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Stabilization of photon-number states via single-photon corrections: a first convergence analysis under an ideal set-up

H. B. Silveira  P. S. Pereira da Silva  P. Rouchon

Abstract—This paper presents a first mathematical convergence analysis of a Fock states feedback stabilization scheme via single-photon corrections. This measurement-based feedback has been developed and experimentally tested in 2012 by the cavity quantum electrodynamics group of Serge Haroche and Jean-Michel Raimond. Here, we consider the infinite-dimensional Markov model corresponding to the ideal set-up where detection errors and feedback delays have been disregarded. In this ideal context, we show that any goal Fock state can be stabilized by a Lyapunov-based feedback for any initial quantum state belonging to the dense subset of finite rank density operators with support in a finite photon-number sub-space. Closed-loop simulations illustrate the performance of the feedback law.

I. INTRODUCTION

In [8], a photon-number states (Fock state) feedback stabilization scheme via single-photon corrections was described and experimentally tested. Such control problem is relevant for quantum information applications [6], [4]. The quantum state $\rho$ corresponds to the density operator of a microwave field stored inside a super-conducting cavity and described as a quantum harmonic oscillator. At each sample time $k \in \mathbb{N}$, a probe atom is launched inside the cavity. The measurement outcome $y_k$ detected by a sensor is the energy-state of this probe atom after its interaction with the microwave field. Each probe atom is considered as a two-level system: either it is detected in the lowest energy state $|g\rangle$, or in the highest energy state $|e\rangle$. Consequently, the measurement outcomes correspond to a discrete-valued output $y_k$ with only two distinct possibilities: $g$ or $e$. Similarly, the control inputs $u_k$ are also discrete-valued with 3 distinct possibilities: $-1, 0, +1$. The open-loop value $u_k = 0$ corresponds to a dispersive atom/field interaction: it achieves in fact a Quantum Non-Demolition measurement of Fock states [2]. The two other values $u_k = \pm 1$ correspond to resonant atom/field interactions where the probe atom and the field exchange energy quanta: these values achieve single-photon corrections.

Although the feedback law proposed and implemented in [8] considered imperfect detections on $y_k$ and delays in the control, here we focus on an ideal-set up, that is, detection errors and control delays have been disregarded. Theorem 2 shows that, by adding an arbitrarily small term to the Lyapunov function used in [8], one ensures almost sure global stabilization of any goal Fock state for the closed-loop ideal set-up. This is achieved by relying on an infinite-dimensional Markov model of the ideal set-up that takes into account the back-action of the measurement outcome $y_k$ on the quantum state $\rho_{k+1}$.

Loosely speaking, in [8], the control value $u_k$ at each sampling step $k$ was chosen so as to minimize the conditional expectation of the Lyapunov function $V(\rho_k) = \text{Tr}(d(N)\rho_k)$, where $N$ is the photon-number operator, $d(n) = (n - \bar{n})^2$ and $\bar{n} = \langle \bar{n} \rangle(\bar{n})$ is the goal Fock state. However, in closed-loop, the difference between such $V$ and its conditional expectation is not strictly positive: such $V$ does not become a strict Lyapunov function in closed-loop and additional arguments have to be considered to prove convergence. These additional arguments are related to Lasalle invariance. They are well established in a smooth context where the control $u$ is a smooth function of the state $\rho$. This cannot be the case here since $u$ is a discrete-valued control. In order to overcome such technical difficulties, we propose, similarly to [1], to add the arbitrarily small term $-\epsilon \sum_{n=0}^{\infty} \langle n|\rho_k|n\rangle^2$ to $V(\rho_k)$, where $\epsilon > 0$. This slightly modified control-Lyapunov function becomes then a strict-Lyapunov function in closed-loop that simplifies notably the convergence analysis. Moreover, the developed convergence analysis is done in the infinite-dimensional setting in the following sense: we show that, for any initial density operator $\rho_0$ with a finite photon-number support (so we have a finite support in the photon-number state $n$ large enough), the closed-loop trajectory $k \mapsto \rho_k$ remains also with a finite photon-number support with a uniform bound on the maximum photon-number. This almost infinite-dimensional behavior simplifies the convergence analysis despite the fact that such condition on $\rho_0$ is met on a dense subset of density operators (Hilbert-Schmidt topology on the Banach space of Hilbert-Schmidt self-adjoint operators).

The paper is organized as follows. Section II presents the ideal Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [8] and precisely formulates the Fock state stabilization problem here treated (see Definition 1). Section III establishes the proposed solution to the control problem in two distinct parts. Firstly, Section III-A considers the case where the initial condition $\rho_0$ is a diagonal density operator (see Theorem 1).
Only the main ideas of the convergence proof are outlined. The technical details are given in Section VII. Afterwards, in Section III-B the main result of the paper is presented: the general solution is obtained from Theorem I for \( \rho_0 \) belonging to a dense subset (see Theorem 2). The simulation results are exhibited in Section V. The proof of some intermediate results and computations required in Sections III and V are presented in Appendices B and C. Finally, the concluding remarks are given in Section VI.

II. IDEAL MARKOV MODEL

Denote by \( \mathcal{H} \) the separable complex Hilbert space \( L_2(\mathbb{C}) \) with orthonormal basis \( \{ |n\rangle, n \in \mathbb{N} \} \) of Fock states (photon-number). Hence, \( \mathcal{H} = \{ \sum_n \psi_n |n\rangle, \langle \psi_1, \psi_2, \ldots \} \in L_2(\mathbb{C}) \}. \) Let \( \mathbb{D} \) be the set of all density operators on \( \mathcal{H} \), that is, the set of trace-class, self-adjoint, non-negative operators on \( \mathcal{H} \) with unit trace. The sample step, corresponding to a sampling period around 100\( \mu \)s, is indexed by \( k \in \mathbb{N} = \{ 0, 1, 2, \ldots \} \) and \( u_k \in \{ -1, 0, 1 \} \) is the control, \( \rho_k \in \mathbb{D} \) the quantum state and \( y_k \in \{ g, e \} \) the measurement outcome.

