Minimum time control of heterodirectional linear coupled hyperbolic PDEs with controls in both sides

Jean Auriol, Florent Di Meglio

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Jean Auriol[1] and Florent Di Meglio[2]

I. Introduction

This article solves the problem of boundary stabilization of two coupled heterodirectional linear first-order hyperbolic Partial Differential Equations (PDEs) in minimum case with one PDE in each direction and with actuation applied on both boundaries.

The main contribution of this paper is a minimum time stabilizing controller. More precisely a proposed boundary feedback law ensures finite-time convergence of the two states to zero in minimum time. The minimum time defined [5] in is the largest time between the two transport times in each direction.

Our approach is the following. Using a Fredholm transformation, the system is mapped to a *target system* with desirable stability properties. This target system is a copy of the original dynamics from which the coupling terms are removed. The well-posedness of the Fredholm transformation is a consequence of a clever choice of the domain on which the kernels are defined. The proof of the invertibility of this transformation is non-trivial and uses an operator-approach inspired by the one developed in [2].

The paper is organized as follow. In Section II we introduce the model equations and the notations. In Section III we present the stabilization result: the target system and its properties are presented in Section III-A. In Section III-B we derive the integral transformation and we present the domains on which the kernels are defined. Some arguments about the well-posedness of the kernels are given in Section III-C. Section IV contains the proof of the invertibility of the Fredholm transformation. In Section V we present the control feedback law and its properties. Finally in Section VI we give some simulation results.

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II. Problem Description

A. System under consideration

We consider the following general linear hyperbolic system

$$u_t(t,x) + \lambda u_x(t,x) = \sigma^{+-} v(t,x) \quad (1)$$
$$v_t(t,x) - \mu v_x(t,x) = \sigma^{-+} u(t,x) \quad (2)$$

with the following linear boundary conditions

$$u(t,0) = U(t), \quad v(t,1) = V(t) \quad (3)$$

with constant coupling terms and constant speeds

$$0 < \lambda \leq \mu \quad (4)$$

The initial conditions are defined by

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x) \quad (5)$$

Remark 1: The coupling terms are assumed constant here but the results of this paper can be adjusted for spatially-varying coupling terms.

Remark 2: System (1)-(2) is equivalent to the following system

$$u_t(t,x) + \lambda u_x(t,x) = \sigma^{++} u(t,x) + \sigma^{+-} v(t,x) \quad (6)$$
$$v_t(t,x) - \mu v_x(t,x) = \sigma^{-+} u(t,x) + \sigma^{--} v(t,x) \quad (7)$$

This can straightforwardly be proved using an exponential transformation.

Remark 3: The assumption $\lambda \leq \mu$ can be done without any loss of generality

B. Well-posedness

Taking the scalar product of (1)-(2) with a smooth test function $\Phi^T = (\phi_1, \phi_2)$ and integrating by parts leads to the following definition of a solution.

Definition 1: Consider system (1)-(2) with initial conditions $u^0, v^0 \in L^2$ and control laws $U(t)$ and $V(t)$. We say that $\begin{pmatrix} u \\ v \end{pmatrix}$ is a (weak) solution if for every $\tau \geq 0$ and every function $\Phi = (\phi_1, \phi_2)^T \in (C^1([0,\tau] \times [0,1]))^2$ such that

$\phi_1(\cdot,1)=\phi_1(\cdot,0)=0$ we have

$$\begin{aligned} 0 =& \int_0^\tau \int_0^1 -(\phi_{1_t}(t,x)+\lambda\phi_{1_x}(t,x)+ \\ & \sigma^{-+}\phi_2(t,x))u(t,x)-(\phi_{2_t}(t,x) \\ & -\mu\phi_{2_x}(t,x)+\sigma^{+-}\phi_1(t,x))v(t,x)dxdt \\ & +\int_0^1 (u(\tau,x)\phi_1(\tau,x)-u(0,x)\phi_1(0,x) \\ & +v(\tau,x)\phi_2(\tau,x)-v(0,x)\phi_2(0,x))dx \\ & -\int_0^\tau \left[\lambda U(t)\phi_1(t,0)+\mu V(t)\phi_2(t,1)\right]dt \end{aligned} \tag{8}$$

We can consequently rewrite the system in the abstract form

$$\frac{d}{dt}\begin{pmatrix}u\\v\end{pmatrix} = A\begin{pmatrix}u\\v\end{pmatrix} + B\begin{pmatrix}U\\V\end{pmatrix} \tag{9}$$

where the operators A and B can be identified through their adjoints. The operator A is thus defined by

$$\begin{aligned} A : & D(A)\subset (L^2(0,1))^2 \to (L^2(0,1))^2 \\ & \begin{pmatrix}u\\v\end{pmatrix} \longmapsto \begin{pmatrix}-\lambda u_x+\sigma^{+-}v\\ \mu v_x+\sigma^{-+}u\end{pmatrix} \end{aligned} \tag{10}$$

