A study on generalized inverses and increasing functions Part I: generalized inverses

Arnaud De La Fortelle

A study on generalized inverses and increasing functions Part I: generalized inverses

Arnaud de La Fortelle

September 3, 2015

Abstract

Generalized inverses of increasing functions are used in several domains such as function analysis, measure theory, probability and fuzzy logic. Several definitions of the generalized inverse are known leading to different properties. This paper aims at giving a precise study of the link between the definitions and the properties. It is shown why the right-continuous generalized inverse is a good choice. This rigorous treatment opens the doors to more focused studies.

1 Introduction

This study was first motivated by theoretical considerations on stochastic processes where some increasing processes are sometimes naturally inverted. One example is the maximum of a Brownian motion B_t on $[0, t]$: this continuous increasing process is the generalized inverse of an increasing Levy process. However it happens that there are many other domains where one can apply this notion of inverse: a generalized version of the inverse of a one-to-one function hence the name *generalized inverse*. In measure theory it is linked to change-of-variables formulæ for Lebesgue–Stieltjes integrals as in [2]; in probability theory it is linked to the distribution function of a real-valued random variable: the generalized inverse appears naturally to transform a uniform random variable into a random variable with a given distribution function and this technique is widely applied to simulation; for the same reason the generalized inverse is sometimes called *quantile function* and the previous operation a quantile transformation: it is used in statistics (and applied e.g. to insurance and finance); generalized inverses are also used in fuzzy logic for building t-norms as in [7, 4]: this is probably where one can found the most general generalized inverses.

The history of the generalized inverse is difficult to draw since it is a well established auxiliary notion that was not considered worth a dedicated study: this is especially true in probability theory. Moreover the name generalized inverse, though the most common, is not stable, changing with the domain: we mentioned *quantile function* but there are other names such as *pseudo-inverse* or *quasi-inverse*. However, with the use of more precise properties, there has been some papers dedicated to generalized inverses. The fuzzy logic domain has made precise definitions such as [4] citing [7] and even earlier work of [6] related to t-norms: in these papers the most general definitions are given at the price of maybe very non-regular generalized inverses. In recent years there was a rising interest to clarify the definitions and the properties of generalized inverses as in [1, 3] who have inspired a large part of this paper.

Though a lot of work has been devoted to the analysis of the generalized inverse, some more work is needed in order to reach the author's initial goal. Indeed the reference above study properties of the transform for a single function (continuity — left or right — at a point, injective and surjective properties, orders...). Now what happens if we apply that to stochastic processes, i.e. families of functions? There is a clear need of continuity properties at the functional level so that we can ensure (or not) that sequences of generalized inverses converge if the original functions converge. Surprisingly enough, when going to the functional level, some interesting properties emerge naturally such as the generalization of the Lebesgue decomposition. However, due to the length these considerations will be developed separately.

This study is aimed at being self-contained so that the paper begins with a detailed section on properties of increasing functions and some descriptors of increasing functions: Lebesgue decomposition, jumps and flat sections that are closely tied with the generalized inverses. Then we deal with the definitions of generalized inverses and their properties: Section 3; one interesting result is to see the generalized inverses as all almost everywhere equals. Finally we concentrate on a specific generalized inverse in Section 4: the right-continuous generalized inverse. It has interesting regularity properties.

Note that lots the properties demonstrated in this paper also hold more generally for functions with bounded variations (that are the difference of two increasing functions). For the sake of conciseness this will not be developed.

2 Increasing functions: definitions and descriptions

Throughout the paper we will use the term *increasing* for non-decreasing functions f in $\mathbb{R}$:

$$x > y \implies f(x) \geq f(y) \tag{2.1}$$

If the right inequality is strict in (2.1) we say f is *strictly increasing*.

Generally we should consider mappings $f : I \mapsto J$ with $I, J \subset \mathbb{R}$. Convexity is implicitly used in the sequel so that I and J should be intervals. Then we need also complete intervals (to take extrema) so that I and J should be chosen closed. For the sake of simplicity we have chosen to take $I = J = \bar{\mathbb{R}} \overset{\text{def}}{=} \mathbb{R} \cup \{-\infty, +\infty\}$ and the usual topology on this compactification of the real numbers. It would be easy to transform the results below for any $I = [a, b]$ and $J = [c, d]$ using conventions like $\inf_I \emptyset = b$, $\sup_I \emptyset = a$ and similarly for J. But it becomes notationally complex without more generality.

Left and right limits always exist for increasing functions and are denoted by

$$f(x-) \overset{\text{def}}{=} \lim_{z \uparrow x} f(z) = \sup_{z<x} f(z), \tag{2.2}$$

$$f(x+) \overset{\text{def}}{=} \lim_{z \downarrow x} f(z) = \inf_{z>x} f(z). \tag{2.3}$$

By definition, f is right-continuous [resp. left-continuous] at x when $f(x) = f(x+)$ [resp. $f(x) = f(x-)$]. And f is continuous at x when it is right and left continuous at x.

