# Not Incompatible Logics


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# Two Incompatible Logics

- Constructivism (BHK)
  - first-order approximation: intuitionistic logic
- Classicism
  - first-order classical logic
- in particuliar, arithmetic:
  - Peano and Heyting versions
  - same axioms, differents inference rules

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  - first-order approximation: intuitionistic logic
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  - first-order classical logic
- in particuliar, arithmetic:
  - Peano and Heyting versions
  - same axioms, differents inference rules
- very pernicious conflict:
  - same syntax, different semantic
  - at least, sound
- two confronting schools for a long time:
  - incompatible properties
  - incompatible persons
- how can we conciliate them ?

# The Root of the Problem

## Disjunction Property

A proof of $\vdash_i A \vee B$ can be turned into a proof of $\vdash_i A$ or a proof of $\vdash_i B$.

- in classical logic $\vdash_c A \vee \neg A$ provable whatever is $A$
- another formulation: $\vdash_c \neg\neg A \Rightarrow A$
- similarly for the $\exists$ quantifier:
  - witness property
  - Drinker's paradox
- lot of solutions
  - depends of what we are expecting
  - as discussed yesterday
  - and may be today

# Double Negation Translations

1. Kolmogorov (1925)

$$\begin{array}{rcll} B^{Ko} & = & \neg\neg B & \text{(atoms)} \\ (B \wedge C)^{Ko} & = & \neg\neg(B^{Ko} \wedge C^{Ko}) & \\ (B \vee C)^{Ko} & = & \neg\neg(B^{Ko} \vee C^{Ko}) & \\ (B \Rightarrow C)^{Ko} & = & \neg\neg(B^{Ko} \Rightarrow C^{Ko}) & \\ (\forall x A)^{Ko} & = & \neg\neg(\forall x A^{Ko}) & \\ (\exists x A)^{Ko} & = & \neg\neg(\exists x A^{Ko}) & \end{array}$$

**Theorem**

$\Gamma \vdash \Delta$ classical provable iff $\Gamma^{Ko}, \neg\Delta^{Ko} \vdash$ intuitionistically provable.

# Double Negation Translations

1. Kolmogorov (1925)
2. Gödel and Gentzen (1931)
   - ⋆ $\vee$ and $\exists$, are the conflicting connective/quantifiers
   - ⋆ leave the rest unchanged

$$
\begin{array}{rcll}
B^{gg} & = & \neg\neg B & \text{(atoms)} \\
(A \wedge B)^{gg} & = & A^{gg} \wedge B^{gg} & \\
(A \vee B)^{gg} & = & \neg(\neg A^{gg} \wedge \neg B^{gg}) & \\
(A \Rightarrow B)^{gg} & = & A^{gg} \Rightarrow B^{gg} & \\
(\forall x A)^{gg} & = & \forall x A^{gg} & \\
(\exists x A)^{gg} & = & \neg\forall x \neg A^{gg} &
\end{array}
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# Double Negation Translations

1. Kolmogorov (1925)
2. Gödel and Gentzen (1931)
   - $\vee$ and $\exists$, are the conflicting connective/quantifiers
   - leave the rest unchanged
3. Glivenko (1929): head negation enough in the propositional case
4. Kuroda (1951): extension to FO: reset after each $\forall$ quantifier

$$
\begin{array}{rcll}
B^{Ku} & = & B & \text{(atoms)} \\
(A \wedge B)^{Ku} & = & A^{Ku} \wedge B^{Ku} & \\
(A \vee B)^{Ku} & = & A^{Ku} \vee B^{Ku} & \\
(A \Rightarrow B)^{Ku} & = & A^{Ku} \Rightarrow B^{Ku} & \\
(\forall x A)^{Ku} & = & \forall x \neg\neg A^{Ku} & \\
(\exists x A)^{Ku} & = & \forall x \; A^{Ku} &
\end{array}
$$

**Theorem**

$\Gamma \vdash \Delta$ classical provable iff $\Gamma^{Ku}, \neg\Delta^{Ku} \vdash$ intuitionistically provable.

# More Refinments

- left intuitionstic and classical sequent rules identical:
  - no need to translate anything on LHS of $\Gamma \vdash_c \Delta$
  - applies to cut-free calculus. Most of the time enough

**LHS**

$$
\begin{aligned}
B^{Ko} &= B \\
(B \wedge C)^{Ko} &= (\neg\neg B^{Ko} \wedge \neg\neg C^{Ko}) \\
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(\forall x A)^{Ko} &= \forall x \neg\neg A^{Ko} \\
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\end{aligned}
$$

**RHS**

$$
\begin{aligned}
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LHS

$$
\begin{aligned}
B^{K+} &= B \\
(B \wedge C)^{K+} &= (B^{K+} \wedge C^{K+}) \\
(B \vee C)^{K+} &= (B^{K+} \vee C^{K+}) \\
(B \Rightarrow C)^{K+} &= (\neg\neg B^{K-} \Rightarrow C^{K+}) \\
(\forall x A)^{K+} &= \forall x A^{K+} \\
(\exists x A)^{K+} &= \exists x A^{K+}
\end{aligned}
$$

