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Abstract: This article proposes a distributed parameters model for a pool of electric hot water tanks (EHWT). EHWT are electric appliances found in numerous homes where they produce hot water for domestic usages. Designing smart piloting for them requires a careful description of several variables of interest and their dynamics. When the number of such devices is large, these dynamics can be lumped into Fokker-Planck equations. In this case, these equations are driven by in-domain control which defines the heating policies in a stochastic manner. The main contribution of this article is the Fokker-Planck model of a pool of EHWT.

Keywords: Water heating, Electric Hot Water Tank, Smart piloting, Fokker-Planck equation

1. INTRODUCTION

The increasing share of intermittent renewable electricity sources in the energy mix (European Commission [2011], Edenhofer et al. [2011]) raises new difficulties in management of the electricity production and equilibrium in distribution networks. Demand Side Management (DSM), which is a portfolio of smart piloting techniques aiming at modifying consumers' demand, is a promising solution for such concerns (Palensky and Dietrich [2011]). A key factor in developing DSM is the ability to find energy storage capacities. In this context, the large pools of electric hot water tanks (EHWT) have a well recognized potential.

An EHWT is a domestic electric appliance which heats a volume of water with an electric thermo-plunger that can be controlled. The home user drains hot water from the EHWT at various times of the day. The literature (Blandin [2010], Zurigat et al. [1991] and the references therein) models EHWT as vertical cylindrical tanks driven by thermo-hydraulic phenomena: heat diffusion, buoyancy effects and induced convection and mixing, forced convection induced by draining and associated mixing, and heat loss at the walls.

To model EHWT, one-dimensional distributed parameter models of the temperature profile in the tank have been developed (see Beeker et al. [2015a,b]). The observed temperature profiles are increasing with height. For smart piloting applications, one can define three variables of interest providing a simplified representation of the temperature profile, under the form of three amounts of energy. From this, advanced control designs can be studied. Among them, optimal control strategies are particularly appealing as they address topics of direct interest both for end-users and electricity producers: cost reduction, comfort constraints, and yield management. Yet, the situation is not that straightforward. In truth, a real challenge lies in solving optimal control problems for large numbers of such EHWT. Coordinated individual control of each tank is feasible for pools of moderate sizes (typically from 2 to 10 EHWT). However, for large pools of EHWT (tens of thousands), this approach is a stalemate. Unfortunately, the real stakes and industrial expectations belong to this range.

Interestingly, it is possible to recast this problem into a distributed parameters approach. This is the path we explore in this article. Following the works of Malhamé and Chong [1985], we consider that the local (individual) control variables of EHWT are each defined according to stochastic processes. Then, we combine *i)* this randomness, *ii)* the diversity in the distribution of the states of the EHWT, *iii)* the randomness of the water consumptions, and we develop a partial differential equation (PDE) for a large pool of EHWT. This takes the form of Fokker-Planck equations (see Risken [1996]) governing the probability distributions of the population of EHWT. The work of Malhamé and Chong was originally focused on a mitigated load represented by a single state, which we need to extend for the smart piloting applications under consideration here. This extension results into a rich system of PDE, which constitutes the main contribution of this paper.

The paper is organized as follows. In the pool, a single EHTW is a macroscopic but small subsystem described by state variables. These variables of interest are defined in Section 2. They have hybrid dynamics by construction. To account for the randomness of water consumption, we propose a single EHWT model as a Markovian stochastic process in Section 3. Then, we introduce probability density function of the population of EHWT and derive the Fokker-Planck equations in Section 4. A summary of the obtained input-output description of the EHWT pool

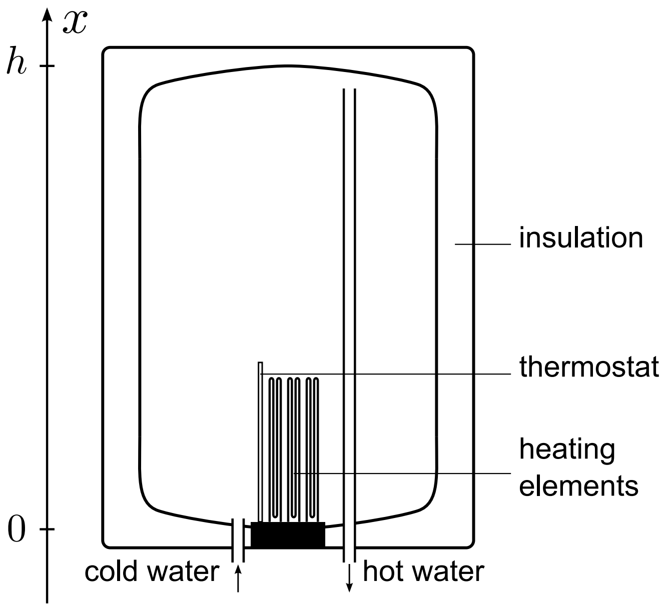


Fig. 1. Schematic view of an electric hot water tank.

is reported in Section 5. Conclusions and perspectives are given in Section 6.