The following result of Lebesgue [5] is fundamental in our analysis.

Theorem 2.1 (Lebesgue decomposition) *Any right-continuous increasing function f may be decomposed as:*

$$f = f_a + f_c + f_j, \tag{2.4}$$

where f_a is absolutely continuous, f_c is a singular continuous function (i.e. $f_c' = 0$ almost everywhere) and f_j is a jump function (its value is a countable sum of jumps). This decomposition is unique up to constants.

Note that the classical definition of a jump function f_j as $f_j(x) = \sum_{y \leq x} f(y+) - f(y-)$ leads to the condition for f to be right-continuous (and so is f_j). If we want to relax it, we need to characterize better the jumps.

2.1 Description of jumps

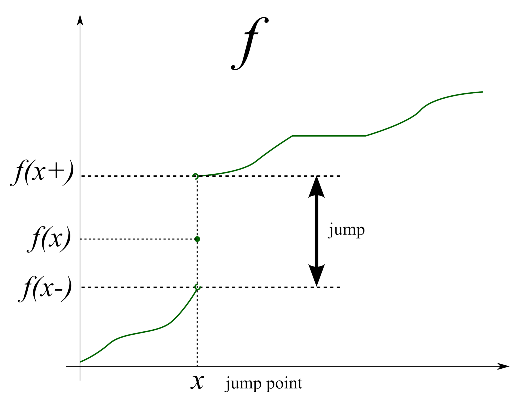

Figure 1: Jumps of an increasing function and the related concepts: jump point, jump, "free" value at a jump point.

Definition 2.2 (Jumps) *We say an increasing function f has a jump at x if $f(x-) < f(x+)$[1]. Whenever f jumps at x we say x is a jump point and we define the jump as the open interval $(f(x-), f(x+))$. The set of all jump points of f is called its jump points set denoted by $J(f)$ and the set of all jumps is called jumps family and denoted by $\mathcal{J}(f)$:*

$$J(f) \stackrel{\text{def}}{=} \left\{x \in \bar{\mathbb{R}} : f(x-) < f(x+)\right\} \tag{2.5}$$

$$\mathcal{J}(f) \stackrel{\text{def}}{=} \bigcup_{x \in J(f)} \left\{ \, (f(x-), f(x+)) \, \right\} = \bigcup_{x \in \bar{\mathbb{R}}} \left\{ \, (f(x-), f(x+)) \, \right\} \tag{2.6}$$

[1]For the special case of infinity, this definition should be adapted to right- or left- jumps. There is no technical difficulty but notational complexity so we skip these special cases.

Note that $f(x) \in [f(x-), f(x+)]$ in general but $f(x) = f(x+)$ [resp. $f(x) = f(x-)$] if, and only if, f is right-continuous [resp. left-continuous] at x. From the above definition, we see that an increasing function also defines a mapping Φ_f between $J(f)$ and $\mathcal{J}(f)$ by:

$$\begin{aligned} \Phi_f : J(f) &\mapsto \mathcal{J}(f) \\ x &\rightarrow (f(x-), f(x+)). \end{aligned} \tag{2.7}$$

Since $f(x) = f(x+)$, it is clear that a right-continuous pure jump function ($f = f_c$ in Lebesgue decomposition (2.4)) is completely characterized by $(J(f), \mathcal{J}(f), \Phi_f)$:

$$f(x) \overset{\text{def}}{=} f(0) + \sum_{y \in (0,x] \cap J(f)} \lambda(\Phi_f(y)) \quad \forall x \geq 0$$

With the classical notation λ for the Lebesgue measure (here it is simply the *length* of the interval $\Phi_f(y)$); and the classical convention that a sum over $(0, x]$ for $x < 0$ is the opposite of the sum over $(x, 0]$. Following the convention of the right term of Equation (2.6), and splitting the positive and the negative summation for the sake of clarity, we can rewrite the above definition as:

$$\begin{aligned} f(x) &\overset{\text{def}}{=} f(0) + \sum_{0<y\leq x} f(y) - f(y-) \quad \forall x \geq 0 && (2.8) \\ &\overset{\text{def}}{=} f(0-) - \sum_{x<y<0} f(y) - f(y-) \quad \forall x < 0 && (2.9) \end{aligned}$$

The above equations avoid the problem of infinities if $f(-\infty) = -\infty$: classically it is assumed — sometimes implicitely — that $f(-\infty) = 0$ as for distribution functions in probability and Equation (2.8) is written $f(x) = \sum_{y \leq x} f(y) - f(y-)$. Moreover it involves only the values of the jumps (i.e. $\lambda(\Phi_f(y)) = f(y) - f(y-)$) as in the classical formula: up to a global constant, this is the minimum information that is needed.

