

On the Benjamin-Bona-Mahony Equation with a Localized Damping

Lionel Rosier

▶ **To cite this version:**

Lionel Rosier. On the Benjamin-Bona-Mahony Equation with a Localized Damping. Journal of Mathematical Study, 2016, 49, pp.195 - 204. 10.4208/jms.v49n2.16.06 . hal-01549581

HAL Id: hal-01549581

https://minesparis-psl.hal.science/hal-01549581

Submitted on 28 Jun 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

arXiv:1605.09574v1 [math.AP] 31 May 2016

On the Benjamin-Bona-Mahony equation with a localized damping

Lionel Rosier *

Abstract

We introduce several mechanisms to dissipate the energy in the Benjamin-Bona-Mahony (BBM) equation. We consider either a distributed (localized) feedback law, or a boundary feedback law. In each case, we prove the global wellposedness of the system and the convergence towards a solution of the BBM equation which is null on a band. If the Unique Continuation Property holds for the BBM equation, this implies that the origin is asymptotically stable for the damped BBM equation.

AMS Subject Classification: 35Q53, 93D15.

Keywords: Benjamin-Bona-Mahony equation, unique continuation property, internal stabilization, boundary stabilization.

1 Introduction

The Benjamin-Bona-Mahony equation

$$v_t - v_{xxt} + v_x + vv_x = 0, \tag{1.1}$$

was proposed in [1] as an alternative to the Korteweg-de Vries (KdV) equation as a model for the propagation of one-dimensional, unidirectional, small amplitude long waves in nonlinear dispersive media. In the context of shallow water waves, $v = v(x,t)$ stands for the displacement of the water surface (from rest position) at location x and time t. In the paper, we shall assume that either $x \in \mathbb{R}$, or $x \in (0,L)$ or $x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ (the one-dimensional torus).

The dispersive term $-v_{xxt}$ produces a strong smoothing effect for the time regularity, thanks to which the wellposedness theory of (1.1) is easier than for KdV (see [4, 9]). Solutions of (1.1) turn out to be analytic in time. On the other hand, the control theory is at his early stage for BBM (for the control properties of KdV, we refer the reader to the recent survey [10]). S. Micu investigated in [5] the boundary controllability of the *linearized* BBM equation, and noticed that the exact controllability fails due to the existence of a *limit point* in the spectrum of the adjoint equation. The author and B.-Y. Zhang introduced in [11] a *moving control* and derived with such a control both the exact controllability and the exponential stability of the full BBM equation. For a distributed control with a *fixed support*, the exact controllability of the linearized BBM equation fails, so that the study of the controllability of the full BBM equation seems hard. However, it is reasonable to expect that some stability properties could be derived by incorporating some dissipation in a fixed subdomain or at the boundary. The aim of this paper is to propose several dissipation mechanisms leading to systems for which one has both the global existence of solutions and

*Centre Automatique et Systèmes, MINES ParisTech, PSL Research University, 60 Boulevard Saint-Michel, 75272 Paris Cedex 06, France Email: Lionel.Rosier@mines-paristech.fr.

a nonincreasing H^1-norm. The conclusion that all the trajectories emanating from data close to the origin are indeed attracted by the origin is valid provided that the following conjecture is true:

Unique Continuation Property (UCP) Conjecture: There exists some number $\delta > 0$ such that for any $v_0 \in H^1(\mathbb{T})$ with $\|v_0\|_{H^1(\mathbb{T})} < \delta$, if the solution $v = v(x,t)$ of

$$\begin{cases} v_t - v_{xxt} + v_x + vv_x = 0, & x \in \mathbb{T}, \\ v(x,0) = v_0(x), & x \in \mathbb{T} \end{cases} \tag{1.2}$$

satisfies

$$v(x,t) = 0 \quad \forall (x,t) \in \omega \times (0,T) \tag{1.3}$$

for some nonempty open set $\omega \subset \mathbb{T}$ and some time $T > 0$, then $v_0 = 0$ (and hence $v \equiv 0$).