2. VARIABLES OF INTEREST IN A EHWT

2.1 General description, stratification, and effects of heating

A typical EHWT is a vertical cylindrical tank filled with water. A heating element is plunged at the bottom end of the tank (see Fig. 1). The heating element is pole-shaped, and relatively lengthy, up to one third of the tank. Cold water is injected at the bottom while hot water is drained from the top at exactly the same flow-rate (under the assumption of pressure equilibrium in the water distribution system). In the tank, layers of water with various temperature coexist (see Fig.2). At rest, these layers are mixed only by heat diffusion which effects are relatively slow compared to the other phenomena (Han et al. [2009]). Existence of a non uniform (increasing with height) quasi-equilibrium temperature profile in the tank is called *stratification* (Dincer and Rosen [2010], Lavan and Thompson [1977]). In practice, this effect is beneficial for the user as hot water available for consumption is naturally stored near the outlet of the EHWT, while the rest of the tank is heated (see Fig.2(b)). Due to this effect and the cylindrical symmetry of the system, one can assume that the water temperature in the tank is homogeneous at each height.

Following the description above, the temperature T of the tank is a continuously increasing function of height (see Fig. 2). The constant inlet temperature T_{in} constitutes a lower bound of the temperature profile. The heating process is driven by turbulence generated by buoyancy effects, which is the cause of a local mixing in the bottom of the tank. We consider that this mixing is perfect on a spatial zone, refereed to as the *plateau* (see Beeker et al. [2015a,b]), and does not affect the temperature profile in the upper part of the tank (see Fig. 2(b)). During the heating process, the plateau grows and gradually covers the whole tank. The user specifies a temperature T_{max} at which the heating has to be stopped to prevent overheating. As a result of the heating process, if the temperature at the bottom of the tank is T_{max}, then the temperature in the tank is uniformly at T_{max} once the heating is finished.

The user can also specify a comfort temperature T_{com}. Water having temperature higher than T_{com} can be blended with cold water to reach T_{com} and is therefore useful, while water having temperature lower than T_{com} is useless.

2.2 Consumption, control and objectives

Each EHWT has two inputs: water consumption and heating power. The user consumes certain quantities of energy each day. For this reason, consumption of hot water is an (uncontrolled) input of our problem. On the other hand, the heat injected via the heating element in the tank is a control variable.

The control design can have various objectives. The most obvious one is individual cost reduction for each single unit in response to a price signal. At larger scales, one can naturally consider a pool of tanks, and aim at reaching a load profile for the aggregate consumption.

2.3 Variables of interest: Available, delay and reserve energies

Describing the exact temperature profile inside the tank is unnecessary for the applications discussed above. Instead, a few (3) variables of interest can be considered.

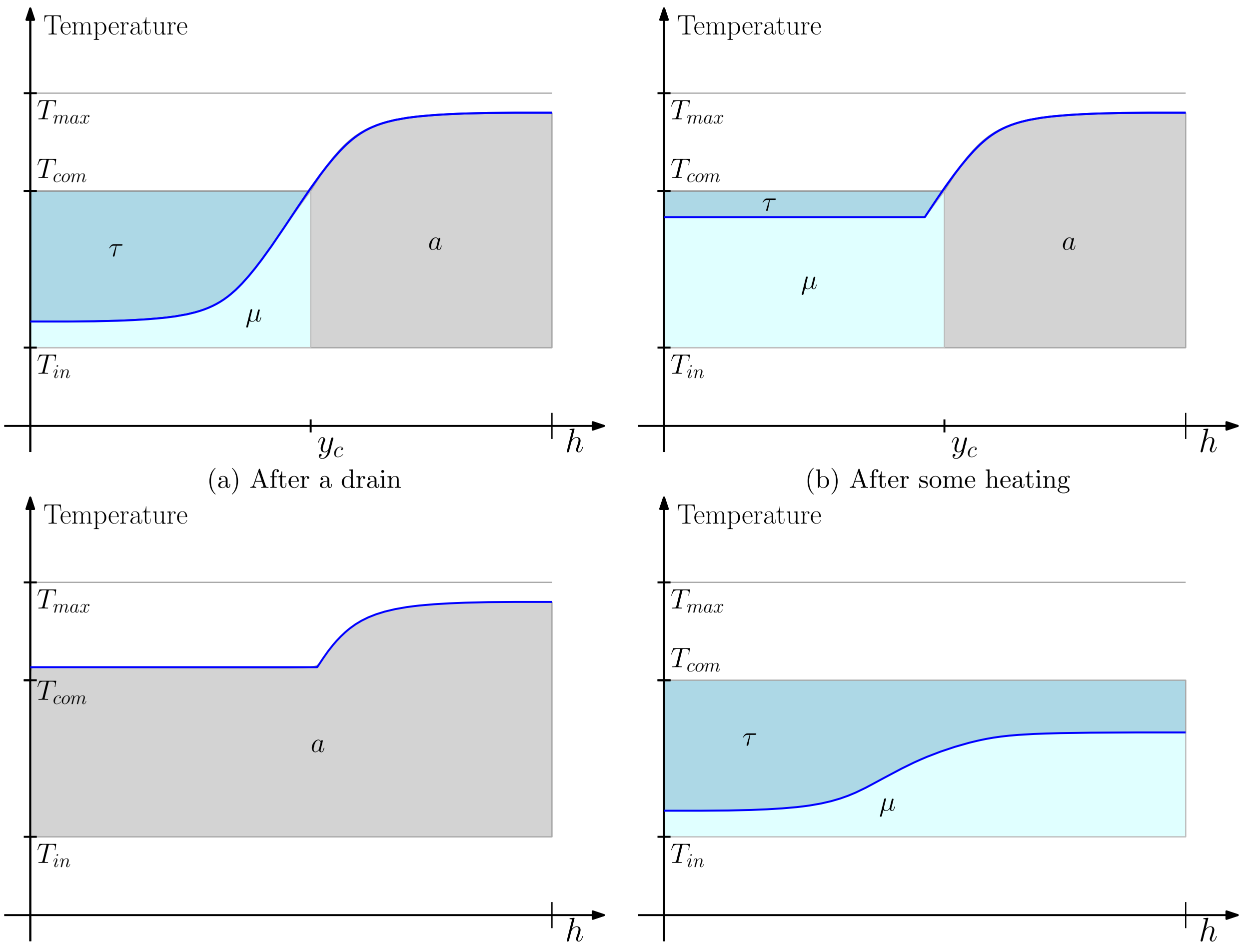
The *available energy* a is defined as the energy contained in the zones having temperature greater than the comfort temperature T_{com}. This constitutes a direct comfort index for the user. If a reaches 0 and a water drain is applied, then the comfort constraints is violated.