If f is not right-continuous (or left-continuous, it would be similar), we can easily adjust the equations but we must add the knowledge of the value of f at the jump points:

$$\begin{aligned} f(x) &\overset{\text{def}}{=} f(0+) + \sum_{0<y<x} f(y+) - f(y-) + (f(x) - f(x-)) \quad \forall x > 0 && (2.10) \\ &\overset{\text{def}}{=} f(0) \quad x = 0 && (2.11) \\ &\overset{\text{def}}{=} f(0-) - \sum_{x<y<0} f(y+) - f(y-) - (f(x+) - f(x)) \quad \forall x < 0 && (2.12) \end{aligned}$$

Note that the sets $J(f)$ and $\mathcal{J}(f)$ are finite or countable hence are of null measure (Lebesgue's measure is always assumed). Since Definition 2.1 only makes use of limits, two increasing functions f_1 and f_2 that would be equal almost everywhere have the same characterization: $(J(f_1), \mathcal{J}(f_1), \Phi_{f_1}) = (J(f_2), \mathcal{J}(f_2), \Phi_{f_2})$. So f_1 and f_2 jump at the same points and are continuous at the same points. This is easy to prove considering that the set where $f_1 = f_2$ is dense.

What we see is that the value of an increasing function at a jump point is "free" and has no influence to most of the properties of jumps. This is why taking a special representative of an increasing function in its equivalence class is rather logic. There are two main choices: left-continuous regularization, by taking $f(x) = f(x-)$ and right-continuous regularization by taking $f(x) = f(x+)$.

Proposition 2.3 (Regularization) *Let f be an increasing function. There exists a unique right-continuous [resp. left-continuous] function g that is almost everywhere equal to f. We call it right-continuous [resp. left-continuous] regularization of f. It the greatest [resp. least] function almost everywhere equal to f.*

Proof : Following the previous characterization of jumps as $(J(f), \mathcal{J}(f), \Phi_f)$ that is invariant within an equivalence class, we build g as follow: $g(x) = f(x+)$ for all $x \in \bar{\mathbb{R}}(f)$. Therefore existence is proved. Since we changed values only at a null measure set, limits of f and g are the same so that g is right-continuous. If f is right-continuous, then $f = g$. In general $g \geq f$ since $f(x) \leq f(x+)$ and there is equality only if f is right-continuous. Therefore $g = f$ if, and only if f is right continuous. Hence uniqueness is proved as well as the maximum property.

The properties for the left-continuous regularization are derived by considering $-f(-x)$. ■

This property allows us to use Lebesgue's decomposition on *any* increasing function f. We apply the decomposition to g, the right-continuous regularization of f: $g = g_a + g_c + g_j$. We set $f_a = g_a$, $f_c = g_c$ and $f_j = g_j$ except at jump points (i.e. on $J(f) = J(g)$) where we set $f_j(x) = g_j(x) + f(x) - f(x+)$. This decomposition exists with all desired properties except for the right-continuity of f_j.

Another consequence of the regularization is the following characterization of an equivalence class of almost everywhere equal increasing functions. Let f be an increasing function, g_r and g_l the right- and left-regularization of f. From Proposition 2.3 we know f is characterized by $g_l \leq f \leq g_r$. This is equivalent to $0 \leq f - g_l \leq g_r - g_l$. Note $g_r - g_l$ is null except on the jump set $J(f)$ where its value is the value of the jump $f(x+) - f(x-)$ (another invariant of the class). Therefore an equivalence class of almost everywhere equal increasing functions is much simpler to describe than an equivalence class of almost everywhere equal functions.

Theses properties of the jumps will be useful in the sequel. Before going to the next point, we would like to conclude this section by some remarks. First we see that the union of jumps of $\mathcal{J}$ ($\mathcal{J}$ is a set of disjoint open intervals, not a subset of $\bar{\mathbb{R}}$) is an open set of $\bar{\mathbb{R}}$ made of at most a countable union of disjoint open intervals; we denote it by O. It is maximal for pure jump increasing functions in the sense that its measure is maximal (this imply it is dense) in the following meaning. For any interval $(a, a + l)$, the measure of $O \cap (a, a + l)$ is l. It means the only increases are made by jumps. It could also serve as a characterization of pure jumps functions though this property is rather technical to demonstrate.