The ideal Markov model of the controlled microwave superconducting cavity used in [8] is given by:

\[
\rho_{k+1} = \begin{cases} 
\rho_{g,k+1}^g & \text{when } y_k = g, \\
\rho_{e,k+1}^e & \text{when } y_k = e,
\end{cases}
\]

where the measurements outcomes \( y_k = g \) and \( y_k = e \) occur with probability \( p_{g,k} = \text{Tr} \left( M_g(u_k) \rho_k M_g^\dagger(u_k) \right) \) and \( p_{e,k} = \text{Tr} \left( M_e(u_k) \rho_k M_e^\dagger(u_k) \right) = 1 - p_{g,k} \), respectively, \( u_k = 0 \) corresponds to a dispersive interaction of the launched atom with the cavity field (Quantum Non-Demolition measurement of photons)

\[
M_g(0) = \cos \left( \frac{\theta_g \sqrt{N}}{2} \right) \rho_{n}\text{, } M_e(0) = \sin \left( \frac{\theta_e \sqrt{N}}{2} \right),
\]

for \( u_k = +1 \) the atom enters the cavity in the state \( |e\rangle \) with a resonant interaction with the cavity field

\[
M_g(+1) = \frac{\sin \left( \frac{\theta_g \sqrt{N}}{\sqrt{N}} \right)}{\sqrt{N}} a^\dagger, \quad M_e(+1) = \cos \left( \frac{\theta_e \sqrt{N}}{\sqrt{N}} \right),
\]

when \( u_k = -1 \) it enters in \( |g\rangle \) with a resonant interaction

\[
M_g(-1) = \cos \left( \frac{\theta_g \sqrt{N}}{\sqrt{N}} \right), \quad M_e(-1) = -a \frac{\sin \left( \frac{\theta_e \sqrt{N}}{\sqrt{N}} \right)}{\sqrt{N}}.
\]

1As usual in quantum physics, it is here assumed that the measurement outcome \( y_k = y \) cannot occur when \( \text{Tr} \left( M_y(u_k) \rho_k M_y^\dagger(u_k) \right) = 0 \), for \( y = g, e \).

2For instance, \( M_g(+1)|n\rangle = \left( \sin \left( \frac{\theta_g \sqrt{N}}{\sqrt{N}} \right) / \sqrt{N} \right) \sqrt{n+1}|n+1\rangle = \sin \left( \frac{\theta_g \sqrt{N}}{\sqrt{N}} \right) / \sqrt{N} = \sin(\theta_g/\sqrt{N}) \sqrt{n+1}|n+1\rangle. \) In order for the definition of \( M_e(-1) \) to be consistent, it is assumed \( \sin(0)/0 = 1. \)

operators. It is clear that \( M_g(u, M_e(u) \) are bounded operators on \( \mathcal{H} \) with \( M_g(u) M_g^\dagger(u) M_e(u) = I \) (identity operator), \( M_e(-1) = M_e^\dagger(+1) = a \sin \left( \frac{\theta_e \sqrt{N}}{\sqrt{N}} \right) / \sqrt{N} \), and \( M_g(-1), M_g(0), M_e(+1) \) are self-adjoint. It is easy to see that if the initial condition \( \rho_0 \) is a density operator then, for all realizations of the ideal Markov process (I-4), \( \rho_k \) is a density operator for \( k \in \mathbb{N} \).

Notice that \( \overline{\sigma} = [\overline{\sigma}] / |\overline{\sigma}| \) is a steady state of the Markov process (I-4) with \( u_k = 0 \), where \( |\overline{\sigma}| \) is arbitrary. The control problem here treated is given as follows:

**Definition 1**: For the ideal Markov process (I-4), the control problem is to find a feedback law \( u_k = f(\rho_k) \) such that, given an initial condition \( \rho_0 \in \mathcal{D} \), the closed-loop trajectory \( \rho_k \) converges almost surely towards the goal Fock state \( \overline{\sigma} = [\overline{\sigma}] / |\overline{\sigma}| \) as \( k \to \infty \).

The almost sure convergence above is with respect to the probabilities amplitudes \( \rho_n = \text{Tr} (|n\rangle \langle n| \rho) = \langle n | \rho | n \rangle \) of \( \rho \), that is, \( \lim_{k \to \infty} \rho_n(\rho_k) = \rho_n(\overline{\sigma}) \) for each \( n \in \mathbb{N} \). In other words, \( \lim_{k \to \infty} P_n(\rho_k) = 1 \) and \( \lim_{k \to \infty} P_n(\rho_k) = 0 \) when \( n \neq \overline{\sigma} \). The solution proposed in this paper for the control problem above is obtained in section IV.

III. STABILIZATION OF FOCK STATES

Given any operator \( A: \mathcal{H} \to \mathcal{H} \), let \( A_{mn} = \langle m | A | n \rangle \) for \( m, n \in \mathbb{N} \). Hence, \( A_{nn} \) is the \( n \)-th diagonal element of \( A \), while \( A_{nn} \) with \( m \neq n \) correspond to its “off-diagonal” elements. One says that the operator \( A \) is diagonal when \( A_{nn} = 0 \) for all \( m, n \in \mathbb{N} \) with \( m \neq n \). One shall begin by solving the control problem given in Definition I in the particular case where the initial condition \( \rho_0 \) is diagonal (see Theorem I in Section III-A). Afterwards, in Section III-B the solution to the general non-commutative case is presented (see Theorem 2); its solution relies essentially on the diagonal case.

A. Diagonal case

For each \( n^* \in \mathbb{N} \), define

\[
D_{n^*} = \{ \rho \in \mathbb{D} \mid \rho \text{ is diagonal and } \rho |n\rangle = 0, \forall n > n^* \}.
\]

Consider the set \( D_{n^*} = \bigcup_{n \in \mathbb{N}} D_{n^*} \subset \mathbb{D} \). Note that \( D_{n^*} \subset D_{n^*+1} \), and that each element \( \rho \) of \( D_{n^*} \) is “finite dimensional” in the following sense: \( \rho \in \mathbb{D} \) is in \( D_{n^*} \), if and only if \( \rho = \sum_{n=0}^{n^*} \rho_{nn} |n\rangle \langle n| \), and \( \rho \in D_{n^*} \) may be considered as an operator from \( \mathcal{H} \) to the finite-dimensional space \( \mathcal{H}_{n^*} = \text{span} \{|0\rangle, \ldots, |n^*\rangle \} \), or as a density matrix on \( \mathcal{H}_{n^*} \). One defines the functions \( n_{\min}: D_{n^*} \to \mathbb{N}, n_{\max}: D_{n^*} \to \mathbb{N}, n_{\text{length}}: D_{n^*} \to \mathbb{N} \) respectively by:

- \( n_{\min}(\rho) \) is the smallest \( n \in \mathbb{N} \) such that \( \rho |n\rangle \neq 0; \)
- \( n_{\max}(\rho) \) is the greatest \( n \in \mathbb{N} \) such that \( \rho |n\rangle \neq 0; \)
- \( n_{\text{length}}(\rho) = n_{\max}(\rho) - n_{\min}(\rho). \)

It is clear that, given \( \rho \in D_{n^*} \), one has \( \rho \in D_{n^*} \) if and only if \( n_{\max}(\rho) \leq n^* \). The next result exhibits the properties of the state \( \rho_k \) of (I-4) with respect to these functions.

\[
\text{Note that if } \rho = |n\rangle \langle n| \text{ for some } n \in \mathbb{N}, \text{ then } \rho \in D_{n^*}.
\]
Proposition 1: For every realization of the ideal Markov process (1–4) with initial condition $\rho_0 \in D_*$, one has that $\rho_k \in D_*$ for all $k \in \mathbb{N}$ with:

- If $u_k = 0$ or $u_k = -1$, then $n_{\text{max}}(\rho_{k+1}) \leq n_{\text{max}}(\rho_k)$ and $n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)$;
- If $u_k = +1$, then $n_{\text{max}}(\rho_{k+1}) \leq n_{\text{max}}(\rho_k) + 1$ and $n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)$.