To the best knowledge of the author, the UCP for BBM as stated in the above conjecture is still open. The main difficulty comes from the fact that the lines $x = 0$ are *characteristic* for BBM, so that the "information" does not propagate well in the x-direction. For some UCP for BBM (with additional assumptions) see [11, 12]. See also [6, 7] for control results for some Boussinesq systems of BBM-BBM type.

The following result is a *conditional* UCP in which it is assumed that the initial data is small in the L^∞-norm and it has a *nonnegative* mean value. Its proof was based on the analyticity in time of the trajectories and on the use of some Lyapunov function.

Theorem 1 *[11] Let $u_0 \in H^1(\mathbb{T})$ be such that*

$$\int_{\mathbb{T}} v_0(x)\,dx \geq 0, \tag{1.4}$$

and

$$\|v_0\|_{L^\infty(\mathbb{T})} < 3. \tag{1.5}$$

Assume that the solution v to (1.2) *satisfies* (1.3)*. Then $v_0 = 0$.*

As it was noticed in [11], the UCP for BBM cannot hold for any state in $L^2(\mathbb{T})$, for any initial data v_0 with values in $\{-2, 0\}$ gives a trivial (stationary) solution of BBM. Thus, either a regularity assumption ($v_0 \in H^1(\mathbb{T})$), or a bound on the norm of the initial data has to be imposed for the UCP to hold.

The paper is outlined as follows. In Section 2, we incorporate a simple localized damping in BBM equation and investigate the corresponding Cauchy problem. In Section 3, we consider another dissipation mechanism involving one derivative. The last section is concerned with the introduction of nonhomogeneous boundary conditions leading again to the dissipation of the H^1-norm.

2 Stabilization of the BBM equation

2.1 Internal stabilization with a simple feedback law

We consider the BBM equation on $\mathbb{T}$ with a localized damping

$$u_t - u_{xxt} + u_x + uu_x + a(x)u = 0 \qquad x \in \mathbb{T},\ t \geq 0, \tag{2.1}$$

$$u(x,0) = u_0(x) \qquad x \in \mathbb{T}, \tag{2.2}$$

where a is a smooth, nonnegative function on $\mathbb{T}$ with $a(x) > 0$ on a given open set $\omega \subset \mathbb{T}$.

We write (2.1)-(2.2) in its integral form

$$u(t) = u_0 - \int_0^t (1 - \partial_x^2)^{-1}[a(x)u + (u + \frac{u^2}{2})_x](\tau)d\tau \tag{2.3}$$

where $(1-\partial_x^2)^{-1}f$ denotes for $f \in L^2(\mathbb{T})$ the unique solution $v \in H^2(\mathbb{T})$ of $(1-\partial_x^2)v = f$. Define the norm $\|\cdot\|_s$ in $H^s(\mathbb{T})$ as

$$\|\sum_{n\in\mathbb{Z}} c_n e^{inx}\|_s^2 = \sum_{n\in\mathbb{Z}} |c_n|^2(1+|n|^2)^s.$$

We have the following result.

Theorem 2 *Let $s \geq 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, there exist $T > 0$ and a unique solution u of (2.3) in $C([0,T], H^s(\mathbb{T}))$. Moreover, the correspondence $u_0 \in H^s(\mathbb{T}) \mapsto u \in C([0,T], H^s(\mathbb{T}))$ is Lipschitz continuous. If $s = 1$, the solution exists for every $T > 0$, and the energy $\|u(t)\|_1^2$ is nonincreasing. Finally, if v_0 denotes any weak limit in $H^1(\mathbb{T})$ of a sequence $u(t_n)$ with $t_n \nearrow +\infty$, then the solution v of system (1.2) satisfies (1.3). In particular, if the UCP conjecture holds, then $v_0 = 0$, so that $u(t) \to 0$ weakly in $H^1(\mathbb{T})$ as $t \to +\infty$, hence strongly in $H^s(\mathbb{T})$ for all $s < 1$.*