with

$$D(A)=\{(u,v)\in (H^1(0,1))^2 | u(0)=v(1)=0\} \tag{11}$$

A is well defined and its adjoint A^* is

$$\begin{aligned} A^* : & D(A^*)\subset (L^2(0,1))^2 \to (L^2(0,1))^2 \\ & \begin{pmatrix}u\\v\end{pmatrix} \longmapsto \begin{pmatrix}\lambda u_x+\sigma^{-+}v\\ -\mu v_x+\sigma^{+-}u\end{pmatrix} \end{aligned} \tag{12}$$

with

$$D(A^*)=\{(u,v)\in (H^1(0,1))^2 | u(1)=v(0)=0\} \tag{13}$$

The operator B is defined by

$$< B\begin{pmatrix}U\\V\end{pmatrix}, \begin{pmatrix}z_1\\z_2\end{pmatrix} >= Uz_1(0)+Vz_2(1) \tag{14}$$

Its adjoint is

$$B^*\begin{pmatrix}z_1\\z_2\end{pmatrix} = \begin{pmatrix}z_1(0)\\z_2(1)\end{pmatrix} \tag{15}$$

C. Control problem

The goal is to design feedback control inputs $U(t)$ and $V(t)$ such that the zero equilibrium is reached in minimum time $t=t_F$, where

$$t_F = \min\left\{\frac{1}{\mu},\frac{1}{\lambda}\right\} \tag{16}$$

III. Control design

The control design is based on a modified backstepping approach: using a specific transformation, we map the system (1)-(3) to a target system with desirable properties of stability. However, unlike the classical backstepping approach where a Volterra transformation is used, we use a Fredholm transformation here.

A. Target system design

We map the system (1)-(3) to the following system

$$\alpha_t(t,x)+\lambda\alpha_x(t,x)=0 \tag{17}$$

$$\beta_t(t,x)-\mu\beta_x(t,x)=0 \tag{18}$$

with the following boundary conditions

$$\alpha(t,0)=0 \quad \beta(t,1)=0 \tag{19}$$

This system is designed as a copy of the original dynamics, from which the coupling terms of (1)-(2) are completely removed.

Lemma 1: The zero equilibrium of (17),(18) with boundary conditions (19) and initial conditions $(\alpha^0,\beta^0)\in\mathcal{L}^2([0,1])$ is exponentially stable in the $\mathcal{L}^2$ sense.

Proof: The proof, using a Lyapunov function, is quite classical and is omitted here. ■

Besides, the following lemma assesses the finite-time convergence of the target system.

Lemma 2: The system (17),(18) reaches its zero equilibrium in finite-time $t_F=\max\{\frac{1}{\lambda},\frac{1}{\mu}\}$

Proof: The proof of this lemma is quite straightforward and uses the same arguments than the proof of [3, Lemma 3.1] ■

Using operators, we can rewrite system (17)-(18) as

$$\frac{d}{dt}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = A_0\begin{pmatrix}\alpha\\\beta\end{pmatrix} \tag{20}$$

The operator A_0 is defined by

$$\begin{aligned} A_0 : & D(A_0)\subset (L^2(0,1))^2 \to (L^2(0,1))^2 \\ & \begin{pmatrix}\alpha\\\beta\end{pmatrix} \longmapsto \begin{pmatrix}-\lambda\alpha_x\\ \mu\beta_x\end{pmatrix} \end{aligned} \tag{21}$$

with

$$D(A_0)=\{(\alpha,\beta)\in (H^1(0,1))^2 | \alpha(0)=\beta(1)=0\} \tag{22}$$

A_0 is well defined and its adjoint A_0^* is

$$\begin{aligned} A_0^* : & D(A_0^*)\subset (L^2(0,1))^2 \to (L^2(0,1))^2 \\ & \begin{pmatrix}\alpha\\\beta\end{pmatrix} \longmapsto \begin{pmatrix}\lambda\alpha_x\\ -\mu\beta_x\end{pmatrix} \end{aligned} \tag{23}$$

with

$$D(A_0^*)=\{(\alpha,\beta)\in (H^1(0,1))^2 | \alpha(1)=\beta(0)=0\} \tag{24}$$

B. Fredholm transformation

1) Definition of the transformation: Without any loss of generality we recall that $\lambda\leq\mu$. In order to map the original system (1)-(3) to the target system (17)-(19), we use the following transformation

$$\alpha(t,x) = u(t,x)$$
$$+ \sum_{i=1}^{D} h_{J_i}(x) \int_x^{-\frac{\mu}{\lambda}x+b_i} (K^i(x,\xi)u(t,\xi) + L^i(x,\xi)v(t,\xi))d\xi$$
$$+ h_{[\frac{\mu}{\mu+\lambda},1]}(x) \int_{\frac{\mu}{\lambda}(1-x)}^{x} (M(x,\xi)u(t,\xi) + N(x,\xi)v(t,\xi))d\xi \tag{25}$$

$$\beta(t,x) = v(t,x)$$
$$+ h_{[0,\frac{\lambda}{\mu+\lambda}[}(x) \int_x^{\frac{\lambda}{\mu}(1-x)} (\bar{K}(x,\xi)u(t,\xi) + \bar{L}(x,\xi)v(t,\xi))d\xi$$
$$+ h_{[\frac{\lambda}{\mu+\lambda},1]}(x) \int_{\frac{\lambda}{\mu}(1-x)}^{x} (\bar{M}(x,\xi)u(t,\xi) + \bar{N}(x,\xi)v(t,\xi))d\xi \tag{26}$$

where $h_I(x)$ (I is an interval) is defined by

$$h_I(x) = \begin{cases} 1 & \text{if} \quad x \in I \\ 0 & \text{else} \end{cases} \tag{27}$$

