total of 5000 realizations were simulated for each \(\epsilon\). Notice that when \(\epsilon\) is relatively large or relatively small in comparison to \(\epsilon = 10^3\), the average settling time \(\bar{k}_s\) deteriorated. Furthermore, although for \(\epsilon = 10^3\) one has that \(k_s\) increased by nearly 22% in comparison to \(\epsilon = 10^3\), the standard deviation \(\sigma\) decreased by nearly 62%. Computer simulations have suggested that a choice of \(\epsilon > 0\) which may perhaps significantly improve \(\bar{k}_s\) generally depends on the initial condition \(\rho_0\) and on the goal Fock state \(\pi = |\pi\rangle\langle\pi|\), and it has to be determined heuristically.

V. PROOF OF THEOREM 1 (DIAGONAL CASE)

Proof of the First Step:
Let \(\epsilon > 0\). Define \(V: D_s \to \mathbb{R}\) and \(W: D_s \to \mathbb{R}\) as
\[V(\rho) = \text{Tr} (d(N)\rho), \quad W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}^2, \quad (7)\]
respectively. Note that \(V_s = V + \epsilon W\). Define:
- \(Q_W(\rho, u) = W(\rho) - \mathbb{E} [W(\rho_{k+1}) | \rho_k = \rho, u_k = u],\)
- \(Q_V(\rho, u) = V(\rho) - \mathbb{E} [V(\rho_{k+1}) | \rho_k = \rho, u_k = u],\)
- \(Q_{V_s}(\rho, u) = V_s(\rho) - \mathbb{E} [V_s(\rho_{k+1}) | \rho_k = \rho, u_k = u],\)
for \( \rho \in D_* \) and \( u \in \{-1, 0, 1\} \). The proof of Theorem 1 is a straightforward consequence of the next proposition:

**Proposition 3:** Let \( \epsilon > 0 \) and \( n_0, r_0, \overline{\pi} \in \mathbb{N} \). There exists an integer \( m_0 > n_0 + r_0 + \overline{\pi} + 1 \) (depending on \( \epsilon, n_0, r_0, \overline{\pi} \)) such that, for each \( \rho \in D_* \) with \( n_{\text{length}}(\rho) \leq n_0 \), if \( n_{\text{max}}(\rho) = m_0 \), then

\[
Q_V(\rho, -1) > \max \{Q_V(\rho, 0), Q_V(\rho, +1)\}.
\]

In fact, given \( \rho_0 \in D_* \), let \( n_0 = n_{\text{length}}(\rho_0) \) and \( r_0 = n_{\min}(\rho_0) \). Note that \( n_{\text{max}}(\rho_0) = m_0 \). By Proposition 1 \( \rho_k \in D_* \) with \( n_{\text{length}}(\rho_k) \leq n_0 \), for all \( k \in \mathbb{N} \). Since \( u = f(\rho) \) maximizes \( Q_V(\rho, f(\rho)) \), Proposition 3 implies that when \( n_{\text{max}}(\rho_k) = m_0 \) for some \( k \in \mathbb{N} \), then the input \( u_k \) will be always be equal to \(-1\), and hence Proposition 1 ensures that \( n_{\text{max}}(\rho_k+1) \leq n_{\text{max}}(\rho_k) = m_0 \). Therefore, \( n_{\text{max}}(\rho_k) \leq m_0 \), \( k \in \mathbb{N} \), showing the First Step.

The following two lemmas are instrumental for showing Proposition 3. Their proofs are given in Appendix D and Appendix E respectively.

**Lemma 1:** Given an arbitrary nonzero \( \theta_0 \in \mathbb{R} \), fix any \( a \in \mathbb{R} \) such that \( 0 < a < 1/2 \). For all nonzero \( N_0, N \in \mathbb{N} \), there exists an integer \( N > N \) big enough such that,

\[
0 < 1/2 - a \leq \sin^2\left(\frac{\theta_0}{2} \sqrt{N}\right) \leq 1/2 + a,
\]

for \( N = N_0, N_0 + 1, \ldots, N_0 + N_0 - 1 \).