Proof. We proceed as in [3] using the estimate

$$||fg||_s \leq C||f||_{s+1}||g||_{s+1} \qquad \forall s \geq -1,\ \forall f, g \in H^{s+1}(\mathbb{T}). \tag{2.4}$$

The estimate (2.4) follows from a similar estimate proved in [3] for functions defined on $\mathbb{R}$, namely

$$||\tilde{f}\tilde{g}||_{H^s(\mathbb{R})} \leq C||\tilde{f}||_{H^{s+1}(\mathbb{R})}||\tilde{g}||_{H^{s+1}(\mathbb{R})} \qquad \forall s \geq -1,\ \forall f, g \in H^{s+1}(\mathbb{R}), \tag{2.5}$$

by letting $\tilde{f}(x) = \varphi(x)f(x)$, $\tilde{g}(x) = \varphi(x)g(x)$, where f and g are viewed as 2π-periodic functions, and $\varphi \in C_0^\infty(\mathbb{R})$ denotes a cut-off function such that $\varphi(x) = 1$ on $[0, 2\pi]$. Indeed, we notice that

$$||f||_{H^s(\mathbb{T})} \leq ||\tilde{f}||_{H^s(\mathbb{R})} \leq C||f||_{H^s(\mathbb{T})}$$

for some constant $C > 0$. Note that for any $s \geq 0$

$$||(1-\partial_x^2)^{-1}\partial_x(fg)||_s \leq C||fg||_{s-1} \leq C||f||_s||g||_s.$$

Pick any $u_0 \in H^s(\mathbb{T})$. Let us introduce the operator

$$(\Gamma u)(t) := u_0 - \int_0^t (1-\partial_x^2)^{-1}[a(x)u + (u + \frac{u^2}{2})_x](\tau)d\tau.$$

Then

$$\begin{aligned}\sup_{0\leq t\leq T} ||(\Gamma u - \Gamma v)(t)||_s &\leq C_2 \int_0^T [||a(u-v)||_{s-2} + ||u-v||_{s-1} + ||u-v||_s||u+v||_s]d\tau \\ &\leq C_3 T(1+2R)||u-v||_{C([0,T],H^s(\mathbb{T}))}\end{aligned}$$

if we assume that u and v are in the closed ball B_R of radius R centered at 0 in $C([0,T], H^s(\mathbb{T}))$. We pick $T := [2C_3(1+2R)]^{-1}$ so that

$$||\Gamma u - \Gamma v||_{C([0,T],H^s(\mathbb{T}))} \leq \frac{1}{2}||u-v||_{C([0,T],H^s(\mathbb{T}))}.$$

On the other hand

$$||\Gamma u||_{C([0,T],H^s(\mathbb{T}))} \leq ||u_0||_s + ||\Gamma u - \Gamma 0||_{C([0,T],H^s(\mathbb{T}))} \leq ||u_0||_s + \frac{R}{2} \leq R$$

for the choice $R = 2||u_0||_s$. It follows that the map Γ contracts in B_R, hence it admits a unique fixed point u in B_R which solves the integral equation (2.3). Furthermore, given any $\rho > 0$ and any $u_0, v_0 \in H^s(\mathbb{T})$ with $||u_0||_s \leq \rho$, $||v_0||_s \leq \rho$, one easily sees that for

$$T = [2C_3(1+4\rho)]^{-1} \tag{2.6}$$

one has

$$||u-v||_{C([0,T],H^s(\mathbb{T}))} \leq 2||u_0-v_0||_s. \tag{2.7}$$

Finally assume that $s=1$. Scaling in (2.1) by u yields

$$\frac{1}{2}||u(T)||_1^2 - \frac{1}{2}||u_0||_1^2 + \int_0^T\int_{\mathbb{T}} a(x)|u(x,t)|^2 dxdt = 0 \tag{2.8}$$