Uniqueness of the mapping Φ_f : It is clear that the jump points set $J(f)$ and the jumps family $\mathcal{J}(f)$ are a necessary information (though we could reduce the jumps family to the set of jumps heights for pure jumps functions). By considering a function with a finite number of jumps, one could wonder if the mapping Φ_f is really necessary in the characterization $(J(f), \mathcal{J}(f), \Phi_f)$ because there is a natural order on $J(f)$ and $\mathcal{J}(f)$. For finite sets the mapping is unnecessary: jump points and jumps can be numbered in increasing order and the mapping is unique. For an infinite jump points set like $J(f) = \mathbb{Z}$, there is a unique shift for the mapping: once the mapping of a jump point is defined, all other can be easily derived by recurrence.

Finally we can build a jump points set and a jumps set such that a single shift is not sufficient to describe the difference between valid mappings. The idea is the following. Consider the jump points $x_n = 2^{n-1}$ for $n \leq 0$ and $x_n = 1 - 2^{-n-1}$ for $n \geq 0$. Jumps are $O_n = (x_n, x_{n+1})$. For any $k \in \mathbb{Z}$, the mappings $x_n \to O_{n+k}$ are valid. Now, the jump points (and jumps) lies within 0 and 1 and we can extend the jump points set (and similarly the jumps family) by adding all jump points $a + x_n$ with corresponding jumps $(a+x_n, a+x_{n+1})$ for any given non null integer a and all $n \in \mathbb{Z}$. We see that we would need several shifts k_a, possibly an infinity. It would be even worse for a dense $J(f)$. This is why the mapping is part of the characterization of the jumps.

♦♦♦

2.2 Flat sections

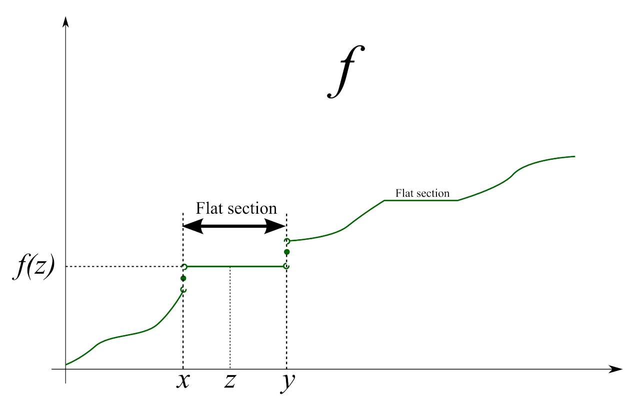

Figure 2: Flat sections of an increasing function and the related concepts: interior (x, y), value of the flat section $f(z)$ and "free" value at the boundaries x and y.

Definition 2.4 (Flat sections) *We say f is flat at z if there exist $x_1 < z < x_2$ such that $f(x_1) = f(x_2) = f(z)$. The flat section at z is defined as the greatest open interval (x, y) containing z such that f is constant on (x, y); this constant value is the flat section value.*

The set of all flat sections is called flat sections family and denoted[2] *by $\mathcal{H}(f)$ and the set of flat sections values $H(f)$.*

The open interval defining a flat section exists and is unique. Consider a point z where f is flat and $E = f^{-1}(\{f(z)\}) \stackrel{\text{def}}{=} \{r \in \bar{\mathbb{R}} : f(r) = f(z)\}$. By definition E contains an open interval (x_1, x_2) so that the flat section cannot be empty. E is also convex so that it is an interval with bounds $x < y$, possibly infinite (i.e. $x = -\infty$ or $y = +\infty$). It always contains its interior, the open interval (x, y). By construction any point $u < x$ has value $f(u) < f(z)$ otherwise it would have value $f(z)$ and belong to E; and symmetrically any point $u > y$ has value $f(u) > f(z)$. Therefore the (x, y) exists, is unique and is the maximal open interval defining the flat section.

As for jumps, there is a natural order defined on the sets $H(f)$ and $\mathcal{H}(f)$. And we can define a one-to-one mapping Φ_f^* between $H(f)$ and $\mathcal{H}(f)$ as in (2.7). The flat sections are completely characterized by $(H(f), \mathcal{H}(f), \Phi_f^*)$. We will see this is dual to the characterization of jumps $(J(f), \mathcal{J}(f), \Phi_f)$ via the generalized inverse.

3 Generalized inverses

There are several ways to define generalized inverses. Let's begin with the most classical definitions following [3] and the notation of [4]:

$$f^{\wedge}(y) \stackrel{\text{def}}{=} \sup\{x \in \bar{\mathbb{R}} : f(x) < y\} \tag{3.1}$$

$$f^{\vee}(y) \stackrel{\text{def}}{=} \inf\{x \in \bar{\mathbb{R}} : f(x) > y\} \tag{3.2}$$

In [3] it is proved that:

$$f^{\wedge}(y) = \inf\{x \in \bar{\mathbb{R}} : f(x) \geq y\} \tag{3.3}$$

$$f^{\vee}(y) = \sup\{x \in \bar{\mathbb{R}} : f(x) \leq x\} \tag{3.4}$$

It is easy to see that both $f^{\wedge}$ and $f^{\vee}$ are increasing. Curiously [3] proves $f^{\vee}(y)$ is right-continuous but not that $f^{\wedge}(y)$ is left-continuous though the sketch of the proof is the same. Note that Equation (3.3) is the most classical definition of the generalized inverse; in probability it is linked to stopping times definition.