**Lemma 2:** Let \( \rho \in D_* \). Then:

\( Q_V(\rho, 0) = 0 \)

\( Q_V(\rho, +1) = -\sum_{n \in \mathbb{N}} \rho_{nn}[2(n - \overline{\pi}) + 1] \sin^2\left(\frac{\theta_0}{2} \sqrt{n + 1}\right) \)

\( Q_V(\rho, -1) = \sum_{n \in \mathbb{N}} \rho_{nn}[2(n - \overline{\pi}) - 1] \sin^2\left(\frac{\theta_0}{2} \sqrt{n + 1}\right) \).

The proof of Proposition 3 is shown in the sequel.

**Proof:** Let \( \epsilon > 0 \) and \( n_0, r_0, \overline{\pi} \in \mathbb{N} \). One has to show that there exists \( m_0 > n_0 + r_0 + \overline{\pi} + 1 \) such that, if \( \rho \in D_* \) with \( n_{\text{length}}(\rho) \leq n_0 \), then \( u = -1 \) always maximizes \( Q_V(\rho, u) \) whenever \( n_{\text{max}}(\rho) = m_0 \). From Lemma 2 and the fact that \( Q_V = Q_V + \epsilon Q_W \), to complete the proof it suffices to show that:

- If \( \rho \in D_* \) is such that \( n_{\text{length}}(\rho) \leq n_0 \) and \( n_{\text{max}}(\rho) \geq n_0 + \overline{\pi} \), then \( Q_V(\rho, +1) \leq 0 \);
- There exists \( m_0 > n_0 + r_0 + \overline{\pi} + 1 \) such that \( Q_V(\rho, -1) > 2 \epsilon \), whenever \( \rho \in D_* \) is such that \( n_{\text{length}}(\rho) \leq n_0 \) and \( n_{\text{max}}(\rho) = m_0 \).

Note that

\[
Q_V(\rho, +1) = -\sum_{n \in \mathbb{N}} \rho_{nn}[2(n - \overline{\pi}) + 1] \sin^2\left(\frac{\theta_0}{2} \sqrt{n + 1}\right),
\]

for any \( \rho \in D_* \). Thus, if \( n_{\text{length}}(\rho) \leq n_0 \) and \( n_{\text{max}}(\rho) \geq \overline{\pi} + n_0 \), then \( n_{\min}(\rho) \geq \overline{\pi} \), and hence the first claim is shown.

Now, fix \( 0 < a < 1/2 \) and let \( N \geq \frac{2(2\pi + \overline{\pi} + 1)}{\sqrt{1/2 - a}} + 2 \pi + 1 \).

Applying Lemma 1 for \( N_0 = n_0 + r_0 + 1 \) and such choice of \( N \), one gets \( N \geq N \) in which \( 0 < 1/2 - a \leq \sin^2\left(\frac{\theta_0}{2} \sqrt{N}\right) \).
Theorem 3: [5, Theorem 1, p. 195] Let $\Omega$ be a probability space and let $W$ be a measurable space. Consider that $X_k: \Omega \to W$, $k \in \mathbb{N}$, is a Markov chain with respect to the natural filtration. Let $Q: W \to \mathbb{R}$ and $V: W \to \mathbb{R}$ be measurable non-negative functions with $V(X_k)$ integrable for all $k \in \mathbb{N}$. If $\mathbb{E}[V(X_{k+1}) | X_k] - V(X_k) = -Q(X_k)$, for $k \in \mathbb{N}$, then $\lim_{k \to \infty} Q(X_k) = 0$ almost surely.