The *delay energy* τ is defined as the energy required by the plateau to reach the temperature T_{com}. When the tank is heated at constant maximum power, in the absence of drains and heat losses, τ is simply proportional to the time necessary to reach a state from which a can effectively be positively impacted by the heating process.

The *reserve energy* μ is defined as the energy contained in the tank that is currently unavailable for consumption, i.e. the energy contained in the water under T_{com}. When, thanks to the heating process, τ reaches the value 0, the energy μ becomes available for consumption: this generates an immediate (discontinuous) increase of a, and μ is reset to 0. [1]

Fig. 2 illustrates the dynamics of these variables. A drain (pictured in (a)) is mainly characterized by a decrease of a and an increase of τ, with a slight raise of μ due to an energy transfer from a. On the other hand, in the heating pictured in (b), τ decreases at the same rate as μ rises, until the plateau reach T_{com}. An energy transfer from μ to a takes place then. This is pictured in (c). A case where comfort constraint is violated is pictured in (d).

[1] The rationale behind these definitions is that to plan the heating, we account for the time left before the energy reserve embodied by a (in the total energy $a + \mu$) is consumed, and the time necessary to provide new hot water, embodied by τ.



(a) After a drain (b) After some heating

(c) High energy (d) Low energy

Fig. 2. Temperature profile, and corresponding states in various situations

2.4 Domains of definition of state variables

Define $\lambda = \frac{T_{conf}-T_{inj}}{T_{max}-T_{inj}}$ and let m be the maximal energy that can be contained in the tank under the temperature T_{max}. Then, by definition, a, τ, μ are subject to the following inequalities :

$$\begin{aligned} &0 \le a \le m, \quad 0 \le \tau \le \lambda m, \quad 0 \le \mu \le \lambda m, \\ &a + \frac{1}{\lambda}(\tau+\mu) \le m, \quad \lambda m \le a+\tau+\mu \end{aligned} \tag{1}$$

from which we define the following open polyhedron of $\mathbb{R}^3$ and its faces (see Fig. 3):

$$\begin{aligned} \Omega_0 =&\{(a,\tau,\mu)|a,\tau,\mu>0, \lambda m < a+\tau+\mu \\ &\text{and } a+\frac{1}{\lambda}(\tau+\mu) < m\} \\ \mathcal{F}_1 =&\bar{\Omega}_0 \cap \{(a,\tau,\mu)|\mu = 0\} \\ \mathcal{F}_2 =&\bar{\Omega}_0 \cap \{(a,\tau,\mu)|\tau = 0\} \\ \mathcal{F}_3 =&\bar{\Omega}_0 \cap \{(a,\tau,\mu)|\lambda m = a+\tau+\mu\} \\ \mathcal{F}_4 =&\bar{\Omega}_0 \cap \{(a,\tau,\mu)|a+\frac{1}{\lambda}(\tau+\mu) = m\}. \end{aligned} \tag{2}$$

The following edges and vertices are considered

$$\begin{aligned} \mathcal{E}_1 =&\mathcal{F}_1 \cap \mathcal{F}_2 & \mathcal{E}_2 =&\mathcal{F}_3 \cap \mathcal{F}_4 \\ \mathcal{V}_1 =&\{(\lambda m, 0, 0)\} & \mathcal{V}_2 =&\{(m,0,0)\} \\ \mathcal{V}_3 =&\{(0,\lambda m, 0)\} & \mathcal{V}_4 =&\{(0,0,\lambda m)\}. \end{aligned} \tag{3}$$

In practice, $x \triangleq (a,\tau,\mu)$ can only belong to $\mathcal{E}_2$ (low energy, e.g. Fig. 2 (d)), Ω_0 (medium energy, e.g. Fig. 2 (a) and (b)) and $\mathcal{E}_1$ (high energy, e.g. Fig. 2 (c)). Faces $\mathcal{F}_1$ to $\mathcal{F}_4$, and vertices $\mathcal{V}_1$ to $\mathcal{V}_4$ constitute boundaries of these three domains. Note that uniformly cold tank can also stack in $\mathcal{V}_3$. In the sequence, we note $\Omega = \Omega_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{V}_3$. Any index $i = 0,1,2,3$ will refer to these domains, respectively.