In [4] there is the following very interesting characterization.

Proposition 3.1 (Regularization) *Let f be an increasing function with generalized inverses $f^{\wedge}$ and $f^{\vee}$ defined in Equations (3.1) and (3.2). Then $f^{\wedge}(y) \leq f^{\vee}(y)$ and*

$$f^{\wedge}(y) = f^{\vee}(y) \quad \Longleftrightarrow \quad \operatorname{Card} f^{-1}(\{y\}) \leq 1.$$

[2]We follow [2] with the notation H for *horizontal* since the flat sections are horizontal in the graph. It seems more readable than $F(f)$.

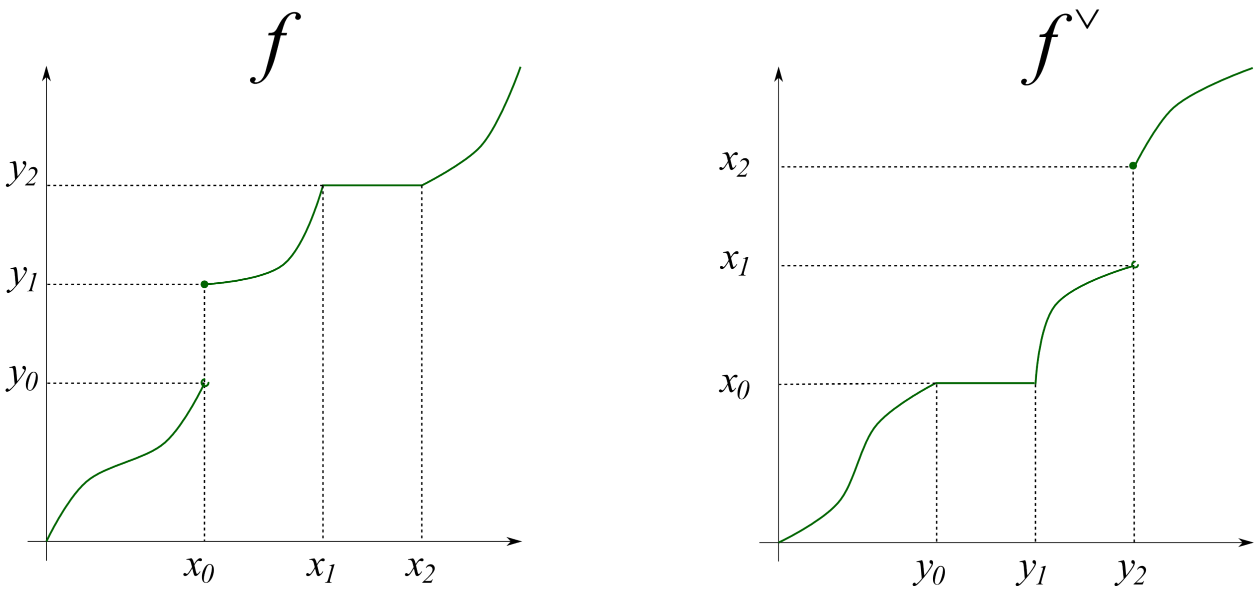

Figure 3: Generalized inverse function (the right-continuous one). Note here both functions are pseudo-inverse of each other since they are right-continuous. The jump of f at x_0 translates into a flat section of $f^\vee$ on $[y_0, y_1]$.

This means that $f^\wedge(y) = f^\vee(y)$ for all $y \notin H(f)$. Since $H(f)$ is at most countable, $f^\wedge = f^\vee$ almost everywhere. Furthermore, Proposition 2.3 implies that $f^\wedge$ is the minimal function in this class of almost everywhere equal increasing functions while $f^\vee$ is the maximal function. This leads to the definition:

Definition 3.2 (Generalized inverses) *Let f be an increasing function with generalized inverses $f^\wedge$ and $f^\vee$ defined in Equations (3.1) and (3.2). Then any increasing function f^* verifying*

$$f^\wedge \leq f^* \leq f^\vee$$

is a generalized inverse of f. These functions are increasing and almost everywhere equal. $f^\vee$ [resp. $f^\wedge$] is the unique right-continuous [resp. left-continuous] generalized inverse of f.