Indeed, let $\mathcal{F}_i$ be the complex Banach space of all trace-class operators on $\mathcal{H}$ with the trace norm $\| \cdot \|_1$, that is, $\| B \|_1 = \text{Tr}([B,B])$, where $[B,B] \triangleq \sqrt{B^*B}$, for $B \in \mathcal{F}_i$. Recall that $\|B\| \leq \|B\|_1$ and $\| \text{Tr}(AB) \| \leq \|A\|B\|_1$, for every $B \in \mathcal{F}_i$ and each bounded operator $A: \mathcal{H} \to \mathcal{H}$, where $\| \cdot \|$ is the usual operator norm (sup norm of bounded operators) [7], [3]. Consider the subspace topology on $D_{mo}$ with respect to $\mathcal{F}_i$. One has that the closed-loop trajectory $\rho_k$, $k \in \mathbb{N}$, is a Markov chain with phase space $D_{mo}$ (with respect to the natural filtration and the Borel algebra on $D_{mo}$). It is clear that $D_{mo}$ is compact, and that $Q_{\alpha} = \text{Tr}(\alpha D_{mo},V_{\rho}(\tilde{\rho}))$. The theorem above implies that $\rho_k$ converges almost surely towards $\tilde{\rho}$ as $k \to \infty$ (with respect to the trace norm). This completes the proof of Theorem 1.

VI. CONCLUDING REMARKS

This paper provided a convergence analysis of Fock states stabilization via single-photon corrections under an ideal set-up, that is, assuming perfect measurement detection and no control delays. In terms of convergence speed, the simulation results here presented have justified the inclusion of the term $-\epsilon \sum_{n \in \mathbb{N}} \rho^2_{mn}$ in the Lyapunov-based feedback law (5)–(6). It is straightforward to verify that the convergence analysis developed in this paper remains valid for: (i) any other function $d(n)$ in (5) satisfying $d(\pi) = 0$, $d(n)$ is increasing for $n > \pi$ and $d(n)$ is decreasing for $n < \pi$; and (ii) $\epsilon > 0$ dependent on $n$, that is, to take the term $-\sum_{n \in \mathbb{N}} \epsilon_n \rho^2_{mn}$. However, it is an open problem how to choose the function $d(n)$ and the gains $\epsilon_n > 0$ so as to achieve the best convergence speed.

Finally, the feedback law used in [8], which corresponds to $\epsilon = 0$, was tailored for an experimental set-up with measurement imperfections and control delays. The convergence analysis of such realistic situation will be investigated in the future.

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APPENDIX

A. Basic properties of the operators $\mathcal{N}$, $\mathcal{a}$ and $\mathcal{a}^\dagger$

Fix $n^* \in \mathbb{N}$ and let $\mathcal{H}_{n^*} = \text{span}\{0, \ldots, n^*\}$. Consider the (linear) operators $\mathcal{N}: \mathcal{H}_{n^*} \to \mathcal{H}_{n^*}$, $\mathcal{a}: \mathcal{H}_{n^*} \to \mathcal{H}_{n^*+1}$ defined respectively as $\mathcal{N}|n\rangle = n |n\rangle$, $\mathcal{a}|0\rangle = 0$, $\mathcal{a}|n\rangle = \sqrt{n} |n-1\rangle$ for $n \geq 1$.

One also recalls that if $A$ is a bounded operator on $\mathcal{H}$ and $B \in \mathcal{F}_i$, then $AB, BA \in \mathcal{F}_i$ with $\text{Tr}(AB) = \text{Tr}(BA)$.

$\mathcal{a}^\dagger|n\rangle = \sqrt{n+1} |n+1\rangle$. Note that these operators cannot be extended to $\mathcal{H}$. Let $f: \mathbb{N} \to \mathbb{R}$ be a function. Define the operator $f(N): \mathcal{H}_{n^*} \to \mathcal{H}_{n^*}$ by $f(N)|n\rangle = f(n)|n\rangle$, for each $n = 0, \ldots, n^*$.

It is clear that $f(N)|n\rangle = f(n)\langle n|$, when $n + m \geq 0$; and $g(n) = 0$, when $n + m < 0$. One uses notation letting $f(N+m)$ stand for $f(N)$. Given two functions $f,g: \mathbb{N} \to \mathbb{R}$, it is clear that $f(N)g(N) = f(N)f(N)$ and $(f+g)(N) = f(N) + g(N)$. Furthermore: $\mathcal{a}^\dagger \mathcal{a} = \mathcal{N}$, $\mathcal{a}f(N) = f(N+1)\mathcal{a}$, $\mathcal{a}^\dagger f(N) = f(N-1)\mathcal{a}^\dagger$.