(Note that $u_t = -(1-\partial_x^2)^{-1}[a(x)u + (u+\frac{u^2}{2})_x] \in C([0,T],H^2(\mathbb{T}))$ so that each term in (2.1) belongs to $C([0,T],L^2(\mathbb{T}))$, and the integrations by parts are valid.) It follows that the map $t \mapsto ||u(t)||_1$ is nonincreasing, hence it admits a nonnegative limit l as $t \to \infty$, and that the solution of (2.1)-(2.2) is defined for all $t \geq 0$. Let T be as in (2.6) with $\rho = ||u_0||_1$. Note that $||u(t)||_1 \leq ||u_0||_1$ for all $t \geq 0$. Let v_0 be any weak limit of $\{u(t)\}_{t\geq 0}$ in $H^1(\mathbb{T})$, i.e. we have that $u(t_n) \to v_0$ weakly in $H^1(\mathbb{T})$ for some sequence $t_n \to +\infty$. Extracting a subsequence if needed, we may assume that

$$t_{n+1} - t_n \geq T \qquad \text{for } n \geq 0. \tag{2.9}$$

From

$$\frac{1}{2}||u(t_{n+1})||_1^2 - \frac{1}{2}||u(t_n)||_1^2 + \int_{t_n}^{t_{n+1}}\int_{\mathbb{T}} a(x)|u(x,t)|^2 dxdt = 0,$$

we obtain that

$$\lim_{n\to+\infty}\int_{t_n}^{t_{n+1}}\int_T a(x)|u(x,t)|^2 dxdt = 0. \tag{2.10}$$

Since $u(t_n) \to v_0$ in $H^s(\mathbb{T})$ for $s<1$, and $||u(t_n)||_s \leq ||u(t_n)||_1 \leq \rho$, we have from (2.7) that for all $s \in [0,1[$

$$u(t_n+\cdot) \to v \qquad \text{in } C([0,T],H^s(\mathbb{T})) \qquad \text{as } n \to +\infty, \tag{2.11}$$

where $v = v(x,t)$ denotes the solution of

$$\begin{aligned} &v_t - v_{xxt} + v_x + vv_x + a(x)v = 0, \qquad x \in \mathbb{T},\ t \geq 0,\\ &v(x,0) = v_0(x), \qquad x \in \mathbb{T}. \end{aligned}$$

Notice that $v \in C([0,T];H^1(\mathbb{T}))$, for $v_0 \in H^1(\mathbb{T})$. (2.10) combined to (2.9) and (2.11) yields

$$\int_0^T\int_{\mathbb{T}} a(x)|v|^2 dxdt = 0. \tag{2.12}$$

Therefore $v \in C([0,T];H^1(\mathbb{T}))$ solves

$$\begin{aligned} &v_t - v_{xxt} + v_x + vv_x = 0, \qquad x \in \mathbb{T},\ t \in (0,T),\\ &v(x,0) = v_0(x) \qquad x \in \mathbb{T},\\ &v(x,t) = 0 \qquad x \in \omega,\ t \in (0,T). \end{aligned}$$

If the UCP conjecture is true, we have that $v \equiv 0$ on $\mathbb{R}\times(0,T)$. It follows that $v_0 \equiv 0$, and that as $t \to +\infty$

$$\begin{aligned} &u(t) \to 0 \qquad \text{weakly in } H^1(\mathbb{T}),\\ &u(t) \to 0 \qquad \text{strongly in } H^s(\mathbb{T}) \text{ for any } s<1. \end{aligned}$$