Proof: See Appendix [I].

Take a goal photon-number $\varpi \in \mathbb{N}$. As in [1], consider the following Lyapunov function $V_\epsilon: D_* \rightarrow \mathbb{R}$ defined as

$$V_\epsilon(\rho) = \text{Tr}(d(N)\rho) - \epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2, \quad \text{for } \rho \in D_*,$$  

where $\epsilon > 0$ is a real number and $d(n) = (n - \varpi)^2$ as defined in [8]. The feedback law $u: D_* \rightarrow \{-1, 0, 1\}$ is given by

$$u = f(\rho) \triangleq \text{Argmin}_{v \in \{-1, 0, 1\}} \mathbb{E}[V_\epsilon(\rho_{k+1}) | \rho_k = \rho, u_k = v].$$

Note that for each $\rho \in D_*$ and $n^* \geq n_{\text{max}}(\rho)$, $d(n)\rho$ in (3) is a well-defined self-adjoint, non-negative, trace-class operator on $H$, by considering $d(N)$ as an operator on $H$, and $d(n)$ as an operator from $H$ to $H$. Indeed, $d(n)\rho = \sum_{n=n_{\text{min}}}^{n_{\text{max}}} \rho_{nn}(n - \varpi)^2/n$ for all $\rho \in D_*$. Thus, (3) is well-defined. Moreover, since $H_\varpi$ is invariant under $\rho \in D_*$ for $n^* \geq n_{\text{max}}(\rho)$, it is clear that $\text{Tr}(d(n)\rho) = \text{Tr}(H_\varpi, d(n)\rho)$, where on the right-hand side one considers $\rho$ as an operator on the finite-dimensional space $H_\varpi$, and the trace taken is over $H_\varpi$.

We have the following convergence result when $\rho_0 \in D_*$.  

Theorem 1: Let $\varpi \in \mathbb{N}$ and $\epsilon > 0$. In (2–4), assume that $\varphi_0/\pi$ and $(\theta_0/\pi)^2$ are irrational numbers, and take $\varphi_R = \pi/2 - \varpi\phi_0$. Consider the closed-loop Markov process (1–4) with $u_k = f(\rho_k)$, where the feedback law $f$ is as in (6). Then, given any initial condition $\rho_0 \in D_*$, one has that $\rho_k$ converges almost surely towards $\varpi = |\varpi|/|\varpi|$ as $k \rightarrow \infty$.

Its proof is decomposed into two steps:  

First Step. Choose $\varpi \in \mathbb{N}$ and $\epsilon > 0$. Let $n_0 = n_{\text{length}}(\rho_0)$, $r_0 = n_{\text{min}}(\rho_0)$. Then, there exists an integer $m_0 > n_0 + r_0 + \varpi + 1$ (depending on $n_0, r_0, \varpi,$ and $\epsilon$) such that, for all closed-loop realizations $\rho_k$, one has $\rho_k \in D_{m_0}$ for $k \in \mathbb{N}$.  

Second Step. Choose irrational numbers $\varphi_0/\pi$ and $(\theta_0/\pi)^2$ in (2–4), and take $\varphi_R = \pi/2 - \varpi\phi_0$. In $D_{m_0}$, $V_\epsilon$ is a strict super-martingale: for all density operators $\rho$ in $D_{m_0}$, one has

$$\mathbb{E}[V_\epsilon(\rho_{k+1}) | \rho_k = \rho, u_k = f(\rho)] - V_\epsilon(\rho) = -QV_\epsilon(\rho, f(\rho)), $$

where $QV_\epsilon(\rho, f(\rho)) \geq 0$, and $QV_\epsilon(\rho, f(\rho)) = 0$ if and only if $\rho = \varpi*$. The almost sure convergence follows then from usual results on strict super-martingales for Markov processes with compact state spaces.

The complete proof of the two steps above is presented in Section [V]. The general case where the initial condition $\rho_0$ is not necessarily diagonal is treated in the next subsection.

B. General case

Consider, for each $n^* \in \mathbb{N}$,

$$D_{n^*} = \{\rho \in D_* | \rho|n\rangle = 0, \forall n > n^*\} \subset D_{n^*+1},$$

and let $D_* = \bigcup_{n \in \mathbb{N}} D_{n^*} \supset D_*$. It is clear that $\rho \in D_*$ if and only if $\rho = \sum_{m,n=0}^{n_{\text{max}}(\rho)} \rho_{mn}|m\rangle\langle n|$. Consequently, $D_*$ is a dense subset of $D$ when $D$ is endowed with the subspace topology induced from the Hilbert-Schmidt norm. Indeed, let $J_\mathbb{D}$ be the complex Banach space of all Hilbert-Schmidt operators on $H$ with the Hilbert-Schmidt norm $\|B\|_2 = \sum_{m,n=0}^{n_{\text{max}}(\rho)} |B_{mn}|^2/2$, for $B \in J_\mathbb{D}$ [7], [3]. Since $D \subset J_\mathbb{D}$ and $\rho \in D_{n^*}$ has the form $\rho = \sum_{m,n=0}^{n_{\text{max}}(\rho)|m\rangle\langle n|$, the density property of $D_*$ in $D$ is clear.

One has that $\rho \in D_{n^*}$ may be considered as an operator from $H$ to the finite-dimensional space $n_{\text{max}}(\rho)$, or as a density matrix on $H_{n^*}$. Hence, $d(N)\rho$ is a well-defined trace-class operator on $H$, by considering $d(N)$ as an operator on $H_{n^*}$, and $\rho \in D_{n^*}$ as an operator from $H$ to $H_{n^*}$. Indeed, $d(N)\rho = \sum_{m,n=0}^{n_{\text{max}}(\rho)} \rho_{mn}(m - \varpi)^2/|m|n|$, and it is trace-class because its range is finite-dimensional [7], [3]. Consequently, the Lyapunov function $V_\epsilon$ in (5), the feedback in (6) and $n_{\text{max}}$ can be extended to $D_*$.  

Define the map $\Delta: \rho \rightarrow H \rightarrow \mathbb{R}$;  

Proposition 2: Let $\rho \in D_*$, $u \in \{-1, 0, 1\}, y = g, e$. Take $\alpha = \text{Tr}(M_\rho^1\rho M_\rho^1(u))$. Then:

- $\text{Tr}(\Delta\rho) = \text{Tr}(\Delta\rho u)$, for every diagonal bounded operator $A: H \rightarrow H$;
- $V_\epsilon(\rho) = V_\epsilon(\Delta\rho), \quad \text{for } \epsilon > 0$;
- $\alpha^{-1}M_\rho^1\rho M_\rho^1(u) \in \text{D}_*$ belongs to $\text{D}_*$ with $\Delta(\alpha^{-1}M_\rho^1\rho M_\rho^1(u)) = \alpha^{-1}M_\rho^1\rho M_\rho^1(u)$;
- $[M_\rho^1(\Delta\rho)\rho M_\rho^1(u)]_{nn} = [M_\rho^1(\rho)\rho M_\rho^1(u)]_{nn}$, for all $n \in \mathbb{N}$. In particular, $\alpha = \text{Tr}(M_\rho^1(\rho)\rho M_\rho^1(u))$.