It is easy to see that any generalized inverse swaps the jumps and the flat sections (see also Figure 3) and the mappings are preserved:

$$J(f^*) = H(f), \qquad \mathcal{J}(f^*) = \mathcal{H}(f) \qquad \text{and} \quad \Phi_{f^*} = \Phi_f^* \tag{3.5}$$

$$H(f^*) = J(f), \qquad \mathcal{H}(f^*) = \mathcal{J}(f) \qquad \text{and} \quad \Phi_{f^*}^* = \Phi_f \tag{3.6}$$

This is easy to prove for $f^* = f^\vee$ (with the properties in the section below); and since it is a property of the class, it holds for all generalized inverses.

The problem of generalized inverses is closely related to the problem of quasi-inverses f^* defined as the increasing functions verifying

$$f \circ f^* \circ f = f \tag{3.7}$$

Klement et Al [4] prove there always exists quasi-inverses of an increasing function and the quasi-inverses are generalized inverses. However a generalized inverse (even regular) is *not* a quasi-inverse in general. E.g. for a right-continuous function f, a quasi-inverse would be the left-continuous generalized inverse $f^\wedge$ but not always the right-continuous $f^\vee$ ([4] demonstrates why). Take for example $f(x) = \lfloor x \rfloor$; then $f^\vee(x) = 1 + \lfloor x \rfloor$ so that $f \circ f^\vee \circ f = f + 1$. This study concentrates on the inversion as a transformation within the set of increasing functions, therefore we will not bring more attention to quasi-inverses (except for Proposition 4.3).

4 A regular generalized inverse

As we have seen above, there are lots of generalized inverses (provided there are flat sections). Moreover they are equal almost everywhere. Therefore we concentrate now on a particular representative of the class: the right-continuous generalized inverse $f^\vee$. We now come back to the usual inverse notation to simplify the reading, being aware of the difference between $f^{-1}(x) = f^\vee(x)$ and $f^{-1}(\{x\}) = \{y \in \bar{\mathbb{R}} : f(y) = x\}$. To emphasize this change we formally define again the generalized inverse. This definition (with singular in "inverse") replaces Definition 3.2.

Definition 4.1 (Generalized inverse) *Let f be an increasing function. The generalized inverse is function f^{-1} defined by*

$$f^{-1}(y) = \inf\{x \in \bar{\mathbb{R}} : f(x) > y\} \tag{4.1}$$

4.1 General properties

The pseudo-inverse has some interesting properties. Lots of them are already demonstrated bu we give a fairly exhaustive list hereafter for the sake of completeness.

Proposition 4.2 (pseudo-inverse properties) *The pseudo-inverse f^{-1} of an increasing function f has following properties:*

1. *f^{-1} is increasing, has left limits and is right continuous (càdlàg);*

2. *the following implications hold for all $x \in \bar{\mathbb{R}}$ and $y \in \bar{\mathbb{R}}$:*

$$f(x) > y \implies x \geq f^{-1}(y), \tag{4.2}$$
$$f(x) = y \implies x \leq f^{-1}(y), \tag{4.3}$$
$$f(x) > y \implies x \leq f^{-1}(y), \tag{4.4}$$
$$f^{-1}(y) > x \implies y \geq f(x), \tag{4.5}$$
$$f^{-1}(y) < x \implies y < f(x); \tag{4.6}$$

3. *for all $x \in \bar{\mathbb{R}}$, $f^{-1}(f(x)) \geq x$;*

4. if f is right continuous at x, then for all $y \in \bar{\mathbb{R}}$

$$\begin{aligned} f^{-1}(y) = x \quad &\Longrightarrow \quad y \le f(x), && (4.7)\\ f(x) > y \quad &\Longleftrightarrow \quad x > f^{-1}(y), && (4.8)\\ f(f^{-1}(y)) \ge y; & && (4.9) \end{aligned}$$

5. f is strictly increasing on $\bar{\mathbb{R}}$ if, and only if, f^{-1} is continuous on $\bar{\mathbb{R}}$;

6. f is continuous on $\bar{\mathbb{R}}$ if, and only if, f^{-1} is strictly increasing on $\bar{\mathbb{R}}$;

7. f is right-continuous if, and only if, $(f^{-1})^{-1} = f$;

8. f is right-continuous if, anf only if, $\{x \in \bar{\mathbb{R}} : f(x) \ge y\}$ is closed for all $y \in \bar{\mathbb{R}}$.

Proof : f^{-1} is increasing since the sets $E_y \overset{\text{def}}{=} \{x \in \bar{\mathbb{R}} : f(x) > y\}$ are decreasing. Left and right limits always exist for increasing functions. Since the inequality in Equation (4.1) is strict:

$$\bigcup_{z>y} E_z = \{x \in \bar{\mathbb{R}} : f(x) > z > y\} = \{x \in \bar{\mathbb{R}} : f(x) > y\} = E_y$$

and we get

$$f^{-1}(y^+) \overset{\text{def}}{=} \inf_{z>y} f^{-1}(z) = \inf_{z>y} \inf E_z = \inf \bigcup_{z>y} E_z = \inf E_y = f^{-1}(y)$$

so that f^{-1} is *always* right continuous (there is no condition on f).