B. Proof of Proposition 2

Fix any $\rho \in D_*$ and let $n \in \mathbb{N}$. In particular, $\rho|n\rangle = \rho_{nn}|n\rangle$. It then follows from (2)–(4) that:

$$M_{g}(0)\rho M_{g}^\dagger(0)|n\rangle = \rho_{nn} \cos^2 \left(\frac{\phi_{nn} + \phi_{nn}}{2}\right)|n\rangle,$$

$$M_{e}(0)\rho M_{e}^\dagger(0)|n\rangle = \rho_{nn} \sin^2 \left(\frac{\phi_{nn} + \phi_{nn}}{2}\right)|n\rangle,$$

$$M_{g}(+1)\rho M_{g}^\dagger(+1)|n\rangle = \left\{ \begin{array}{ll} 0, & \text{for } n = 0, \\ \rho_{n-1,n-1} \sin^2 \left(\frac{\phi_{nn}}{2}\right)|n\rangle, & \text{if } n \geq 1. \end{array} \right.$$
Take $u = 0$. From (8)–(9) in Appendix B, one has
\[
\mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = 0] = \text{Tr} \left( d(N) M_g(0) \rho M_g^\dagger(0) + \text{Tr} \left( d(N) M_e(0) \rho M_e^\dagger(0) \right) \right)
\]
In particular,
\[
Q_{V}(\rho, 0) = 0.
\] (15)

Now, take $u = +1$. Then, (14) above and (10)–(11) in Appendix B provide that
\[
\mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = +1] = \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N + 1) \rho \right) + \text{Tr} \left( \cos^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N) \rho \right).
\]
By summing and subtracting $\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N) \rho \right)$,
\[
\mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = +1] = \text{Tr} \left( d(N) \rho \right) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N + 1) \rho \right) - \text{Tr} \left( \cos^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N) \rho \right).
\]
In particular,
\[
Q_{V}(\rho, +1) = -\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N + 1} \right) d(N + 1) \rho \right) - \sum_{n \in \mathbb{N}} \rho_{nn} \left[ 2(n - \eta) + 1 \right] \sin^2 \left( \frac{\theta_0}{2} \sqrt{n + 1} \right).
\] (16)

Finally, take $u = -1$. Using (14) above and (12)–(13) in Appendix B, one has
\[
\mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = -1] = \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N - 1) \rho \right) + \text{Tr} \left( \cos^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N) \rho \right).
\]
By summing and subtracting $\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N) \rho \right)$,
\[
\mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = -1] = \text{Tr} \left( d(N) \rho \right) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N - 1) \rho \right) - \text{Tr} \left( \cos^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N) \rho \right).
\]
In particular,
\[
Q_{V}(\rho, -1) = -\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{N} \right) d(N - 1) \rho \right) - \sum_{n \in \mathbb{N}} \rho_{nn} \left[ 2(n - \eta) + 1 \right] \sin^2 \left( \frac{\theta_0}{2} \sqrt{n} \right).
\] (17)