The sequence b_i with $i \in N$ is defined by

$$b_{n+1} = (1 + \frac{\mu}{\lambda})a_n \tag{28}$$
$$a_n = \frac{\lambda}{\mu}(b_n - 1) \tag{29}$$
$$b_1 = \frac{\mu}{\lambda} \tag{30}$$

The parameter D is defined as follow

$$D = \min\{n | a_n < 0\} \tag{31}$$

One can readily prove that D is well defined.
Finally the sequence J_i is defined by

$$J_i = [\max\{a_i, 0\}, a_{i-1}[\tag{32}$$

with the convention $a_0 = \frac{\mu}{\lambda+\mu}$. All the J_i are disjoint and are a partition of $[0, \frac{\mu}{\lambda+\mu}[$. We define the following domains:

$$\mathcal{T}_0 = \{(x,\xi) | x \in [\frac{\mu}{\lambda+\mu}, 1], \quad \frac{\mu}{\lambda}(1-x) < \xi \leq x\} \tag{33}$$

$$\mathcal{T}_i = \{(x,\xi) | x \in J_i, \quad x \leq \xi < \frac{\mu}{\lambda}x + b_i\} \tag{34}$$

$$\bar{\mathcal{T}}_0 = \{(x,\xi) | x \in [0, \frac{\lambda}{\lambda+\mu}[, \quad x \leq \xi < \frac{\lambda}{\mu}(1-x)\} \tag{35}$$

$$\bar{\mathcal{T}}_1 = \{(x,\xi) | x \in [\frac{\lambda}{\lambda+\mu}, 1], \quad \frac{\lambda}{\mu}(1-x) < \xi \leq x\} \tag{36}$$

The kernels K^i, L^i are defined on $\mathcal{T}_i$, M, N are defined on $\mathcal{T}_0$. The kernels $\bar{K}, \bar{L}$ are defined on $\bar{\mathcal{T}}_0$ and $\bar{M}, \bar{N}$ defined on $\bar{\mathcal{T}}_1$. They all have yet to be defined.

Remark 4: This transformation is a Fredholm transformation and can be rewritten using integrals between 0 and 1:

$$\alpha(t,x) = u(t,x) + \int_0^1 Q_{11}(x,\xi)u(t,\xi) + Q_{12}(x,\xi)v(t,\xi)d\xi \tag{37}$$

$$\beta(t,x) = v(t,x) + \int_0^1 Q_{21}(x,\xi)u(t,\xi) + Q_{22}(x,\xi)v(t,\xi)d\xi \tag{38}$$

with

$$Q_{11}(x,\xi) = \sum_{i=1}^{D} K^i(x,\xi)h_{[x,-\frac{\mu}{\lambda}x+b_i]}(\xi)h_{J_i}(x) + M(x,\xi)h_{[\frac{\mu}{\lambda}(1-x),x]}(\xi)h_{[\frac{\mu}{\lambda+\mu},1]}(x) \tag{39}$$

$$Q_{12}(x,\xi) = \sum_{i=1}^{D} L^i(x,\xi)h_{[x,-\frac{\mu}{\lambda}x+b_i]}(\xi)h_{J_i}(x) + N(x,\xi)h_{[\frac{\mu}{\lambda}(1-x),x]}(\xi)h_{[\frac{\mu}{\lambda+\mu},1]}(x) \tag{40}$$

$$Q_{21}(x,\xi) = \bar{K}(x,\xi)h_{[x,\frac{\lambda}{\mu}(1-x)]}(\xi)h_{[0,\frac{\lambda}{\lambda+\mu}[}(x) + \bar{M}(x,\xi)h_{[\frac{\lambda}{\mu}(1-x),x]}(\xi)h_{[\frac{\lambda}{\lambda+\mu},1]}(x) \tag{41}$$

$$Q_{22}(x,\xi) = \bar{L}(x,\xi)h_{[x,\frac{\lambda}{\mu}(1-x)]}(\xi)h_{[0,\frac{\lambda}{\lambda+\mu}[}(x) + \bar{N}(x,\xi)h_{[\frac{\lambda}{\mu}(1-x),x]}(\xi)h_{[\frac{\lambda}{\lambda+\mu},1]}(x) \tag{42}$$

Remark 5: Since $\alpha(0) = \beta(1) = 0$ the two control laws U and V can be computed as functions of (u, v).