Proof: See Appendix [C].

Now, let $\epsilon > 0$ and $\varpi = |\varpi|/|\varpi|$, where $\varpi \in \mathbb{N}$. Assume that $\rho_0 \in D_*$. Let $\rho_k, k \in \mathbb{N}$, be the corresponding closed-loop trajectory for a fixed realization of (1–4) with feedback $u_k = f(\rho_k)$, where $f$ is as in (6). It is immediate from the proposition above that:

- $\rho_k \in \text{D}_*$, for $k \in \mathbb{N}$;
- $\Delta \rho_k \in \text{D}_*$, $k \in \mathbb{N}$, is the corresponding closed-loop trajectory of (1–4) for the initial condition $\rho_0$, the same realization (and with the same transition probabilities $\rho_{e,k}$ and $\rho_{e,k}$), as well as the same feedback $u_k = f(\rho_k)$.

- $\text{Tr}(|n\rangle\langle n|) = \text{Tr}(|n\rangle\langle n|\Delta\rho_k)$, for any $n \in \mathbb{N}$.

From these arguments, Theorem [I] and the fact that $\Delta \varpi = \varpi$, one immediately obtains the following generic solution to the control problem, that is, when the initial condition $\rho_0$ belongs to the dense subset $D_* = D$:

Theorem 2: Let $\varpi \in \mathbb{N}$ and $\epsilon > 0$. In (2–4), assume that $\varphi_0/\pi$ and $(\theta_0/\pi)^2$ are irrational numbers, and take $\varphi_R = \pi/2 - \varpi\phi_0$. Consider the closed-loop Markov process (1–4) with $u_k = f(\rho_k)$, where the feedback law $f$ is as in (6). Then, given any initial condition $\rho_0 \in D_*$, one has that $\rho_k$ converges almost surely towards $\varpi = |\varpi|/|\varpi|$ as $k \rightarrow \infty$. 


IV. SIMULATION RESULTS

This section presents the closed-loop simulation results concerning the application of Theorem 2 above to the ideal Markov process (1)–(4). The quantum experimental results exhibited in [8] used the following control parameter values in (2)–(4): $\phi_0/\pi = 0.252$ and $\theta_0/\pi \approx 2/\sqrt{3} + 1$. However, according to the assumptions in Theorem 2, $\phi_0/\pi$ and $(\theta_0/\pi)^2$ should be irrational numbers. Hence, here one chooses $\phi_0/3.14 = 0.252$ and $\theta_0/3.14 = 2/\sqrt{3} + 1$. One takes $\rho_0 = \sum_{n=0}^{15} |n\rangle\langle n|/16 \in \mathbb{D}$ as the initial condition, $\pi = 10$ for the goal Fock state $\bar{\rho} = |\pi\rangle\langle \pi|$, and $\epsilon = 10^3$ as the gain for the feedback $u_k = f(\rho_k)$ in (5)–(6). Figure 1 exhibits the simulation results for one closed-loop realization with such choices and a final sample step of 120. It shows: the dynamics of the populations of $\rho_k$ (top), the controls $u_k$ (middle) and the simulated outcomes $y_k$ (bottom). The populations of $\rho_k$ correspond to the following observables: $A_1 = \sum_{n=0}^{3} |n\rangle\langle n| (n < \bar{n})$, $A_2 = |\bar{n}\rangle\langle \bar{n}| (n = \bar{n})$, $A_3 = \sum_{n>\bar{n}} |n\rangle\langle n| (n > \bar{n})$. Therefore, one sees from the dynamics of the populations that $\rho_k$ converges to $\bar{\rho}$ as $k \to \infty$, which is in accordance with Theorem 2. Note that $\langle |\bar{n}\rangle\langle \bar{n}| \rangle \approx 1$ and $u_k = 0$ for all $k > 45$.

Recall that Theorem 2 assumes that $\epsilon > 0$. In order to further analyze the performance of the Lyapunov-based feedback law here proposed, we now make a comparison with the one used experimentally in [8], which corresponds to take $\epsilon = 0$ in (5), i.e. to disregard the term $-\epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2$. Figure 2 presents the simulation results for one closed-loop realization of such case. The control parameters, $\rho_0$ and $\bar{n} = 10$ are the same as above. Note that $\langle |\bar{n}\rangle\langle \bar{n}| \rangle \approx 1$ and $u_k = 0$ for all $k > 78$. In order to make a comparison in terms of the speed of convergence, define the settling time $k_s$ to be the smallest $k \in \mathbb{N}$ such that $\langle |\bar{n}\rangle\langle \bar{n}| \rangle > 0.9$ for all $k \geq k_s$. One has $k_s = 45$ for the case $\epsilon = 10^3$ above, and $k_s = 78$ for $\epsilon = 0$. Therefore, in the two realizations here simulated, the choice of $\epsilon = 10^3$ reduced the settling time $k_s$ by nearly 42% with respect to $\epsilon = 0$. This behavior is typical on an average basis, thereby justifying the term $-\epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2$ in (5). Table I shows the average value $\bar{k}_s$ and the standard deviation $\sigma$ of $k_s$ for $\epsilon \in \{0, 0.1, 1, 10, 10^2, 10^3, 10^4, 10^5\}$, of a total of 5000 realizations were simulated for each $\epsilon$. Notice that when $\epsilon$ is relatively large or relatively small in comparison to $\epsilon = 10^3$, the average settling time $\bar{k}_s$ deteriorated. Furthermore, although for $\epsilon = 10^5$ one has that $\bar{k}_s$ increased by nearly 22% in comparison to $\epsilon = 10^3$, the standard deviation $\sigma$ decreased by nearly 62%. Computer simulations have suggested that a choice of $\epsilon > 0$ which may perhaps significantly improve $\bar{k}_s$ generally depends on the initial condition $\rho_0$ and on the goal Fock state $\bar{\rho} = |\bar{n}\rangle\langle \bar{n}|$, and it has to be determined heuristically.