2.5 Transient behavior

Heating mostly induces a continuous variation of x. If $\tau, \mu > 0$ (i.e. $x \in \Omega_0$), then the heating yields a decrease of τ and an increase of μ. Under certain conditions, a threshold effect can be observed: when τ reaches 0 (i.e. when x reaches $\mathcal{F}_2$), then suddenly, all the unavailable energy μ becomes available, μ takes the value 0 and all its energy is transferred into a. This induces a discontinuity, transferring x to $\mathcal{E}_1$ in which the heating has again a continous effect on x, increasing the available energy a.

Heat losses also mostly induce a continuous variation of x, where a and μ decrease while τ increases. The reverse threshold effect can also be observed: when x reaches $\mathcal{F}_3$, an entire layer of water goes under the temperature T_{com}, which causes a to take the value 0 and x to jump in $\mathcal{E}_2$ in which it will again vary continuously.

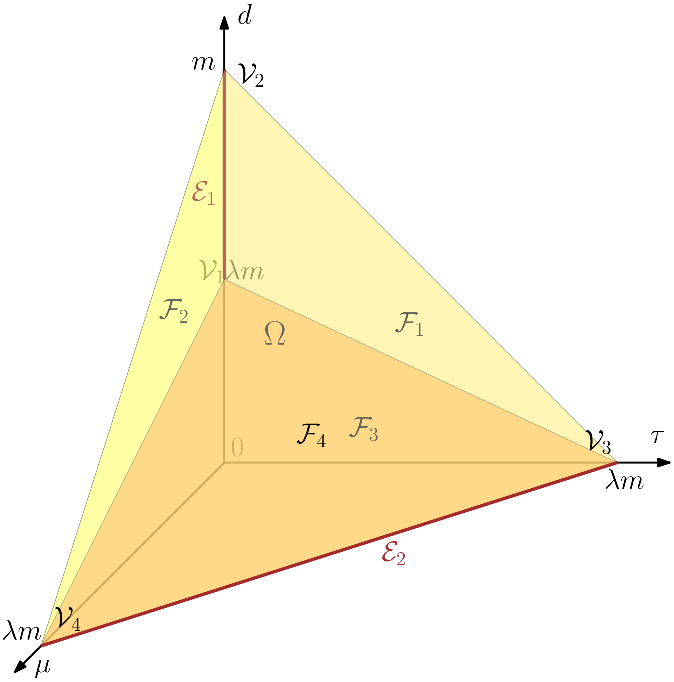


Fig. 3. Domains of definition of the state variables

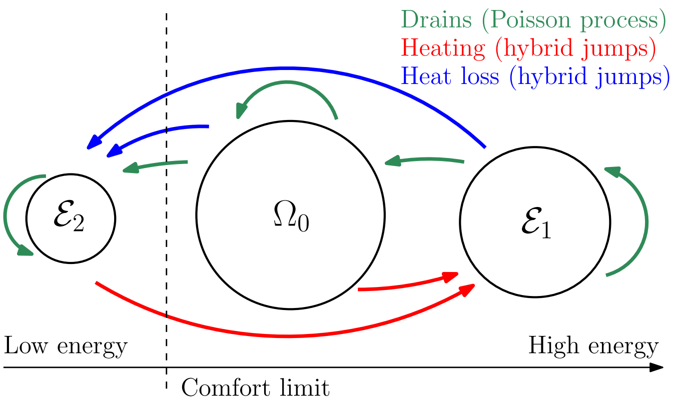


Fig. 4. Transient between sub-domains

The drains have a very short duration compared to heating and heat loss. They yield quasi-instantaneous decrease of a and μ and increase of τ, having time of occurrence and magnitude depending on the user's habits. We model them as jumps only. Very large decreases can cause x to jump from $\mathcal{E}_1$ to Ω_0 (or even to $\mathcal{E}_2$), or from Ω_0 to $\mathcal{E}_2$.

The transients are pictured in Fig. 2. A summary of jump dynamics is pictured in Fig. 4.

3. EHWT AS A HYBRID-STATE STOCHASTIC PROCESS

3.1 Stochastic process representation

The times of occurence and magnitude of the drains are user-dependent. We model this dependency under the form of a stochastic process. This results in a model similar in spirit to the single-dimensional representation originally proposed by Malhamé and Chong [1985].

In our model, each EHWT is defined by the state vector

$$X_t = \begin{bmatrix} a_t \\ \tau_t \\ \mu_t \end{bmatrix} \in \Omega \tag{4}$$

and its heating status $S_t \in \{r, h\}$, which is also of stochastic nature. From now-on, the indexes $\{r, h\}$ refer to "rest" and "heating", respectively.

The dynamics of X_t is governed by the phenomena described in Section 2.5. In each domain Ω_0, $\mathcal{E}_1$, $\mathcal{E}_2$, the state vector X_t continuously (due to heating and heat loss) and discontinuously (due to drains) changes, with respect to a stochastic differential equation, constituting the *flow map*, that will be stated below. A jump of hybrid nature appears when X_t reaches $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{V}_1$ or $\mathcal{V}_4$. This discontinuity, according to the terminology of Goebel et al. [2012], constitutes the *jump map*. A correspondence between phenomena and stochastic/hybrid representation can be found in Table 1.