2) Operator formulations and properties: In this subsection we rewrite the previous Fredholm transformation using operators. This will lead to some relations verified by the adjoint operators.
The Fredholm transformation (37)-(38) can be written as an operator P acting on $\begin{pmatrix} u \\ v \end{pmatrix}$. More precisely we have

$$P = Id_2 - Q \tag{43}$$
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix} \tag{44}$$

where $Q : (L^2(0,1))^2 \to (L^2(0,1))^2$ is the integral operator defined by

$$Q \begin{pmatrix} u \\ v \end{pmatrix} = \int_0^1 \begin{pmatrix} Q_{11}(x,\xi)u(t,\xi) + Q_{12}(x,\xi)v(t,\xi) \\ Q_{21}(x,\xi)u(t,\xi) + Q_{22}(x,\xi)v(t,\xi) \end{pmatrix} d\xi \tag{45}$$

Its adjoint is:

$$Q^* \begin{pmatrix} u \\ v \end{pmatrix} = \int_0^1 \begin{pmatrix} Q_{11}(\xi,x)u(t,\xi) + Q_{21}(\xi,x)v(t,\xi) \\ Q_{12}(\xi,x)u(t,\xi) + Q_{22}(\xi,x)v(t,\xi) \end{pmatrix} d\xi \tag{46}$$

One can easily check that:

$$K^*(D(A^*)) \subset D(A^*) \tag{47}$$

The control $\begin{pmatrix} U \\ V \end{pmatrix}$ can also be rewritten using operators

$$\begin{pmatrix} U \\ V \end{pmatrix} = \Gamma \begin{pmatrix} u \\ v \end{pmatrix} \tag{48}$$

with

$$\Gamma \begin{pmatrix} u \\ v \end{pmatrix} = \int_0^1 \begin{pmatrix} Q_{11}(0,\xi)u(t,\xi) + Q_{12}(0,\xi)v(t,\xi) \\ Q_{21}(1,\xi)u(t,\xi) + Q_{22}(1,\xi)v(t,\xi) \end{pmatrix} d\xi \tag{49}$$

Using (20) and (44) yields

$$\frac{d}{dt}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A_0 P \begin{pmatrix} u \\ v \end{pmatrix} \tag{50}$$

Moreover using (9) and (44) we get

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \frac{d}{dt}(P\begin{pmatrix} u \\ v \end{pmatrix}) \\ &= PA\begin{pmatrix} u \\ v \end{pmatrix} + PB\Gamma\begin{pmatrix} u \\ v \end{pmatrix} \end{aligned} \tag{51}$$

Consequently P and Γ satisfy the following relation:

$$A_0 P = PA + PB\Gamma \tag{52}$$

Taking the adjoints, this is equivalent to

$$P^* A_0^* = A^* P^* + \Gamma^* B^* P^* \tag{53}$$

3) Kernel equations: We now differentiate the Fredholm transformation (25)-(26) with respect to time and space to compute the equations satisfied by the kernels.

<u>if $x \geq \frac{\lambda}{\mu+\lambda}$:</u> Differentiating (26) with respect to space and using the Leibniz rule yields

$$\begin{aligned} \beta_x(t,x) &= v_x(t,x) + \bar{M}(x,x)u(t,x) + \bar{N}(x,x)v(t,x) \\ &\quad + \frac{\lambda}{\mu}\bar{M}(x,\frac{\lambda}{\mu}(1-x))u(t,\frac{\lambda}{\mu}(1-x)) \\ &\quad + \frac{\lambda}{\mu}\bar{N}(x,\frac{\lambda}{\mu}(1-x))v(t,\frac{\lambda}{\mu}(1-x)) \\ &\quad + \int_{\frac{\lambda}{\mu}(1-x)}^x \bar{M}_x(x,\xi)u(t,\xi) + \bar{N}_x(x,\xi)v(t,\xi)d\xi \end{aligned} \tag{54}$$

Differentiating (26) with respect to time, using (1), (2) and integrating by parts yields

$$\begin{aligned} &\beta_t(t,x) = \mu v_x(t,x) + \sigma^{-+}u(t,x) + \\ &\mu\bar{N}(x,x)v(t,x) - \mu\bar{N}(x,\frac{\lambda}{\mu}(1-x))v(t,\frac{\lambda}{\mu}(1-x)) \\ &- \lambda\bar{M}(x,x)u(t,x) + \lambda\bar{M}(x,\frac{\lambda}{\mu}(1-x))u(t,\frac{\lambda}{\mu}(1-x)) \\ &+ \int_{\frac{\lambda}{\mu}(1-x)}^x \lambda\bar{M}_\xi(x,\xi)u(t,\xi) - \mu\bar{N}_\xi(x,\xi)v(t,\xi) \\ &\quad + \sigma^{-+}\bar{N}(x,\xi)u(t,\xi) + \sigma^{+-}\bar{M}(x,\xi)v(t,\xi)d\xi \end{aligned} \tag{55}$$