V. PROOF OF THEOREM 1 (DIAGONAL CASE)

Proof of the First Step:

Let $\epsilon > 0$. Define $V: D_+ \to \mathbb{R}$ and $W: D_+ \to \mathbb{R}$ as

$$V(\rho) = \text{Tr} (d(N) \rho), \quad W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}^2, \quad (7)$$

respectively. Note that $V_r = V + \epsilon W$. Define:

- $Q_W(\rho, u) = W(\rho) - \mathbb{E}[W(\rho_{k+1}) | \rho_k = \rho, u_k = u]$,
- $Q_V(\rho, u) = V(\rho) - \mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = u]$,
- $Q_{V_r}(\rho, u) = V_r(\rho) - \mathbb{E}[V_r(\rho_{k+1}) | \rho_k = \rho, u_k = u]$,
for $\rho \in D_*$ and $u \in \{-1, 0, 1\}$. The proof of Theorem 1 is a straightforward consequence of the next proposition:

**Proposition 3.** Let $\epsilon > 0$ and $n_0, r_0, \pi \in \mathbb{N}$. There exists an integer $n_0 > n_0 + r_0 + \pi + 1$ (depending on $\epsilon, n_0, r_0, \pi$) such that, for each $\rho \in D_*$ with $n_{\text{length}}(\rho) \leq n_0$, if $n_{\text{max}}(\rho) = m_0$, then

$$Q_V(\rho, -1) > \max \{Q_V(\rho, 0), Q_V(\rho, +1)\}.$$  

In fact, given $\rho_0 \in D_*$, let $n_0 = n_{\text{length}}(\rho_0)$ and $r_0 = n_{\text{min}}(\rho_0)$. Note that $n_{\text{max}}(\rho_0) = m_0 < r_0$. By Proposition 1 $\rho_k \in D_*$ with $n_{\text{length}}(\rho_k) \leq n_0$, for all $k \in \mathbb{N}$. Since $u = f(\rho)$ maximizes $Q_V(\rho, f(\rho))$, Proposition 3 implies that when $n_{\text{max}}(\rho_k) = m_0$ for some $k \in \mathbb{N}$, then the input $u_k$ will always be equal to $-1$, and hence Proposition 1 ensures that $n_{\text{max}}(\rho_{k+1}) \leq n_{\text{max}}(\rho_k) = m_0$. Therefore, $n_{\text{max}}(\rho_k) \leq m_0$, $k \in \mathbb{N}$, showing the First Step.

The following two lemmas are instrumental for showing Proposition 3. Their proofs are given in Appendix D and Appendix E respectively.  

**Lemma 1:** Given an arbitrary nonzero $\theta_0 \in \mathbb{R}$, fix any $a \in \mathbb{R}$ such that $0 < a < 1/2$. For all nonzero $N_0, N \in \mathbb{N}$, there exists an integer $N > N_0$ big enough such that,

$$0 < 1/2 - a \leq \sin^2 \left(\frac{\pi}{\sqrt{N}}\right) \leq 1/2 + a,$$

for $n = \sqrt{N}$, $N + 1, \ldots, N + N_0 - 1$.

**Lemma 2:** Let $\rho \in D_*$: Then,

- $|Q_V(\rho, u)| \leq 1$, for each $u \in \{-1, 0, 1\}$;
- $Q_V(\rho, 0) = 0$;
- $Q_V(\rho, +1) = -\sum_{n \in \mathbb{N}} \rho_{nn}[2(n - \pi) + 1] \sin^2 \left(\frac{\rho_n}{\sqrt{N}}\right) + 1$;
- $Q_V(\rho, -1) = \sum_{n \in \mathbb{N}} \rho_{nn}[2(n - \pi) - 1] \sin^2 \left(\frac{\rho_n}{\sqrt{N}}\right)$.

The proof of Proposition 3 is shown in the sequel.  

**Proof:** Let $\epsilon > 0$ and $n_0, r_0, \pi \in \mathbb{N}$. One has to show that there exists $m_0 > n_0 + r_0 + \pi + 1$ such that, if $\rho \in D_*$ with $n_{\text{length}}(\rho) \leq n_0$, then $u = -1$ always maximizes $Q_V(\rho, u)$ whenever $n_{\text{max}}(\rho) = m_0$. From Lemma 2 and the fact that $Q_V = Q_V + \epsilon Q_W$, to complete the proof it suffices to show that:

- If $\rho \in D_*$ is such that $n_{\text{length}}(\rho) \leq n_0$ and $n_{\text{max}}(\rho) \geq n_0 + \pi$, then $Q_V(\rho, +1) \leq 0$;
- There exists $m_0 > n_0 + r_0 + \pi + 1$ such that $Q_V(\rho, -1) > 2\epsilon$, whenever $\rho \in D_*$ is such that $n_{\text{length}}(\rho) \leq n_0$ and $n_{\text{max}}(\rho) = m_0$.  

Note that

$$Q_V(\rho, +1) = -\sum_{n = n_{\text{min}}(\rho)}^{n_{\text{max}}(\rho)} \rho_{nn}[2(n - \pi) + 1] \sin^2 \left(\frac{\rho_n}{\sqrt{N}}\right),$$

for any $\rho \in D_*$. Thus, if $n_{\text{length}}(\rho) \leq n_0$ and $n_{\text{max}}(\rho) \geq \pi + n_0$, then $n_{\text{min}}(\rho) \geq \pi$, and hence the first claim is shown.

Now, fix $0 < a < 1/2$ and let

$$N \geq \frac{\pi}{\sqrt{N}} \geq \frac{\pi}{\sqrt{N + 1}}.$$  

Applying Lemma 1 for $N_0 = n_0 + r_0 + 1$ and such choice of $N$, one gets $N > N$ in which $0 < 1/2 - a \leq \sin^2 \left(\frac{\pi}{\sqrt{N}}\right)$.

4As $N$ is an integer, it follows that $N \geq \pi + 1$.
Theorem 3: [5, Theorem 1, p. 195] Let $\Omega$ be a probability space and let $W$ be a measurable space. Consider that $X_k: \Omega \rightarrow W, k \in \mathbb{N}$, is a Markov chain with respect to the natural filtration. Let $Q: W \rightarrow \mathbb{R}$ and $V: W \rightarrow \mathbb{R}$ be measurable non-negative functions with $V(X_k)$ integrable for all $k \in \mathbb{N}$. If $E[V(X_{k+1}) | X_k] - V(X_k) = -Q(X_k)$, for $k \in \mathbb{N}$, then $\lim_{k \to \infty} Q(X_k) = 0$ almost surely.

Indeed, let $J_1$ be the complex Banach space of all trace-class operators on $\mathcal{H}$ with the trace norm $\| \cdot \|_1$, that is, $\|B\|_1 = \text{Tr}(|B|)$, where $|B| \triangleq \sqrt{B^*B}$, for $B \in J_1$. Recall that $\|B\| \leq \|B\|_1$, and $\text{Tr}(AB) \leq \|A\|\|B\|_1$, for every $B \in J_1$ and each bounded operator $A: \mathcal{H} \rightarrow \mathcal{H}$, where $\| \cdot \|$ is the usual operator norm ($sup$ norm of bounded operators) [7], [3]. Consider the subspace topology on $D_{ma}$ with respect to $J_1$. One has that the closed-loop trajectory $\rho_k, k \in \mathbb{N}$, is a Markov chain with phase space $D_{ma}$ (with respect to the natural filtration and the Borel algebra on $D_{ma}$). It is clear that $D_{ma}$ is compact, and that $Q_e$ and $V_e - \alpha_e$ are non-negative and continuous on $D_{ma}$, for all $e > 0$, where $\alpha_e \triangleq \min_{\rho \in D_{ma}} V_e(\rho)$. The theorem above implies that $\rho_k$ converges almost surely towards $\bar{\rho}$ as $k \to \infty$ (with respect to the trace norm). This completes the proof of Theorem 1.