3.2 Flow map: Stochastic process dynamics on each domain

The stochastic differential equations of X_t and S_t are

$$\begin{aligned} dX_t &= v(X_t)dt + dJ_t + \sigma(X_t)dW_t \\ dS_t &= dN_t \end{aligned} \tag{5}$$

where

- $v(X_t)dt$ is the drift component which represents the heat losses and/or heating effects.
- dJ_t is the infinitesimal integration with respect to a 3-dimensional compound Poisson process J_t representing the jump effects of drains on a_t, τ_t and μ_t.
- Uncertainties are lumped into a variance term $\sigma(X_t)$, integrated with respect to a Wiener process (or standard Brownian motion) W_t.
- dN_t is the infinitesimal integration with respect to a 1-dimensional Poisson process N_t representing the status switch between h and r.

Expressions of v, σ and of the random characteristics of J_t depend on the domain and the status of the EHWT. We now detail them.

Heat loss and heating modeling as a drift Under the assumption that the ambient temperature T_a is equal to T_{inj}, the heat loss per unit of time for a_t is equal to $-\frac{k}{S\rho c_p}a_t$, where S is the cross-section of the tank, ρ and c_p are the density and heat capacity of water, and k is the heat loss coefficient of the tank per unit of height (see Fig. 2).

Likewise, for μ_t, the heat loss is equal to $-\frac{k}{S\rho c_p}\mu_t$. This heat loss generates a positive effect on τ_t which therefore increases with the rate $\frac{k}{S\rho c_p}\mu_t$. In summary,

$$v^r(X_t) = \frac{k}{S\rho c_p}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} X_t. \tag{6}$$

To this heat loss drift is added a drift $v^h(X_t)$ due to power injection for tanks subject to heating. In Ω_0 and $\mathcal{E}_2$, power injection p lowers τ_t and increases μ_t, so that one has $v^h(X_t) = p\,[0\ \ -1\ \ 1]^T$. In $\mathcal{E}_1$, the injected power only affects a_t, and therefore $v^h(X_t) = [p\ \ 0\ \ 0]^T$. Finally, when the tank is heating

$$v(X_t) = v^r(X_t) + v^h(X_t) \tag{7}$$

and otherwise $v(X_t) = v^r(X_t)$.

Drain as a Poisson process The drains appear as a sequence of quasi-instantaneous events of various magnitudes, so that we choose to model them as a compound Poisson process J_t (see e.g. Applebaum [2004]).

Then, the time between jumps follows an exponential law of parameter $\theta(t)$, and the magnitude of jumps is characterized by a probability density function $\omega : \Omega^2 \times \mathbb{R}_+ \to \mathbb{R}_+$. In words, a jump from y at time t maps to Ω with distribution characterized by $\omega(y, \cdot, t)$.

EHWT variability as a Wiener process Some phenomenon are not taken into account in the description above (e.g. diffusion). We choose to lump these into into a variance parameter $\sigma(X_t)$ acting through a Wiener process. However, the closer X_t is to the boundary of Ω, the smaller the uncertainty should be on some of variables. Thus, certain components of $\sigma(X_t)$ vanish at the boundaries of domains. Additionally, the heating increases uncertainty on the dynamics. Therefore, the variance takes the form

$$\sigma(X_t) = \sigma^r(X_t) + \sigma^h(X_t) \tag{8}$$

for heating tanks, and $\sigma(X_t) = \sigma^r(X_t)$ otherwise.

On/off heating switch modeled with a Poisson process The switching between the two statuses h and r constitute a sequence of instantaneous events that can be controlled. We choose to also model it with a Poisson process of intensity $\alpha(X_t, t, S_t) = \alpha^{S_t}(X_t, t)$ (indexed on S_t for transition from S_t to the opposite one). This means that instead of exactly setting the switching times, two functions α^r, α^h define a probability to switch from one status to another, depending on the state X_t and time t.

3.3 Jump map: Hybrid modeling of the domain switch

The threshold jump when heating or with heat losses constitute hybrid deterministic jumps. When reaching a certain boundary, it maps a domain to another, depending on the status. The transition, in the framework of Goebel et al. [2012], gives x^+ (the value after the jump) as a function of x (the value before the jump). For the sake of simplicity, a summary is given in Table 2. The maximal energy m that can be contained in the tank is attained at point $\mathcal{V}_2$, when heating. We assume that the heating automatically switches off at this point for security reasons, which is characterized by a hybrid jump for the status from h to r. This jump is also presented in the jump map.

4. FOKKER-PLANCK PDE FOR A LARGE POOL OF EHWT

4.1 EHWT pool population representation

Representing a large pool of tanks each having a 3-dimensional state leads to an unnecessarily large finite-dimensional system, which can be difficult to design controllers for. Rather, a probability density functions representation can be employed.

The main idea is to define seven function f_0^r, f_0^h, f_1^r, f_1^h, f_2^r, f_2^h, f_3^r (for each domain Ω_0, $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{V}_3$, one for each status r or h at the exception of $\mathcal{V}_3$ in which only resting tanks can stack) which represent the population density of the tanks of a given status in a certain domain. These positive functions are subject to the balance consistency

$$\iiint_{\Omega_0} (f_0^h + f_0^r) + \int_{\mathcal{E}_1} (f_1^h + f_1^r) + \int_{\mathcal{E}_2} (f_2^h + f_2^r) + f_3^r = 1. \tag{9}$$

The dynamics governing these probability functions are obtained from the preceding dynamics. We now detail them.