Plugging these expressions into the target system (17)-(18) yields the following system of kernel equations

$$0 = -\mu\bar{M}_x(x,\xi) + \lambda\bar{M}_\xi(x,\xi) + \sigma^{-+}\bar{N}(x,\xi) \tag{56}$$

$$0 = -\mu\bar{N}_x(x,\xi) - \mu\bar{N}_\xi(x,\xi) + \sigma^{+-}\bar{M}(x,\xi) \tag{57}$$

$$0 = \bar{M}(x,x) - \frac{\sigma^{-+}}{\lambda+\mu} \tag{58}$$

$$0 = \bar{N}(x,\frac{\lambda}{\mu}(1-x)) \tag{59}$$

<u>if $x < \frac{\lambda}{\mu+\lambda}$:</u> Similarly we get

$$0 = -\mu\bar{K}_x(x,\xi) + \lambda\bar{K}_\xi(x,\xi) + \sigma^{-+}\bar{L}(x,\xi) \tag{60}$$

$$0 = -\mu\bar{L}_x(x,\xi) - \mu\bar{L}_\xi(x,\xi) + \sigma^{+-}\bar{K}(x,\xi) \tag{61}$$

$$0 = \bar{K}(x,x) + \frac{\sigma^{-+}}{\lambda+\mu} \tag{62}$$

$$0 = \bar{L}(x,\frac{\lambda}{\mu}(1-x)) \tag{63}$$

The corresponding domains, characteristic lines and boundary conditions in Figure 1 We now focus on the alpha-transformation

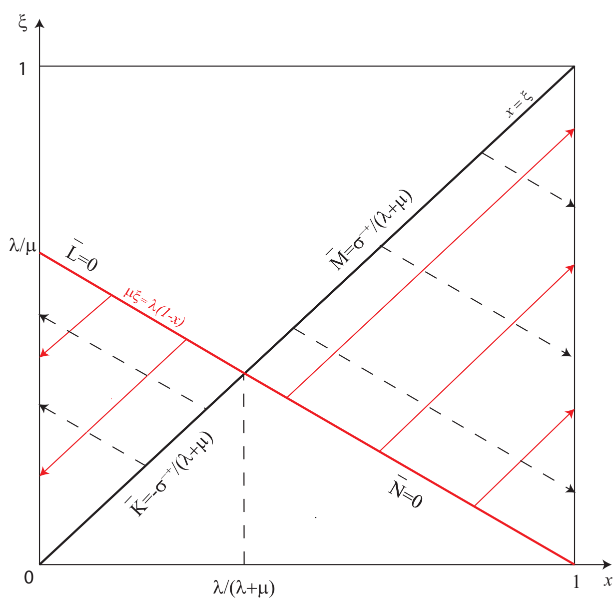

Fig. 1. Representation of the beta-kernels

<u>if $x \geq \frac{\mu}{\mu+\lambda}$:</u> As above, differentiating (25) with respect to space and time and then plugging into the target system (17)-(18) yields the following system of kernel equations

$$0 = \lambda N_x(x,\xi) - \mu N_\xi(x,\xi) + \sigma^{+-}M(x,\xi) \tag{64}$$

$$0 = \lambda M_x(x,\xi) + \lambda M_\xi(x,\xi) + \sigma^{-+}N(x,\xi) \tag{65}$$

$$0 = N(x,x) + \frac{\sigma^{+-}}{\lambda+\mu} \tag{66}$$

$$0 = M(x,\frac{\mu}{\lambda}(1-x)) \tag{67}$$

if $x \in J_i$ As above we get the following kernel equations

$$0 = \lambda L_x^i(x,\xi) - \mu L_\xi^i(x,\xi) + \sigma^{+-} K^i(x,\xi) \quad (68)$$

$$0 = \lambda K_x^i(x,\xi) + \lambda K_\xi^i(x,\xi) + \sigma^{-+} L^i(x,\xi) \quad (69)$$

$$0 = L^i(x,x) - \frac{\sigma^{+-}}{\lambda+\mu} \quad (70)$$

$$0 = K^i(x, -\frac{\mu}{\lambda}x + b_i) \quad (71)$$

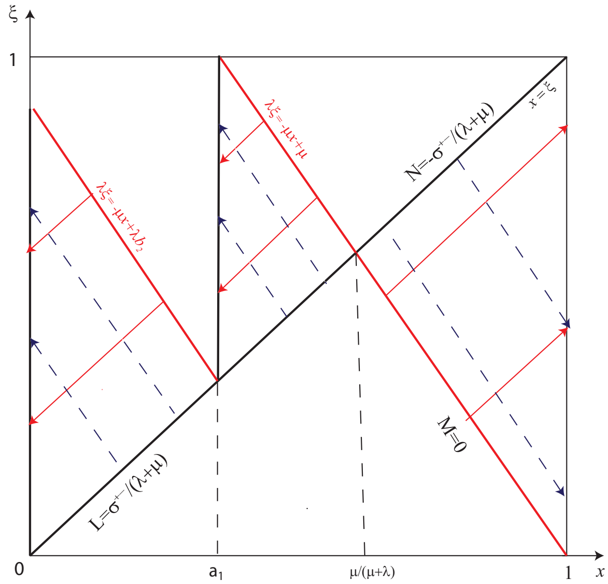

Fig. 2. Representation of the alpha-kernels

C. Well-posedness of the kernel equations

Theorem 1: Consider systems (56)-(59), (60)-(63), (64)-(67), (68)-(71). There exists a unique solution K_i, L_i (defined on $L^\infty(\mathcal{T}_i)$), M_i, N_i (defined on $L^\infty(\mathcal{T}_0)$), $\bar{K}, \bar{L}$ (defined on $L^\infty(\bar{\mathcal{T}}_0)$), $\bar{M}, \bar{N}$ (defined on $L^\infty(\bar{\mathcal{T}}_1)$)
Classically (see [3], [4] and [6]) the proof of this theorem consists in transforming the kernel equations into integral equations using the method of the characteristics. These integral equations are then solved using the method of successive approximations.