VI. CONCLUDING REMARKS

This paper provided a convergence analysis of Fock states stabilization via single-photon corrections under an ideal set-up, that is, assuming perfect measurement detection and no control delays. In terms of convergence speed, the simulation results here presented have justified the inclusion of the term $-\epsilon \sum_{n \in \mathbb{N}} n^2 \rho_{nn}$ in the Lyapunov-based feedback law 5–6. It is straightforward to verify that the convergence analysis developed in this paper remains valid for: (i) any other function $d(n)$ in 5 satisfying $d(\bar{n}) = 0, d(n)$ is increasing for $n > \bar{n}$ and $d(n)$ is decreasing for $n < \bar{n}$; and (ii) $\epsilon > 0$ dependent on $n$, that is, to take the term $-\sum_{n \in \mathbb{N}} \epsilon_n \rho_{nn}$. However, it is an open problem how to choose the function $d(n)$ and the gains $\epsilon_n > 0$ so as to achieve the best convergence speed.

Finally, the feedback law used in [8], which corresponds to $\epsilon = 0$, was tailored for an experimental set-up with measurement imperfections and control delays. The convergence analysis of such realistic situation will be investigated in the future.

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APPENDIX

A. Basic properties of the operators $N, a$ and $a^+$

Fix $n^* \in \mathbb{N}$ and let $\mathcal{H}_{n^*} = \text{span}\{0, \ldots, [n^*]\}$. Consider the (linear) operators $N: \mathcal{H}_{n^*} \rightarrow \mathcal{H}_{n^*}, a: \mathcal{H}_{n^*} \rightarrow \mathcal{H}_{n^*-1}, a^+: \mathcal{H}_{n^*} \rightarrow \mathcal{H}_{n^*+1}$ defined respectively as $N(n) = n |n\rangle, a|0\rangle = 0, a|n\rangle = \sqrt{n} |n-1\rangle$, for $n \geq 1$.\footnote{One also recalls that if $A$ is a bounded operator on $\mathcal{H}$ and $B \in J_1$, then $AB, BA \in J_1$ with $\text{Tr}(AB) = \text{Tr}(BA)$.}

Note that these operators cannot be extended to $\mathcal{H}$. Let $f: \mathbb{N} \rightarrow \mathbb{R}$ be a function. Define the operator $f(N): \mathcal{H}_{n^*} \rightarrow \mathcal{H}_{n^*}$ by $f(N)|n\rangle = f(n)|n\rangle$, for each $n = 0, \ldots, n^*$. It is clear that $f(N)$ can be extended to $\mathcal{H}$ whenever $f$ is a bounded function. Given $f: \mathbb{N} \rightarrow \mathbb{R}$ and an integer $m$, one defines $g: \mathbb{N} \rightarrow \mathbb{R}$ as: $g(n) = f(n + m)$, when $n + m \geq 0$; and $g(n) = 0$, when $n + m < 0$. One abuses notation letting $f(N + m)$ stand for $g(N)$. Given two functions $f, g: \mathbb{N} \rightarrow \mathbb{R}$, it is clear that $f(N)g(N) = g(N)f(N) = (fg)(N)$ and $(f + g)(N) = f(N) + g(N)$. Furthermore: $aa^+ = N^+I, a^+a = N, af(N) = f(N + 1)a, a^+f(N) = f(N - 1)a^+$.\footnote{By assumption, $\rho_0 \in D_*$. Then, \ref{5}, \ref{8}–\ref{13} above and induction on $k$ show the assertions in Proposition 1.}

B. Proof of Proposition 2

Fix any $\rho \in D_*$ and let $\rho \in \mathbb{N}$. In particular, $\rho|n\rangle = \rho_{nn}|n\rangle$. It then follows from \ref{2}–\ref{4} that:

\begin{align}
M_g(0)|\rho M_g(0)\rangle &= \rho_{nn} \cos^2 \left(\frac{\phi_{n+\rho}}{2}\right)|n\rangle, \quad \text{for } \rho > 0, \\
M_e(0)|\rho M_e(0)\rangle &= \rho_{nn} \sin^2 \left(\frac{\phi_{n+\rho}}{2}\right)|n\rangle, \\
M_g(1)|\rho M_g(1)\rangle &= \left\{ \begin{array}{ll}
0, & \text{for } n = 0, \\
\rho_{nn} \cos^2 \left(\frac{\phi_{n+\rho}}{2}\right) |n\rangle, & \text{for } n \geq 1,
\end{array} \right. \\
M_e(1)|\rho M_e(1)\rangle &= \rho_{nn} \cos^2 \left(\frac{\phi_{n+\rho}}{2}\right) |n\rangle, \\
M_g(-1)|\rho M_g(-1)\rangle &= \rho_{nn} \sin^2 \left(\frac{\phi_{n+\rho}}{2}\right) |n\rangle, \\
M_e(-1)|\rho M_e(-1)\rangle &= \rho_{nn} \sin^2 \left(\frac{\phi_{n+\rho}}{2}\right) |n\rangle.
\end{align}

Therefore:

\begin{align}
M_g(0)|\rho M_g(0)\rangle &= \sum_{n=0}^{\rho_{nn}} \rho_{nn} \cos^2 \left(\frac{\phi_{n+\rho}}{2}\right)|n\rangle, \\
M_e(0)|\rho M_e(0)\rangle &= \sum_{n=0}^{\rho_{nn}} \rho_{nn} \sin^2 \left(\frac{\phi_{n+\rho}}{2}\right)|n\rangle, \\
M_g(1)|\rho M_g(1)\rangle &= \sum_{n=0}^{\rho_{nn}} \rho_{nn} \cos^2 \left(\frac{\phi_{n+\rho}}{2}\right)|n\rangle, \\
M_e(1)|\rho M_e(1)\rangle &= \sum_{n=0}^{\rho_{nn}} \rho_{nn} \sin^2 \left(\frac{\phi_{n+\rho}}{2}\right)|n\rangle, \\
M_g(-1)|\rho M_g(-1)\rangle &= \sum_{n=0}^{\rho_{nn}} \rho_{nn} \sin^2 \left(\frac{\phi_{n+\rho}}{2}\right)|n\rangle, \\
M_e(-1)|\rho M_e(-1)\rangle &= \sum_{n=0}^{\rho_{nn}} \rho_{nn} \sin^2 \left(\frac{\phi_{n+\rho}}{2}\right)|n\rangle.
\end{align}

By assumption, $\rho_0 \in D_*$. Then, \ref{5}, \ref{8}–\ref{13} above and induction on $k$ show the assertions in Proposition 1.