D. Proof of Lemma 7

Assume that $N_0$ is even (otherwise one may take $N_0 + 1$ instead of $N_0$ in this proof). Define the function $\eta : \mathbb{N} \to \mathbb{R}$ by
\[
\eta(\ell) = \left[ \frac{2}{\theta_0} (\ell \pi + \pi/4) \right]^2.
\] (18)
By definition, one has $\frac{\theta_0}{2} \sqrt{\eta(\ell)} = \ell \pi + \frac{\pi}{4}$ for all $\ell \in \mathbb{N}$. Let $h = \pi/4 - \arcsin \left( \sqrt{1/2 - a} \right)$. Using the definition of $h$ and the symmetric of the function $\sin^2(\cdot)$, it is easy to show that
\[
1/2 - a \leq \sin^2(x + \pi/4) \leq a + 1/2, \quad \forall x \in [-h, h].
\] (19)
Let $\bar{T} \in \mathbb{N}$ be even and big enough such that the following two conditions are simultaneously met:
\[
\eta(\bar{T}) > N_0/2 + N, \quad \frac{1}{8} \theta_0 N_0/\sqrt{\eta(\bar{T}) - N_0/2} \leq h.
\] (20)
Now, take $\bar{N} = [\eta(\bar{T}) - N_0/2 + 1 \leq N_0$, where $[\eta]$ denotes the greatest integer which is less or equal to $\eta$. By construction, $\eta(\bar{T})$ is in-between the points $N_0/2 - 1$ and $N_0 + N_0/2$, and hence it is in the interval $[N_0, N_0 + N_0 - 1]$. Then, for $n = N_0, \ldots, N_0 + N_0 - 1$, one has that $|n - \eta(\bar{T})| < N_0/2$. Consider the function $\phi(x) = 0 = \frac{\theta_0}{2} \sqrt{x}$. From the fact that $\phi'(x) = \frac{\theta_0}{4 \sqrt{x}}$, by the mean value theorem applied to the function $\phi$ and the second inequality in (20), one obtains
\[
\left| \frac{\theta_0}{2} \sqrt{n - \frac{\theta_0}{2} \sqrt{\eta(\bar{T})}} \right| < h, \quad \text{for } n = N_0, \ldots, N_0 + N_0 - 1.
\]
Then, the proof follows easily from (13), (19) and the fact that $\sin^2(x - \ell \pi/2) = \sin^2(x)$, for every even $\ell \in \mathbb{N}$.

E. Proof of Lemma 2

Proof of the first claim: Let $u \in \{-1, 0, 1\}, \rho \in D_u$. Recall that $W(\rho) = -\sum_{n \in N} \rho_{nn}$. Since $\text{Tr} (\rho) = \sum_{n \in N} \rho_{nn} = 1$, then $-1 = -\sum_{n \in N} \rho_{nn} \leq W(\rho) \leq 0$. Now, by (1),
\[
\mathbb{E}[W(\rho_{k+1}) \mid \rho_k = \rho, u_k = u] = p_{g,k} W(\rho_{k+1}^g) + p_{e,k} W(\rho_{k+1}^e),
\]
where $p_{g,k}, p_{e,k} \geq 0$ with $p_{g,k} + p_{e,k} = 1$. Thus $-1 \leq \mathbb{E}[W(\rho_{k+1}) \mid \rho_k = \rho, u_k = u] \leq 0$. Since $Q_W(\rho, u)$ is the difference of two numbers that are in-between $-1$ and $0$, one concludes that $|Q_W(\rho, u)| \leq 1$.

The second, third and fourth claims, are immediate from (15), (16) and (17) in Appendix C, respectively.

F. Proof of Lemma 3

Proof of the first claim: Let $\rho \in D_{m_g}$. By (8)–(9) in Appendix B, $M_g(0)\rho M_g^\dagger(0) + M_e(0)\rho M_e^\dagger(0)$ is $\rho$. Taking $\rho_k = \rho$ in $u_k = 0$ in (1), define
\[
\rho^g \triangleq \rho_{k+1}^g = \frac{M_g(0)\rho M_g^\dagger(0)}{\text{Tr} \left( M_g(0)\rho M_g^\dagger(0) \right)}. \quad \text{for } g = g, e.
\]
Hence, $\alpha \rho^g + (1 - \alpha)\rho^e = \rho$, where $\alpha \triangleq p_{g,k} = \text{Tr} \left( M_g(0)\rho M_g^\dagger(0) \right)$. In particular, $\alpha \rho_{nn}^g + (1 - \alpha)\rho_{nn}^e = \rho_{nn}$, for $n \in \mathbb{N}$. Note that, if $\alpha = 0$, then $M_g(0)\rho M_g^\dagger(0) = 0$, and so $\rho^g = \rho$. Similarly, $\alpha = 1$ implies $\rho^g = \rho$. Thus, the identity $\alpha \rho_{nn}^g + (1 - \alpha)\rho_{nn}^e = \rho_{nn}$, for $n \in \mathbb{N}$, still holds when $\alpha = 0$ or $\alpha = 1$. From (1), (7) and $\alpha = p_{g,k}$, one has
\[
Q_W(\rho, 0) = W(\rho) - [p_{g,k} W(\rho_{k+1}^e) + p_{e,k} W(\rho_{k+1}^e)]
\]
\[
= \sum_{n \in \mathbb{N}} \left[ (\rho_{nn}^g)^2 + (1 - \alpha)(\rho_{nn}^e)^2 - [\alpha \rho_{nn}^g + (1 - \alpha)\rho_{nn}^e]^2 \right]
\]
\[
= \alpha (1 - \alpha) \sum_{n \in \mathbb{N}} [\rho_{nn}^g - \rho_{nn}^e]^2 \geq 0.
\] (21)