IV. Invertibility of the Fredholm Transformation

Unlike the Volterra transformation, the Fredholm transformation is not always invertible. In [2], the authors prove the invertibility of such a transformation in the case of a first-order integro-differential hyperbolic equation. In this section we use similar arguments to prove the invertibility of our transformation. We give first the following useful lemmas:

Lemma 3: ker $P^* \subset D(A_0^*) = D(A^*)$

Proof: Let us consider $z \in$ ker P^*. Consequently we have $P^* z = 0$. We can rewrite it

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \int_0^1 \begin{pmatrix} Q_{11}(\xi,x)z_1(t,\xi) + Q_{21}(\xi,x)z_2(t,\xi) \\ Q_{12}(\xi,x)z_1(t,\xi) + Q_{22}(\xi,x)z_2(t,\xi) \end{pmatrix} d\xi \quad (72)$$

If we evaluate the first line for $x = 1$ and the second one for $x = 0$, using the fact that $Q_{11}(\xi,0) = Q_{21}(\xi,0) = Q_{12}(\xi,1) = Q_{22}(\xi,1) = 0$, we get

$$z_1(1) = z_2(0) = 0 \quad (73)$$

Consequently $z \in D(A_0^*)$ and we can write

$$\text{ker} P^* \subset D(A_0^*) \quad (74)$$

■

Lemma 4: ker $P^* \subset$ ker B^*

Proof: Let us consider $z \in$ ker P^*. Consequently we have $P^* z = 0$. We can rewrite it

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \int_0^1 \begin{pmatrix} Q_{11}(\xi,x)z_1(t,\xi) + Q_{21}(\xi,x)z_2(t,\xi) \\ Q_{12}(\xi,x)z_1(t,\xi) + Q_{22}(\xi,x)z_2(t,\xi) \end{pmatrix} d\xi \quad (75)$$

If we evaluate the first line for $x = 0$ and the second one for $x = 1$, using the fact that $Q_{11}(\xi,1) = Q_{21}(\xi,1) = Q_{12}(\xi,0) = Q_{22}(\xi,0) = 0$, we get

$$z_1(0) = z_2(1) = 0 \quad (76)$$

Consequently $z \in$ ker B^* and we can write

$$\text{ker} P^* \subset \text{ker} B^* \quad (77)$$

■

Lemma 5: $\forall \lambda \in \Re$ ker$(\lambda Id_2 - A_0^*) \cap$ ker $B^* = \{0\}$

Proof: Let us consider $\lambda \in \Re$ and $z \in$ ker$(\lambda Id_2 - A_0^*) \cap$ ker $B^* = \{0\}$. Consequently we have

$$\begin{pmatrix} \lambda z_{1_x}(t,x) + \sigma^{++} z_1(t,x) - \lambda z_1(t,x) \\ -\mu z_{2_x}(t,x) + \sigma^{--} z_2(t,x) - \lambda z_2(t,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (78)$$

with the boundary conditions

$$z_1(0) = z_2(0) = 0 \quad (79)$$

Consequently we have $z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ■

We can now state the following theorem

Theorem 2: The map $P^* = Id_2 - Q^*$ is invertible

Proof: Since Q^* is a compact operator we can use the Fredholm alternative (e.g [1]): $Id_2 - Q^*$ is either non-injective or surjective. Consequently it suffices to prove that P^* is injective. In addition, the Fredholm alternative also gives [1]

$$\text{dim ker}(Id - Q^*) < +\infty \quad (80)$$

By contradiction we assume that ker $P^* \neq \{0\}$. We first prove that ker P^* is stable by A_0^*. We have ker $P^* \subset A_0^*$. Let then consider $z \in$ ker P^*. Using (53) we can obtain

$$P^* A_0^* z = 0 \tag{81}$$

We thus have $A_0^* Z \in$ ker P^*. Consequently the restriction $A^*_{0_{|\ker P^*}}$ of A_0^* to ker P^* is a linear operator from ker P^* to ker P^*. Since the dimension of ker P^* is finite we can find at least one eigenvalue λ. Let $e \in$ ker P^* be a corresponding eigenvector (by definition $e \neq 0$). We have $e \in$ ker P^* and so $e \in$ ker B^*. Moreover we have $A_0^* e = \lambda e$. Consequently

$$e \in \ker(\lambda - A_0^*) \cap \ker B^* \tag{82}$$

which contradicts Lemma 5 and concludes the proof. ■

V. Control Law and Main Results

We now state the main stabilization result as follows:

Theorem 3: System (1)-(2) with the following feedback control laws

$$U(t) = \int_0^{-\frac{\mu}{\lambda}x + b_D} (K^D(0,\xi)u(t,\xi) + L^D(0,\xi)v(t,\xi))d\xi \tag{83}$$

$$V(t) = \int_{\frac{\mu}{\lambda}(1-x)}^1 (\bar{M}(1,\xi)u(t,\xi) + \bar{N}(1,\xi)v(t,\xi))d\xi \tag{84}$$

reaches its zero equilibrium in finite time t_F, where t_F is given by (16). The zero equilibrium is exponentially stable in the L^2-sense.

Proof: Notice that evaluating (25) at $x = 0$ yields (83) and evaluating (26) at $x = 1$ yields (84). Since the kernels are invertible, there exists a unique function $\mathcal{S}$ such that

$$\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = \begin{pmatrix} \alpha(t,x) \\ \beta(t,x) \end{pmatrix} - \int_0^1 \mathcal{S}(x,\xi) \begin{pmatrix} \alpha(t,\xi) \\ \beta(t,\xi) \end{pmatrix} d\xi \tag{85}$$

Applying Lemma 2 implies that (α, β) go to zero in finite time t_F, therefore (u, v) converge to zero in finite time t_F ■

Remark 6: One can notice that the control laws use only four kernels. Consequently from a practical point of view it is not necessary to compute all the kernels.

VI. Simulation Results

In this section we illustrate our results with simulations on a tou problem. The numerical values of the parameters are as follow.

$$\lambda = 0.5, \quad \mu = 1, \quad \sigma^{+-} = 0.5, \quad \sigma^{-+} = 1 \tag{86}$$

Figure 3 pictures the $\mathcal{L}^2$-norm of the state (u, v) in open loop and using the control law (83)-(84) presented in this paper. While the system in open loop is unstable the ($\mathcal{L}^2$-norm diverges), it converges in minimum time $t_F = \max\{\frac{1}{\lambda}, \frac{1}{\mu}\} = 2$ when controller (83)-(84) is applied as expected from Theorem 3

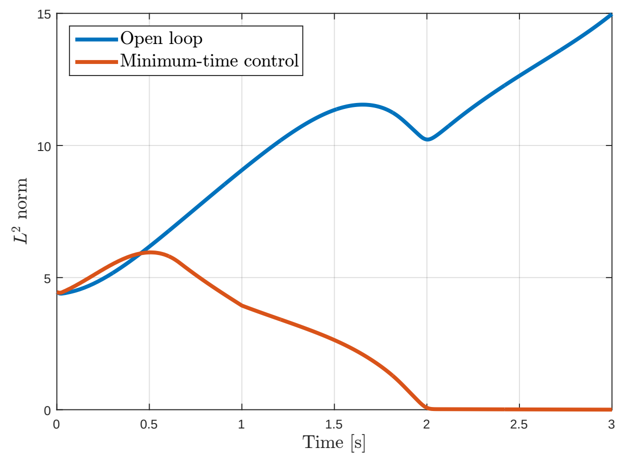

Fig. 3. Time evolution of the L2-norm in open loop and using the controler

VII. Uncollocated Observer Design and Output Feedback Controller

In this section we design an observer that relies on the measurements of u at the right boundary and of v at the left boundary, i.e we measure

$$y(t) = \begin{pmatrix} u(t,1) \\ v(t,0) \end{pmatrix} \tag{87}$$

Then using the estimates given by our observer and the control law (83)-(84), we derive an output feedback controller.

A. Observer design

The observer equations read as follows

$$\hat{u}_t(t,x) + \lambda \hat{u}_x(t,x) = \sigma^{+-}\hat{v}(t,x) - p_{11}(x)(\hat{u}(t,1) - u(t,1)) - p_{12}(x)(\hat{v}(t,0) - v(t,0)) \tag{88}$$

$$\hat{v}_t(t,x) - \mu \hat{v}_x(t,x) = \sigma^{-+}\hat{u}(t,x) - p_{21}(x)(\hat{u}(t,1) - u(t,1)) - p_{22}(x)(\hat{v}(t,0) - v(t,0)) \tag{89}$$

with the boundary conditions

$$\hat{u}(t,0) = U(t), \quad \hat{v}(t,1) = V(t) \tag{90}$$

where p_{11}, p_{12}, p_{21} and p_{22} are the observer gains and have yet to be designed. This yields the following error system

$$\tilde{u}_t(t,x) + \lambda \tilde{u}_x(t,x) = \sigma^{+-}\tilde{v}(t,x) - p_{11}(x)\tilde{u}(t,1) - p_{12}(x)\tilde{v}(t,0) \tag{91}$$

$$\tilde{v}_t(t,x) - \mu \tilde{v}_x(t,x) = \sigma^{-+}\tilde{u}(t,x) - p_{21}(x)\tilde{u}(t,1) - p_{22}(x)\tilde{v}(t,0) \tag{92}$$