C. Computation of $Q_V(\rho, u)$

Fix any $\rho \in D_*$ and $\bar{n} \in \mathbb{N}$. Recall that $V(\rho) = \text{Tr}(d(N)\rho)$, where $d: \mathbb{N} \rightarrow \mathbb{R}$ be given by $d(n) = (n - \bar{n})^2$. Note that \ref{5} implies that, for each $u \in \{-1, 0, 1\}$,

\begin{align}
E[V(\rho_{k+1}) | \rho_k = \rho, u_k = u] &= \text{Tr} \left[d(N)M_g(u)\rho M_g(u) + d(N)M_e(u)\rho M_e(u)\right].
\end{align}
Take $u = 0$. From (3)-(9) in Appendix B, one has
$$\mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = 0] = \text{Tr} \left( d(N)M_g(0)\rho M_g(0) \right) + \text{Tr} \left( d(N)M_c(0)\rho M_c(0) \right)$$
$$= \text{Tr} \left( d(N) \left[ M_g(0)\rho M_g(0) + M_c(0)\rho M_c(0) \right] \right)$$
$$= \text{Tr} (d(N)\rho) = V(\rho).$$

In particular,
$$Q_V(\rho, 0) = 0. \quad (15)$$

Now, take $u = +1$. Then, (14) above and (10)-(11) in Appendix B provide that
$$\mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = +1] = \text{Tr} \left( \sin^2 \left( \frac{\theta}{2} \sqrt{N + 1} \right) d(N + 1) \rho \right) + \text{Tr} \left( \cos^2 \left( \frac{\theta}{2} \sqrt{N + 1} \right) d(N) \rho \right).$$

By summing and subtracting $\text{Tr} (\sin^2 (\frac{\theta}{2} \sqrt{N + 1}) d(N) \rho)$,
$$\mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = +1] = \text{Tr} (d(N)\rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta}{2} \sqrt{N + 1} \right) d(N + 1) - d(N) \right) \rho.$$\hspace{1cm} (16)

Finally, take $u = -1$. Using (14) above and (12)-(13) in Appendix B,
$$\mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = -1] = \text{Tr} \left( \sin^2 \left( \frac{\theta}{2} \sqrt{N + 1} \right) d(N + 1) - d(N) \right) \rho = \rho - \sum_{n \in \mathbb{N}} \rho_{nn} \left( n + 1 \right) \sin^2 \left( \frac{\theta}{2} \sqrt{n + 1} \right). \quad (17)$$

D. Proof of Lemma 7

Assume that $N_0$ is even (otherwise one may take $N_0 + 1$ instead of $N_0$ in this proof). Define the function $\eta: \mathbb{N} \rightarrow \mathbb{R}$ by
$$\eta(\ell) = \left[ \frac{2}{\theta_0} \left( \frac{\ell}{2} + \frac{\pi}{4} \right) \right]^2. \quad (18)$$

By definition, one has $\frac{\theta}{2} \sqrt{\eta(\ell)} = \frac{\ell}{2} + \frac{\pi}{4}$ for all $\ell \in \mathbb{N}$. Let $h = \pi/4 - \arcsin \left( \sqrt{1/2 - a} \right)$.

E. Proof of Lemma 2

Proof of the first claim: Let $u \in \{-1, 0, 1\}, \rho \in D_\lambda$. Recall that $W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}^2$. Since $\text{Tr} (\rho) = \sum_{n \in \mathbb{N}} \rho_{nn} = 1$, then $1 - \sum_{n \in \mathbb{N}} \rho_{nn} \leq W(\rho) \leq 0$. Now, by (1), $\mathbb{E}[W(\rho_{k+1}) | \rho_k = \rho, u_k = u] = p_{g,k} W(\rho_{k+1}^g) + p_{c,k} W(\rho_{k+1}^c)$, where $p_{g,k}, p_{c,k} \geq 0$ with $p_{g,k} + p_{c,k} = 1$. Thus $-1 \leq \mathbb{E}[W(\rho_{k+1}) | \rho_k = \rho, u_k = u] \leq 0$. Since $Q_W(\rho, u)$ is the difference of two numbers that are in-between $-1$ and $0$, one concludes that $|Q_W(\rho, u)| \leq 1$.

The second, third and fourth claims, are immediate from (15), (16) and (17) in Appendix C, respectively.

Proof of the first claim: Let $\rho \in D_m$. By (3)-(9) in Appendix B, $M_g(0)\rho M_g(0) + M_c(0)\rho M_c(0) = \rho$. Taking $\rho_k = \rho$ in $u_k = 0$ in (1), define
$$p^\rho = \frac{\rho_{k+1}^y}{\text{Tr} \left( M_g(0)\rho M_g(0) \right)} \quad \text{for } g, e.$$

Hence, $\alpha p^\rho + (1 - \alpha)\rho^\rho = \rho$, where $\alpha \triangleq p_{g,k} = \text{Tr} \left( M_g(0)\rho M_g(0) \right)$. In particular, $\alpha \rho_{nn}^\rho + (1 - \alpha)\rho_{nn}^c = \rho_{nn}$, for $n \in \mathbb{N}$. Note that, if $\alpha = 0$, then $M_g(0)\rho M_g(0) = 0$, and so $\rho^\rho = \rho$. Similarly, $\alpha = 1$ implies $\rho^\rho = \rho$. Thus, the identity $\alpha \rho_{nn}^\rho + (1 - \alpha)\rho_{nn}^c = \rho_{nn}$, for $n \in \mathbb{N}$, still holds when $\alpha = 0$ or $\alpha = 1$. From (1) and (7) and $\alpha \triangleq p_{g,k}$, one has
$$Q_W(\rho, 0) = W(\rho) - \left[ p_{g,k} W(\rho_{k+1}^g) + p_{c,k} W(\rho_{k+1}^c) \right]$$
$$= \sum_{n \in \mathbb{N}} \left( \rho_{nn}^\rho + (1 - \alpha)\rho_{nn}^c \right)^2 - \left[ \alpha \rho_{nn}^\rho + (1 - \alpha)\rho_{nn}^c \right]^2$$
$$\geq 0, \quad (21)$$
\hspace{1cm} \text{(More precisely, } \sin^2(\pi/2 - x) = 1 - \cos^2(\pi/2 - x) = 1 - \sin^2(x). \text{)}
thereby showing the first part of the first claim.