RHS

$$
\begin{aligned}
B^{K-} &= B \\
(B \wedge C)^{K-} &= (\neg\neg B^{K-} \wedge \neg\neg C^{K-}) \\
(B \vee C)^{K-} &= (\neg\neg B^{K-} \vee \neg\neg C^{K-}) \\
(B \Rightarrow C)^{K-} &= (B^{K+} \Rightarrow \neg\neg C^{K-}) \\
(\forall x A)^{K-} &= \forall x \neg\neg A^{K-} \\
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- left intuitionstic and classical sequent rules identical:
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  - ★ applies to cut-free calculus. Most of the time enough
- Gilbert: left/right + Kuroda + Gödel-Gentzen.
  - ★ Minimal. End of Story ?

| RHS (gg) | LHS | RHS (Ku) |
|---|---|---|
| $\varphi(P) = \neg\neg P$ | $\chi(P) = P$ | $\psi(P) = P$ |
| $\varphi(B \wedge C) = \varphi(B) \wedge \varphi(C)$ | $\chi(B \wedge C) = \chi(B) \wedge \chi(C)$ | $\psi(B \wedge C) = \psi(B) \wedge \psi(C)$ |
| $\varphi(B \vee C) = \neg\neg(\psi(B) \vee \psi(C))$ | $\chi(B \vee C) = \chi(B) \vee \chi(C)$ | $\psi(B \vee C) = \psi(B) \vee \psi(C)$ |
| $\varphi(B \Rightarrow C) = \chi(B) \Rightarrow \varphi(C)$ | $\chi(B \Rightarrow C) = \psi(B) \Rightarrow \chi(C)$ | $\psi(B \Rightarrow C) = \chi(B) \Rightarrow \psi(C)$ |
| $\varphi(\neg B) = \neg\chi(B)$ | $\chi(\neg B) = \neg\psi(B)$ | $\psi(\neg B) = \neg\chi(B)$ |
| $\varphi(\forall xA) = \forall x\varphi(A)$ | $\chi(\forall xA) = \forall x\chi(A)$ | $\psi(\forall xA) = \forall x\varphi(A)$ |
| $\varphi(\exists xA) = \neg\neg\exists x\psi(A)$ | $\chi(\exists xA) = \exists x\chi(A)$ | $\psi(\exists xA) = \exists x\psi(A)$ |

**Theorem**

$\Gamma \vdash C$ classically iff $\chi(\Gamma) \vdash \varphi(C)$ intuitionistically.

# More Insights

- Chaudhuri, Clerc, Ilik, Miller:
  - bijections between proofs of focused calculi
  - generating a particular translation by choosing a polarity
- Friedman:
  - generalize: replace "¬" with "$\Rightarrow A$" in translations
  - theorem:

**Theorem**

$\Gamma \vdash \Delta$ classical provable iff $\Gamma^{Ku}, \neg\Delta^{Ku} \vdash \bot$ provable.

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  - generalize: replace “$\neg$” with “$\Rightarrow A$” in translations
  - theorem:

**Theorem**

$\Gamma \vdash \Delta$ classical provable iff $\Gamma^A, \Delta^A \Rightarrow A \vdash A$ provable.

  - equiprovability of certain statements ($\Pi^0_2$)
    - require decidability of some class of formulas
    - “Friedman’s trick”: take as $A$ the statement itself.

# Mixed Logics

- "*On the Unity of Logic*", Girard (1993)
- not the *logic* is classical/intuitionistic/...
- ... but the connectives
- problem:
  - ★ usual translations negate atoms (no connective here)

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- problem:
  - usual translations negate atoms (no connective here)
  - "light" translations negate the whole (no connective there either)

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# Mixing Logics

- Dowek's translation goes double

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\begin{aligned}
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- gain: no negated atoms, no negated formulas
- definition of classical connectives and quantifiers

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(\forall_c x A) &= \neg\neg\forall_i x \neg\neg A \\
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- intuitionistic calculus as a basis

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- intuitionistic calculus as a basis
- can be made lighter (De Morgan + Gödel-Gentzen ideas)

# We don't care about theorems

We care about proofs!

- naïve translations look at inference steps
- less naïve translations permute/gather inference rules (cf. focusing)
- ee also Friedman's translation
- that apply to all proofs
- Reverse Mathematics

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- Gilbert's work
  - analyse every proof, encode in Dedukti
  - 54% of Zenon's proofs are constructive
- Cauderlier's work
  - encode in Dedukti, rewrite proofs terms with higher-order rewrite rules
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  - 62% of Zenon’s proofs
- both combined: 82%

# Still Unsatisfied ?

- if we cannot be shallow, go **deeper**!
- express provability in logic X as a first-order theory, reason about it in a constructively

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- if we cannot be shallow, go **deeper**!
- express provability in logic X as a first-order theory, reason about it in a constructively
- Or **change the rules of the game**!

**Definition**

A constructive proof is a proof from which we can extract a program.

- Control operators and classical realizability
- Classical logic is a constructive logic:
  - ★ but can change mind
  - ★ "I say $\neg A / A(0)$ and I defy you to show that I am wrong"