with the boundary conditions

$$\tilde{u}(t,0) = 0, \quad \tilde{v}(t,1) = 0 \tag{93}$$

B. Target system

We map the system (91)-(92) to the following system

$$\tilde{\alpha}_t(t,x) + \lambda \tilde{\alpha}_x(t,x) = 0 \tag{94}$$

$$\tilde{\beta}_t(t,x) - \mu \tilde{\beta}_x(t,x) = 0 \tag{95}$$

with the following boundary conditions

$$\tilde{\alpha}(t,0)=0, \quad \tilde{\beta}(t,1)=0 \tag{96}$$

Lemma 6: The system (94)-(95) reaches its zero equilibrium in a finite time t_F where t_F is defined by (16).

Proof: The proof of this lemma is straightforward and is omitted here. ■

C. Fredholm transformation

We use the following transformation

$$\begin{aligned}\tilde{u}(t,x) &= \tilde{\alpha}(t,x)\\ &+ h_{[0,\frac{1}{2}]}(x)\int_0^x K(x,\xi)\tilde{\alpha}(t,\xi)+L(x,\xi)\tilde{\beta}(t,\xi)\\ &+ h_{]\frac{1}{2},1]}(x)\int_x^1 M(x,\xi)\tilde{\alpha}(t,\xi)+N(x,\xi)\tilde{\beta}(t,\xi)d\xi \end{aligned}\tag{97}$$

$$\begin{aligned}\tilde{v}(t,x) &= \tilde{\beta}(t,x)\\ &+ h_{[0,\frac{1}{2}]}(x)\int_0^x \bar{K}(x,\xi)\tilde{\alpha}(t,\xi)+\bar{L}(x,\xi)\tilde{\beta}(t,\xi)d\xi\\ &+ h_{]\frac{1}{2},1]}(x)\int_x^1 \bar{M}(x,\xi)\tilde{\alpha}(t,\xi)+\bar{N}(x,\xi)\tilde{\beta}(t,\xi)d\xi \end{aligned}\tag{98}$$

Remark 7: We do have $\tilde{\alpha}(t,0)=\tilde{u}(t,0)=\tilde{v}(t,1)=\tilde{\beta}(t,1)$. Moreover $\tilde{\alpha}(t,1)=\tilde{u}(t,1)$ and $\tilde{v}(t,0)=\tilde{\beta}(t,0)$

if $x\leq\frac{1}{2}$ Differentiating the equations (97)-(98) with respect to time and space yields

$$0=\lambda K_x(x,\xi)+\lambda K_\xi(x,\xi)-\sigma^{+-}\bar{K}(x,\xi) \tag{99}$$
$$0=-\mu\bar{K}_x(x,\xi)+\lambda\bar{K}_\xi(x,\xi)-\sigma^{-+}K(x,\xi) \tag{100}$$
$$0=\lambda L_x(x,\xi)-\mu L_\xi(x,\xi)-\sigma^{+-}\bar{L}(x,\xi) \tag{101}$$
$$0=-\mu\bar{L}_x(x,\xi)-\mu\lambda\bar{L}_\xi(x,\xi)-\sigma^{-+}L(x,\xi) \tag{102}$$
$$0=+(\lambda+\mu)L(x,x)-\sigma^{+-} \tag{103}$$
$$0=-(\mu+\lambda)\bar{K}(x,x)-\sigma^{-+} \tag{104}$$
$$0=-\mu L(x,0)+p_{12}(x) \tag{105}$$
$$0=p_{11}(x) \tag{106}$$
$$0=p_{21}(x) \tag{107}$$
$$0=-\mu\bar{L}(x,0)+p_{22}(x) \tag{108}$$

if $x>\frac{1}{2}$ Similarly we get

$$0=\lambda M_x(x,\xi)+\lambda M_\xi(x,\xi)-\sigma^{+-}\bar{M}(x,\xi) \tag{109}$$
$$0=-\mu\bar{M}_x(x,\xi)+\lambda\bar{M}_\xi(x,\xi)-\sigma^{-+}M(x,\xi) \tag{110}$$
$$0=\lambda N_x(x,\xi)-\mu N_\xi(x,\xi)-\sigma^{+-}\bar{N}(x,\xi) \tag{111}$$
$$0=-\mu\bar{N}_x(x,\xi)-\mu\lambda\bar{N}_\xi(x,\xi)-\sigma^{-+}N(x,\xi) \tag{112}$$
$$0=+(\lambda+\mu)\bar{M}(x,x)-\sigma^{-+} \tag{113}$$
$$0=-(\mu+\lambda)N(x,x)-\sigma^{+-} \tag{114}$$
$$0=-\lambda\bar{M}(x,1)+p_{21}(x) \tag{115}$$
$$0=-\lambda M(x,1)+p_{11}(x) \tag{116}$$
$$0=p_{12}(x) \tag{117}$$
$$0=p_{22}(x) \tag{118}$$

All those kernels equations seem well-posed.
Question : Is the transformation invertible ?

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