More precisely, $\sin^2(\pi/2 - x) = 1 - \cos^2(\pi/2 - x) = 1 - \sin^2(x)$. 

thereby showing the first part of the first claim.

If \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) with \( 0 < \alpha < 1 \), then (8)–(10) in Appendix B imply that \( \rho^0 = \rho^0 = \rho \), and so \( Q_W(0, \rho) = 0 \). Now, one shows that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) whenever \( Q_W(0, \rho) = 0 \). Suppose \( Q_W(0, \rho) = 0 \). Then, (21) implies that \( \alpha = 0 \), or \( \alpha = 1 \), or \( \rho_{nn}^0 = \rho_{nn} \) for all \( n \in \mathbb{N} \) with \( 0 < \alpha < 1 \). Assume that \( \alpha = 0 \). Hence, \( M_g(0)\rho M_g^\dagger(0) = \sum_{X \in \mathbb{N}} \rho_{m,n} \cos(\frac{\omega_{m,n} + \phi}{2}) |m\rangle\langle n| = 0 \) by (8) in Appendix B. Suppose that \( \rho \neq |m\rangle\langle m| \) for every \( m \in \mathbb{N} \). Hence, \( \rho \neq |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) whenever \( \alpha = 0 \). If \( \alpha = 1 \), or \( \rho_{nn}^0 = \rho_{nn} \) for all \( n \in \mathbb{N} \) with \( 0 < \alpha < 1 \), then similar arguments and computations one also concludes that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \).

**Proof of the second claim:** Let \( m \in \mathbb{N} \) and take \( \rho = |m\rangle\langle m| \in D_* \). It is clear that \( W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}^2 = -1 \). From (10)–(13) in Appendix B one has that:

\[
(M_g(+1)\rho M_g^\dagger(+1))_{nn} = \delta(n, m + 1) \sin^2 \left(\frac{\theta_0}{2} \sqrt{m + 1}\right),
\]

\[
(M_e(+1)\rho M_e^\dagger(+1))_{nn} = \delta(n, m) \cos^2 \left(\frac{\theta_0}{2} \sqrt{m + 1}\right),
\]

\[
(M_g(-1)\rho M_g^\dagger(-1))_{nn} = \delta(n, m) \cos^2 \left(\frac{\theta_0}{2} \sqrt{m}\right),
\]

\[
(M_e(-1)\rho M_e^\dagger(-1))_{nn} = \delta(n + 1, m) \sin^2 \left(\frac{\theta_0}{2} \sqrt{m}\right),
\]