4.2 Fokker-Planck equation for a stochastic process

A useful tool for the population distribution computation is the Fokker-Planck equation (see Risken [1996]). For a set of independent Markov process in a state space Ω following the same generic stochastic equation

$$dX_t = v(X_t, t)dt + \sigma(X_t, t)dW_t + dJ_t \tag{10}$$

where $v(X_t, t), \sigma(X_t, t) \in \mathbb{R}^6$, dW_t is the integration with respect to a 1-dimensional Wiener process W_t, and dJ_t is the integration with respect to a compound Poisson process of intensity $\theta(t)$ and whose compound distribution is represented with the probability density function ω (i.e. when a jump occurs on state x, the probability density function of transition to state y is represented with $\omega(x, y, t)$ at time t), the probability density function (when the number of stochastic process tends to infinity) is given by the Fokker-Planck equation (see e.g. Applebaum [2004])

$$\begin{aligned} \partial_t f(x,t) = {} & -\nabla_x \cdot [v(x,t) f(x,t)] + \nabla_x \cdot [D(x,t) \nabla_x f(x,t)] \\ & + \theta(t) \int_\Omega (f(y,t) - f(x,t)) \omega(x,y,t) dy \end{aligned} \tag{11}$$

for $(x,t) \in \Omega \times \mathbb{R}_+$, where

$$D(x,t) = \frac{1}{2} \sigma(x,t) \cdot \sigma^t(x,t) \in \mathbb{R}^{3\times 3}. \tag{12}$$

4.3 Detailed expression of the PDE

Several observations can be made on the stochastic model presented in Section 3. First, for each tank, the stochastic process X_t defined by (5) constitutes a Markov process. Moreover, given a pool of tanks, the independence of the stochastic process of each tank appears as a reasonable hypothesis, given the assumption that hot water consumptions of distinct households are not related.

Therefore, one can directly follow the work of Malhamé and Chong [1985], and derive the Fokker-Planck equations. In our case, for each domain and each status, this equation takes the form of a parabolic PDE. The hybrid character of the stochastic process appears in the boundary conditions and yields an additional integral source term.

On each domain $i = 0, 1, 2$ (for Ω_0, $\mathcal{E}_1$, and $\mathcal{E}_2$), f_i^r and f_i^h are driven by a system of the form

$$\begin{aligned} \partial_t f_i^r + \nabla_x \cdot [v^r f_i^r] = {} & \nabla_x \cdot [D_i^r \nabla_x f_i^r] \\ & - (\alpha_i^r + \theta) f_i^r + \alpha_i^h \cdot f_i^h + S_i^r(f(\cdot,t), x, t) \\ \partial_t f_i^h + \nabla_x \cdot [(v^r + v_i^h) f_i^h] = {} & \nabla_x \cdot [(D_i^r + D_i^h) \nabla_x f_i^h] \\ & - (\alpha_i^h + \theta) f_i^h + \alpha_i^r \cdot f_i^r + S_i^h(f(\cdot,t), x, t) \end{aligned} \tag{13}$$

while f_3^r follows an ordinary differential equation that will be stated later.

4.4 Parameters specification on each domain

Each term of the PDE derives from the infinitesimal generator defining the parabolic operator used in the Fokker-Planck equation. Each term in the stochastic differential equation has a matching term in the partial differential equation (see e.g. Ethier and Kurtz [2005], Sato [2011]). For the sake of clarity, we present in Table 1 each stochastic term in our case and its corresponding term in the PDE. The necessary steps of computations are omitted for brevity.

The heat loss drift in the PDE has the same form as the one in the stochastic equation, i.e.

$$v^r(x) = \frac{k}{S\rho c_p}\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 1 & 0\end{bmatrix} x \in \mathbb{R}^3. \tag{14}$$

The drift $v_i^h \in \mathbb{R}^3$, the diffusion $D_i^r(x,t), D_i^h(x,t) \in \mathbb{R}_+^{3\times 3}$, and the source terms $S_i^s(f(\cdot,t),x,t) \in \mathbb{R}_+$ for $s = r,h$ have to be defined in each domain. The term caused by the Poisson process leads to source terms of various integral forms, depending on the probability density function ω_i on each domain (e.g. $\omega_0 = \omega_{|\Omega_0}$). Source terms can also appear due to hybrid transfer from other domains in the form of an integral flow. For that purpose, on $\mathcal{F}_2$ and $\mathcal{F}_3$, we introduce the functions η_2 and η_3 s.t.: $\eta_2(y,x) = 1$ if $[a_x \tau_x \mu_x]^T = [a_y + \mu_y \tau_y 0]^T$ and 0 otherwise; and $\eta_3(y,x) = 1$ if $[a_x \tau_x \mu_x]^T = [0 \tau_y a_y + \mu_y]^T$ and 0 otherwise.

Details are reported in Table 3. Finally, the exchange terms $\alpha_i^s(x,t)$ can be chosen as they are control-dependent, while $\theta(t)$ does not depend on space.

4.5 Boundary conditions

The domain Ω_0 has 4 boundaries ($\mathcal{F}_1$ to $\mathcal{F}_4$), while $\mathcal{E}_1$ and $\mathcal{E}_2$ have 2 boundaries, each in the form of vertices. Boundary conditions stems from the behavior of the stochastic process. A special case is the boundary $\mathcal{V}_3$ where uniformly cold tanks stack.