The implication (4.2) is a direct consequence of the definition and Equation (4.1). The implication (4.3) and (4.4) can be demonstrated together; assume $f(x) \ge y$. Then we have $f(z) > y \implies f(z) > f(x) \implies z > x$ or equivalently

$$\{z \in \bar{\mathbb{R}} : f(z) > y\} \subset \{z \in \bar{\mathbb{R}} : f(z) > f(x)\} \subset \{z \in \bar{\mathbb{R}} : z > x\}$$

and taking the infimum $f^{-1}(y) \ge x$. The implication (4.5) is the contrapositive of implication (4.2) and (4.6) the contrapositive of the aggregation of (4.3) and (4.4) gathered so that there is only $f(x) \ge y$ in the left part of the implication. Therefore the second point is demonstrated.

The third point is a direct consequence of implication (4.3). The special case $f^{-1}(y) = \infty$ is obvious since by completion $f(\infty) = y$ so that there is even equality. Note that this inequality can be strict when f is constant on an open interval.

In the fourth point we assume f is right-continuous in x i.e. $f(x) = \inf_{z>x} f(z)$. Take $f^{-1}(y) = x$. From the definition of $f^{-1}(y)$, for any $\varepsilon > 0$ there exists z such that $f(z) > y$ and $z \le f^{-1}(y) + \varepsilon = x + \varepsilon$. For any $\eta > 0$, since $f(x) = \inf_{z>x} f(z)$, we can chose $\varepsilon > 0$ such that $f(z) \le f(x) + \eta$. Therefore $f(x) \ge f(z) - \eta > y - \eta$. Because this inequality is true for all $\eta > 0$ we get $f(x) \ge y$ and implication (4.7) is demonstrated.

Combining this implication with implication (4.6) we get $f^{-1}(y) \leq x \implies y \leq f(x)$ whose contrapositive is $f(x) > y \implies x < f^{-1}(y)$. Combining with implication (4.6), we get the equivalence (4.8). Inequality (4.9) is a straightforward consequence of implication (4.7), similar to the demonstration of fourth point.

Let demonstrate the fifth point by contrapositive. Assume f is discontinous at point x. Then $y_1 \stackrel{\text{def}}{=} f(x-) < y_2 \stackrel{\text{def}}{=} f(x+)$ and $y_0 \stackrel{\text{def}}{=} f(x) \in [y_1, y2]$. Either (y_1, y_0) or (y_0, y_2) is non-empty so that we can find $y_3 < y_4$ such that $[y_3, y_4]$ lies in one of these sets. Therefore there is no $z \in \bar{\mathbb{R}}$ such that $f(z) \in (y_3, y_4]$ and it is straightforward to see that $f^{-1}(y_3) = f^{-1}(y_4)$. Hence f^{-1} is not strictly increasing. Assume now that f^{-1} is not strictly increasing. This means there exists $x \in \bar{\mathbb{R}}$ and $y_1 < y_2$ such that $f^{-1}(y) = x$ for all $y \in [y_1, y_2]$. Take $\varepsilon > 0$; by implication (4.5), $f(x-\varepsilon) \leq f(x) = y_1$; by implication (4.6), $f(x) = y_2 \leq f(x+\varepsilon)$; hence $f(x-) \leq y_1 < y_2 \leq f(x+)$ and f is discontinuous.

The proof of the sixth statement is very similar to the proof of the fifth statement and is left to the reader. It is obvious if f is right-continuous following the next statement.

The seventh statement is easy in the reverse direction: if f is a pseudo-inverse, by statement 1 it is right-continuous. Now assume f is right-contiunous. Take $x \in \bar{\mathbb{R}}$ and $y \stackrel{\text{def}}{=} (f^{-1})^{-1}(x)$. By definition of the pseudo-inverse (equation 4.1), for all $\varepsilon_1 > 0$

$$f^{-1}(y + \varepsilon_1) \stackrel{\text{def}}{=} x + \eta_1 > x \tag{4.10}$$

$$f^{-1}(y - \varepsilon_1) \leq x \tag{4.11}$$

Therefore $\eta_1 > 0$. Applying the equivalent (for f^{-1}) of inequality (4.11) to the equality (4.10), yield, for any $\varepsilon_2 > 0$:

$$f(x + \eta_1 - \varepsilon_2) \leq y + \varepsilon_1$$

Taking $\varepsilon_2 \stackrel{\text{def}}{=} \eta_1/2 > 0$ yields $y \geq f(x+\eta_1/2)-\varepsilon_1 \geq f(x)-\varepsilon_1$. Since this is true for any $\varepsilon_1 > 0$, we get $y \geq f(x)$. Now, because f is right-continuous, we can combine implications (4.6) and (4.7) to the inequality (4.11) which gives $y - \varepsilon_1 \leq f(x)$ hence $y \leq f(x)$. Finally we proved $y = f(x)$ and the seventh statement is demonstrated.