(22)

where \( \delta(n, m) \) is the usual Kronecker delta: \( \delta(n, m) = 0 \) if \( n \neq m \), and \( \delta(n, m) = 1 \) if \( n = m \). In particular:

\[
\text{Tr} \left( M_g(+1)\rho M_g^\dagger(+1) \right) = \sin^2 \left(\frac{\theta_0}{2} \sqrt{m + 1}\right),
\]

\[
\text{Tr} \left( M_e(+1)\rho M_e^\dagger(+1) \right) = \cos^2 \left(\frac{\theta_0}{2} \sqrt{m + 1}\right),
\]

\[
\text{Tr} \left( M_g(-1)\rho M_g^\dagger(-1) \right) = \cos^2 \left(\frac{\theta_0}{2} \sqrt{m}\right),
\]

\[
\text{Tr} \left( M_e(-1)\rho M_e^\dagger(-1) \right) = \sin^2 \left(\frac{\theta_0}{2} \sqrt{m}\right),
\]

\[
\sum_{n \in \mathbb{N}} \left( \frac{M_g(u)\rho M_g^\dagger(u)}{\text{Tr}(M_g(u)\rho M_g^\dagger(u))} \right)^2_{nn} = 1, \text{ for } u = \pm 1, y = g, e
\]

(assuming no division by 0). Now, using (1) and the above computations, one gets

\[
\mathbb{E} \left[ W(p_{k+1}) \mid p_k = \rho, u_k = \pm 1 \right] = p_{g,k} W(p_{k+1}^0) + p_{e,k} W(p_{k+1}^1)
\]

\[
= -\sum_{y = g, e} \left[ \text{Tr} \left( M_g(\pm 1)\rho M_g^\dagger(\pm 1) \right) \times \sum_{n \in \mathbb{N}} \left( \frac{M_g(\pm 1)\rho M_g^\dagger(\pm 1)}{\text{Tr}(M_g(\pm 1)\rho M_g^\dagger(\pm 1))} \right)^2_{nn} \right]
\]

\[
= -1 = W(\rho).
\]

Therefore, \( Q_W(|m\rangle\langle m|, \pm 1) = 0 \).

**G. Proof of Proposition 2**

Fix \( \rho \in D_* \). Since \( \text{Tr}(d(N)|\rho) = \text{Tr}(d(N)\Delta\rho) \) and \( \rho_{nn} = (\Delta\rho)_{nn} \) for \( n \in \mathbb{N} \), the first two assertions are immediate from the definitions. As for the third and fourth assertions, let \( |\psi\rangle = \sum_{n \in \mathbb{N}} \langle m|\psi\rangle |m\rangle \in \mathcal{H} \). Note that \( |\psi\rangle = \sum_{n \in \mathbb{N}} \rho_{m,n} |m\rangle \), for \( m \in \mathbb{N} \). Using (2)–(4):

\[
M_g(0)\rho M_g^\dagger(0)|\psi\rangle = \sum_{m, n = 0} \rho_{mn} \cos \left(\frac{\theta_0 + \phi}{2} \right) \cos \left(\frac{\theta_0 + \phi}{2} \right) \langle m|\psi\rangle |n\rangle,
\]

\[
M_e(0)\rho M_e^\dagger(0)|\psi\rangle = \sum_{m, n = 0} \rho_{mn} \sin \left(\frac{\theta_0 + \phi}{2} \right) \sin \left(\frac{\theta_0 + \phi}{2} \right) \langle m|\psi\rangle |n\rangle,
\]

\[
M_g(+1)\rho M_g^\dagger(+1)|\psi\rangle = \sum_{m = 1, n = 0} \rho_{m-1,n} \sin \left(\frac{\theta_0 + \phi}{2} \right) \sin \left(\frac{\theta_0 + \phi}{2} \right) \langle m|\psi\rangle |n+1\rangle,
\]

\[
M_e(+1)\rho M_e^\dagger(+1)|\psi\rangle = \sum_{m = 0, n = 0} \rho_{m,n+1} \sin \left(\frac{\theta_0 + \phi}{2} \right) \sin \left(\frac{\theta_0 + \phi}{2} \right) \langle m|\psi\rangle |n\rangle.
\]

Since \( \Delta\rho \in D_* \subset D_* \), \( n_{\text{max}}(\Delta\rho) = n_{\text{max}}(\rho) \) and \( (\Delta\rho)_{nn} = \rho_{nn} \), the proof is straightforward from (8)–(13) in Appendix B.

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