On the borders, we have :

- $d_\tau(x), d_\mu(x) \to 0$ when $x \to \mathcal{F}_1$ or $\mathcal{F}_2$,
- $d_a(x) \to 0$ when $x \to \mathcal{F}_3$ or $\mathcal{F}_4$,
- $d_1(x) \to 0$ when $x \to \mathcal{V}_1$ or $\mathcal{V}_2$,
- $d_2(x) \to 0$ when $x \to \mathcal{V}_3$ or $\mathcal{V}_4$.

This allows to define boundary conditions of the Dirichlet or free boundary types, except in $\mathcal{V}_3$. Their definitions stem from exchange between the domains: free boundary corresponds to the case where tanks flowing outside the domain flow inside another domain as a source term (hybrid jumps). On the contrary, the zero Dirichlet boundary conditions correspond to the fact that no new tank can enter the system (the population is fixed). A summary is presented in Table 4.

The vertex $\mathcal{V}_3$ constitutes a free boundary for f_2^r, in which the population of completely cold tanks can stack.

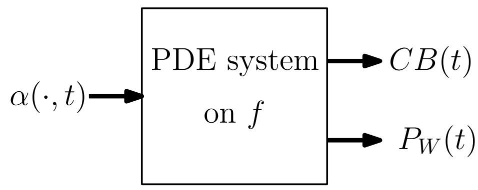


Fig. 5. Input/output representation of the pool of EHWT.

The population $f_3^r(t)$ at this point can be heated and constitutes the input flow for f_2^h through the boundary condition $(v_2^h + v_2^r(\mathcal{V}_3))f_2^h(\mathcal{V}_3,t) = \alpha_2^r(\mathcal{V}_3,t)f_3^r(t)$. Therefore, $f_3^r(t)$ is driven by

$$\begin{aligned}\dot{f}_3^r(t) = &-\alpha_2^r(\mathcal{V}_3,t)f_3^r(t) + v_2^r(\mathcal{V}_3)f_2^r(\mathcal{V}_3,t)\\ &+ \int_{\mathcal{E}_2} f_2^r(y)\omega_2(y,\mathcal{V}_3,t)dy\end{aligned} \tag{15}$$

5. INPUT-OUTPUT DESCRIPTION

The input of the system is the set of functions (over space and time) $\alpha = [\alpha_0^r, \alpha_0^h, \alpha_1^r, \alpha_1^h, \alpha_2^r, \alpha_2^h]^T$, which determine the intra-domain migration between the populations of heating and resting tanks.

To control the pool of tanks, several indicators can be interesting. They are the output of the proposed model. Among them, the mass (number) of tanks breaking the comfort constraints is

$$CB(t) = \int_{\mathcal{E}_2} (f_2^h + f_2^r)(x,t)dx + f_3^r(t) \tag{16}$$

and the total power demand is

$$P_W(t) = \sum_{i=0}^{2} \iiint_\Omega f_i^h(x,t)dx \tag{17}$$

They constitute valuable performance indexes for the system. The system can be seen with the input/output representation depicted in Fig. 6.

A natural goal is then to design controls $(\alpha(\cdot,t) : \Omega \to \mathbb{R}_+^6)$ s.t. CB is as low as possible, while the total power demand P_W follows a given objective function.

As a mean of illustration, the probability density functions on $\mathcal{E}_1$, $\mathcal{E}_2$, and a representative segment of Ω_0 from the middle of $\mathcal{E}_2$ to $\mathcal{V}_2$ are shown in Fig. 5. Two profiles are shown. A fictional initial one, and the one subsequent to the following heating policy. We choose to promote heating (i.e. α_i^r high and α_i^h low) on $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{V}_3$, and let the tanks rest (i.e. α_i^r low and α_i^h high) on Ω_0. The profile varies as is shown in the figure, and tends to spread due to diffusion and integral drains. After some time, due to diffusion effects, a stationary profile representative of the cycle $\mathcal{E}_2 \to \mathcal{E}_1 \to \Omega_0 \to \mathcal{E}_2$ should take place.

6. CONCLUSION AND PERSPECTIVES

In this article we have explained the derivation of a model for a large pool of EHWT. The input is a parameter defining the stochastic process of heating of each indidividual EHWT in the pool. The outputs are the overall comfort variable defined in (16) and the total power demand (17). The dynamics are a collection of Fokker-Planck partial differential equations.

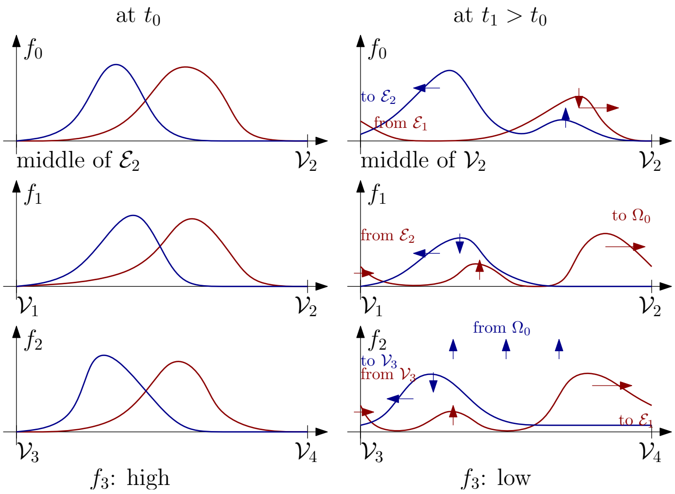


Fig. 6. Evolution of probability density function, given a heating policy (red: heating, blue: resting)

The next steps should address the control problems based on this input-output description. A question to be solved can be formulated as follows: how to design $\alpha(\cdot, t)$ so that the power demand matches some desirable history while limiting or minimizing the discomfort? This problem belongs to the class of optimal control (tracking) of distributed parameters systems with in-domain actuation and is the topic of current investigations.