For the eighth statement, first we see that if $z \in \{x \in \bar{\mathbb{R}} : f(x) \geq y\}$, then $[z, \infty] \in \{x \in \bar{\mathbb{R}} : f(x) \geq y\}$. Therefore the set is a segment that is either $[x_0, \infty]$ or $(x_0, \infty]$. The first one is closed and the second one not. In any case, $f(x_0+) \geq y$. If f is right continuous, $f(x_0+) = f(x_0) \geq y$ hence $x_0 \in \{x \in \bar{\mathbb{R}} : f(x) \geq y\}$ and it is closed. Assume f is not right continuous: then there exists x_0 such that $f(x_0) < (f(x_0+)$ and $x_0 \notin \{x \in \bar{\mathbb{R}} : f(x) \geq y\}$ that is not closed. Hence the eighth statement is proved. ■

4.2 Jumps and flat sections

This proposition recalls some earlier results and adds an interesting equivalence.

Proposition 4.3 *The pseudo-inverse f^{-1} of an increasing and right-continuous function f swaps jumps and flat sections as in Equations (3.5)-(3.5). Moreover*

1. *$f^{-1}(f(x)) = x$ if, and only if, $f(x) \notin H(f)$. Otherwise $f^{-1}(f(x)) > x$;*

2. *$f(f^{-1}(y)) = y$ if, and only if, $y \notin J(f)$. Otherwise $f(f^{-1}(y)) > y$;*

Proof : The proof of property 1. is an adaptation of Lemma 2 of [7] that proves a similar result for the left-continuous generalized inverse. And the second property derives from the first one by the property 7. of Proposition 4.2. ■

The first item of Proposition 4.3 implies that f is bijective if, and only if, $H(f) = \emptyset$. Now it is a bijection between $\bar{\mathbb{R}}$ and the range of f. It is only injective as a function into $\bar{\mathbb{R}}$. The second item deals with the surjective property. Hence there is a true bijection if, and only if, $J(f) = H(f) = \emptyset$. However this statement is not precise enough since there are technicalities at infinity (we must impose $\lim_{-\infty} f = -\infty$ and $\lim_{\infty} f = \infty$ and consider jumps at infinity). What holds for sure if $J(f) = H(f) = \emptyset$ is that f and f^{-1} are continuous and strictly increasing.

5 Conclusion

This study shows deep connections between several characteristics of increasing functions when applying the generalized inverse: jumps and flat section, right-continuity, almost everywhere equal functions... This study is the first part of a larger effort to understand these connnections, considering only the generalized inverse of a single function.

When applying the generalized inverse as a *transform* of the set of càdlàg increasing functions (right-continuous with left limits), more questions appear naturally. This transform is an involution: f and f^{-1} are the transform of each other. But the transform is stable on special subsets of functions: this leads to an extended Lebesgue decomposition. A second direction of research is the continuity property of this transform. And these two considerations give space to precise statements about time changes and the operations that can be realized with time changes. Therefore we believe the precise study of increasing function is worth some more work.

References

[1] Embrechts, P., and Hofert, M. A note on generalized inverses. *Mathematical Methods of Operations Research 77*, 3 (2013), 423–432.

[2] Falkner, N., and Teschl, G. On the substitution rule for lebesgue-stieltjes integrals. *Expositiones Mathematicae 30*, 4 (2012), 412 – 418.

[3] Feng, C., Wang, H., Tu, X. M., and Kowalski, J. A note on generalized inverses of distribution function and quantile transformation. *Applied Mathematics 3* (December 2012), 2098–2100.

[4] Klement, E. P., Mesiar, R., and Pap, E. Quasi- and pseudo-inverses of monotone functions, and the construction of t-norms. *Fuzzy Sets and Systems 104*, 1 (1999), 3 – 13. Triangular Norms.

[5] Lebesgue, H. *Leçons sur l'intégration et la recherche des fonctions primitives, 2e éd., 1928.* Gauthier-Villars, 1928.

[6] Schweizer, B., and Sklar, A. *Probabilistic Metric Spaces.* Elsevier Science, New York, 1983.

[7] Viceník, P. A note to a construction of t-norms based on pseudo-inverses of monotone functions. *Fuzzy Sets and Systems 104*, 1 (1999), 15 – 18. Triangular Norms.