If \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) with \( 0 < \alpha < 1 \), then (8) and (9) in Appendix B imply that \( \rho^\theta = \rho^\theta = \rho \), and so \( Q_W(\rho, 0) = 0 \). Now, one shows that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) whenever \( Q_W(\rho, 0) = 0 \). Suppose \( Q_W(\rho, 0) = 0 \). Then, (21) implies that \( \alpha = 0 \), or \( \alpha = 1 \), or \( \rho^\theta_{m,n} = \rho^\theta_{m,n} \) for all \( m \in \mathbb{N} \) with \( 0 < \alpha < 1 \). Assume that \( \alpha = 0 \). Hence, \( M_g(0)M_g^\dagger(0) = \sum_{n,m \in \mathbb{N}} \rho_{m,n} \cos^2(\frac{\phi_{m,n} + \phi_{n,m}}{2}) |n\rangle\langle n| = 0 \) by (8) in Appendix B. Suppose that \( \rho \neq |m\rangle\langle m| \) for every \( m \in \mathbb{N} \). Thus, there exists \( n_1, n_2 \in \mathbb{N} \) with \( n_1 \neq n_2 \), \( \rho_{n_1,n_2} > 0 \), \( \rho_{n_2,n_1} > 0 \). Recall that \( \sin(x_1) = \pm \sin(x_2) \) if and only if \( x_1 + x_2 = \ell \pi \) or \( x_2 - x_1 = \ell \pi \), where \( \ell \) is an integer. Therefore, \( \sin(\frac{\phi_{m,n} + \phi_{n,m}}{2}) = \pm \sin(\frac{\phi_{m,n} + \phi_{n,m}}{2}) \), which contradicts the assumptions that \( \phi_0/\pi \) is an irrational number and \( \phi_R = \pi/2 - \tilde{n}\phi_0 \). One has shown that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) whenever \( \alpha = 0 \). If \( \alpha = 1 \), or \( \rho^\theta_{m,n} \neq \rho^\theta_{m,n} \) for all \( m \in \mathbb{N} \) with \( 0 < \alpha < 1 \), then from similar arguments and computations one also concludes that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \).

**Proof of the second claim:** Let \( m \in \mathbb{N} \) and take \( \rho = |m\rangle\langle m| \in D_\alpha \). It is clear that \( W(\rho) = \sum_{n \in \mathbb{N}} \rho^\theta_{m,n} = -1 \). From (10)–(13) in Appendix B one has that:

\[
\begin{align*}
(M_g(+1)ρM_g^\dagger(+1))_{m,n} &= \delta(n,m+1) \sin^2 \left( \frac{n\delta_0}{2} \sqrt{m+1} \right), \\
(M_e(+1)ρM_e^\dagger(+1))_{m,n} &= \delta(n,m) \cos^2 \left( \frac{n\delta_0}{2} \sqrt{m+1} \right), \\
(M_g(-1)ρM_g^\dagger(-1))_{m,n} &= \delta(n,m) \cos^2 \left( \frac{n\delta_0}{2} \sqrt{m} \right), \\
(M_e(-1)ρM_e^\dagger(-1))_{m,n} &= \delta(n+1,m) \sin^2 \left( \frac{n\delta_0}{2} \sqrt{m} \right),
\end{align*}
\] (22)

where \( \delta(n,m) \) is the usual Kronecker delta: \( \delta(n,m) = 0 \) if \( n \neq m \), and \( \delta(n,m) = 1 \) if \( n = m \). In particular:

\[
\begin{align*}
\text{Tr} \left( M_g(+1)ρM_g^\dagger(+1) \right) &= \sin^2 \left( \frac{n\delta_0}{2} \sqrt{m+1} \right), \\
\text{Tr} \left( M_e(+1)ρM_e^\dagger(+1) \right) &= \cos^2 \left( \frac{n\delta_0}{2} \sqrt{m+1} \right), \\
\text{Tr} \left( M_g(-1)ρM_g^\dagger(-1) \right) &= \cos^2 \left( \frac{n\delta_0}{2} \sqrt{m} \right), \\
\text{Tr} \left( M_e(-1)ρM_e^\dagger(-1) \right) &= \sin^2 \left( \frac{n\delta_0}{2} \sqrt{m} \right),
\end{align*}
\]

\[
\sum_{n \in \mathbb{N}} \left( \frac{M_g(u)ρM_g^\dagger(u)}{\text{Tr}(M_g(u)ρM_g^\dagger(u))} \right)^2_{m,n} = 1, \quad \text{for } u = g,e
\]

(assuming no division by 0). Now, using (14) and the above computations, one gets:

\[
\mathbb{E} [W(p_{k+1}) | p_k = ρ, u_k = ±1] = p_{g,k} W(p_{k+1}^g) + p_{e,k} W(p_{k+1}^e),
\]

\[
= - \sum_{y = g,e} \left[ \text{Tr} \left( M_g(±1)ρM_g^\dagger(±1) \right) \times \sum_{n \in \mathbb{N}} \left( \frac{M_g(±1)ρM_g^\dagger(±1)}{\text{Tr}(M_g(±1)ρM_g^\dagger(±1))} \right)^2_{m,n} \right]
\]

\[
= -1 = W(ρ).
\]

Therefore, \( Q_W(|m\rangle\langle m|, ±1) = 0 \).

**G. Proof of Proposition 2**

Fix \( ρ \in D_α \). Since \( \text{Tr}(d(N)ρ) = \text{Tr}(d(N)Δρ) \) and \( ρ_{m,n} = (Δρ)_{m,n} \) for \( n \in \mathbb{N} \), the first two assertions are immediate from the definitions. As for the third and fourth assertions, let \( |ψ⟩ = \sum_{m,n} (m|ψ⟩)|m⟩ \) \( \in \mathcal{H} \). Note that \( ρ|m⟩ = \sum_{n=0}^{n_{max}(m)} ρ_{m,n}|n⟩ \), for \( m \in \mathbb{N} \). Using (2)–(4):

\[
M_g(0)ρM_g^\dagger(0)|ψ⟩ = \sum_{m,n=0}^{n_{max}(ρ)} ρ_{m,n} \cos \left( \frac{\phi_{m,n} + \phi_{n,m}}{2} \right) |m⟩\langle n| |ψ⟩,
\]

\[
M_e(0)ρM_e^\dagger(0)|ψ⟩ = \sum_{m,n=0}^{n_{max}(ρ)} ρ_{m,n} \sin \left( \frac{\phi_{m,n} + \phi_{n,m}}{2} \right) |m⟩\langle n| |ψ⟩,
\]

\[
M_g(+1)ρM_g^\dagger(+1)|ψ⟩ = \sum_{m=1,n=0}^{n_{max}(ρ)+1} ρ_{m-1,n} \sin \left( \frac{n\delta_0}{2} \sqrt{m+1} \right) |m⟩\langle n| |ψ⟩ + 1,
\]

\[
M_e(+1)ρM_e^\dagger(+1)|ψ⟩ = \sum_{m=0,n=1}^{n_{max}(ρ)} ρ_{m,n} \cos \left( \frac{n\delta_0}{2} \sqrt{m} \right) |m⟩\langle n| |ψ⟩,
\]

\[
M_g(-1)ρM_g^\dagger(-1)|ψ⟩ = \sum_{m=0,n=1}^{n_{max}(ρ)} ρ_{m,n} \cos \left( \frac{n\delta_0}{2} \sqrt{m} \right) |m⟩\langle n| |ψ⟩ - 1.
\]

Since \( Δρ \in D_α \subset D_σ \), \( n_{max}(Δρ) = n_{max}(ρ) \) and \( (Δρ)_{m,n} = ρ_{m,n} \), the proof is straightforward from (8)–(13) in Appendix B.

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