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Phenomenon	**Stochastic representation**	**Fokker-Planck equation**
Heat loss	Drift	Convection
Heating	Drift	Convection
Drain	Poisson process	Integral source term
Threshold effect on the comfort	Hybrid jump	Free boundary condition, and source term
Control	Poisson process	Coupling source term
Uncertainty	Brownian motion	Diffusion

Table 1. Correspondance between: phenomenon, hybrid stochastic counterpart, and PDE term

In	Ω_0		$\mathcal{E}_1$		$\mathcal{E}_2$
If reaches	$\mathcal{F}_2 \times \{h\}$	$\mathcal{F}_3 \times \{r, h\}$	$\mathcal{V}_1 \times \{r\}$	$\mathcal{V}_2 \times \{h\}$	$\mathcal{V}_4 \times \{h\}$
Jump to	$\mathcal{E}_1 \times \{h\}$	$\mathcal{E}_2 \times \{r, h\}$	$\mathcal{E}_2 \times \{r\}$	$\mathcal{V}_2 \times \{r\}$	$\mathcal{E}_1 \times \{h\}$
According to the transition	$\begin{bmatrix} a^+ \\ \tau^+ \\ \mu^+ \end{bmatrix} = \begin{bmatrix} a+\mu \\ \tau \\ 0 \end{bmatrix}$	$\begin{bmatrix} a^+ \\ \tau^+ \\ \mu^+ \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \\ a+\mu \end{bmatrix}$	$\begin{bmatrix} a^+ \\ \tau^+ \\ \mu^+ \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \\ a+\mu \end{bmatrix}$	$s^+ = r$	$\begin{bmatrix} a^+ \\ \tau^+ \\ \mu^+ \end{bmatrix} = \begin{bmatrix} a+\mu \\ \tau \\ 0 \end{bmatrix}$

Table 2. Jump map

Domain	**Drift** v_i^h	**Diffusion** $D_i^r(x)$ **and** $D_i^h(x)$	**Source terms** $S_i^r(f(\cdot), x, t)$ **and** $S_i^h(f(\cdot), x, t)$
Ω_0	$p \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} d_a(x) & 0 & 0 \\ 0 & d_\tau(x) & 0 \\ 0 & 0 & d_\mu(x) \end{bmatrix}$	$S_0^r = \theta(t) \iiint_{\Omega_0} f_0^r(y,t)\omega_0(y,x,t)dy + \theta(t) \int_{\mathcal{E}_1} f_1^r(y,t)\omega_1(y,x,t)dy$ $S_0^h = \theta(t) \iiint_{\Omega_0} f_0^h(y,t)\omega_0(y,x,t))dy + \theta(t) \int_{\mathcal{E}_1} f_1^h(y,t)\omega_1(y,x,t))dy$
$\mathcal{E}_1$	$p \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$d_1(x) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$S_1^r = 0$ $S_1^h = \iint_{\mathcal{F}_2} \eta_2(y,x)(v_0(y) + v_0^h) f_0^h(y,t)dy$
$\mathcal{E}_2$	$p \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$	$\frac{1}{2} d_2(x) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$	$S_2^r = \theta(t) \iiint_{\Omega_0} f_0^r(y)\omega_0(y,x,t)dy + \theta(t) \int_{\mathcal{E}_2} f_2^r(y)\omega_2(y,x,t)dy$ $S_2^h = \theta(t) \iiint_{\Omega_0} f_0^h(y)\omega_0(y,x,t)dy + \iint_{\mathcal{F}_3} \eta_3(y,x)v_0(y) f_0^h(y,t)dy$ $+\theta(t) \int_{\mathcal{E}_2} f_2^h(y)\omega_2(y,x,t)dy$

Table 3. Definition if distributed parameters equation term on each domain

Domain	**Boundary**	**Boundary condition**
Ω_0	$\mathcal{F}_1$	$f_0^r(x,t) = 0$ and $f_0^h(x,t) = 0$
	$\mathcal{F}_2$	$f_0^r(x,t) = 0$ and $f_0^h(x,t)$ free
	$\mathcal{F}_3$	$f_0^r(x,t)$ free and $f_0^h(x,t)$ free
	$\mathcal{F}_4$	$f_0^r(x,t) = 0$ and $f_0^h(x,t) = 0$
$\mathcal{E}_1$	$\mathcal{V}_1$	$f_1^r(x,t)$ free and $f_1^h(x,t) = f_2^h(\mathcal{V}_4, t)$
	$\mathcal{V}_2$	$f_1^r(x,t) = f_1^h(\mathcal{V}_2, t)$ and $f_1^h(x,t)$ free
$\mathcal{E}_2$	$\mathcal{V}_3$	$f_2^r(x,t)$ free and $(v_2^h + v^r(x)) f_2^h(x), t) = \alpha_2^r(x), t) f_3^r(t)$
	$\mathcal{V}_4$	$f_2^r(x,t) = 0$ and $f_2^h(x,t)$ free

Table 4. Boundary conditions