■

2.2 Internal stabilization with one derivative in the feedback law

We now pay attention to the stabilization of the BBM equation by means of a "stronger" feedback law involving one derivative. More precisely, we consider the system

$$u_t - u_{xxt} + u_x + uu_x - (a(x)u_x)_x = 0, \qquad x \in \mathbb{T},\ t \geq 0, \tag{2.13}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{T}. \tag{2.14}$$

where $a = a(x)$ denotes again any nonnegative smooth function on $\mathbb{T}$ with $a(x) > 0$ on a given open set $\omega \subset \mathbb{T}$. Scaling in (2.13) yields (at least formally)

$$\frac{1}{2}||u(T)||_1^2 - \frac{1}{2}||u_0||_1^2 + \int_0^T\int_{\mathbb{T}} a(x)|u_x(x,t)|^2 dxdt = 0. \tag{2.15}$$

The decay of the energy is quantified by an integral term involving the square of a localized H^1-norm in (2.15). By contrast, the integral term in (2.8) involved the square of a localized L^2-norm. This suggests that the damping mechanism involved in (2.13) acts in a *much stronger way* than in (2.1). As a matter of fact, in the trivial situation when $a(x) \geq C > 0$ for all $x \in \mathbb{T}$ in (2.13), it is a simple exercise to establish the exponential stability in $H^1(\mathbb{T})$ for both the linearized equation and the nonlinear BBM equation for states with zero means. In the general case when the function $a(x)$ is supported in a subdomain of $\mathbb{T}$, we obtain the following result.

Theorem 3 *Let $s \geq 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, there exist $T > 0$ and a unique solution u of (2.13)-(2.14) in $C([0,T], H^s(\mathbb{T}))$. Moreover, the correspondence $u_0 \in H^s(\mathbb{T}) \mapsto u \in C([0,T], H^s(\mathbb{T}))$ is Lipschitz continuous. If $s = 1$, one can pick any $T > 0$ and $\|u(t) - [u_0]\|_{H^1(\mathbb{T})}$ is nondecreasing. If the UCP conjecture is valid by replacing (1.3) by $v = C$ on $\omega \times (0,T)$, then $u(t) \to [u_0] = (2\pi)^{-1}\int_{\mathbb{T}} u_0(x)dx$ weakly in $H^1(T)$ hence strongly in $H^s(\mathbb{T})$ for $s < 1$ as $t \to \infty$.*

Proof. As the proof is very similar to those of Theorem 2, we limit ourselves to pointing out the only differences. For the wellposedness, we use the estimate valid for $s \geq 0$

$$||(1 - \partial_x^2)^{-1}(au_x)_x||_s \leq C||u||_s.$$

For $s = 1$, we claim that (2.15) is justified by noticing that for $u \in H^1(\mathbb{T})$

$$\langle -(au_x)_x, u\rangle_{H^{-1},H^1} = \langle au_x, u_x\rangle_{L^2,L^2}.$$

Then the wellposedness statement is established as in Theorem 2. Let us proceed to the asymptotic behavior. Pick any $u_0 \in H^1(\mathbb{T})$ and any $v_0 \in H^1(\mathbb{T})$ which is the weak limit in $H^1(\mathbb{T})$ of a sequence $u(t_n)$ with $t_n \to \infty$ and $t_{n+1} - t_n \geq T$. Let us still denote by v the solution of (1.2). Equation (2.12) has to be replaced by

$$\int_0^T\int_{\mathbb{T}} a(x)|v_x(x,t)|^2 dxdt = 0. \tag{2.16}$$

To prove (2.16), we notice first that $u(t_n + \cdot) \to v$ in $C([0,T], H^s(\mathbb{T}))$ for $s < 1$, and that $||u(t_n + \cdot)||_{L^2(0,T;H^1(\mathbb{T}))} \leq const$, so that, extracting a subsequence if needed, we have that

$$u(t_n + \cdot) \to v \qquad \text{weakly in } L^2(0,T;H^1(\mathbb{T})).$$

This yields, with (2.15),

$$\int_0^T\int_{\mathbb{T}} a(x)|v_x|^2 dxdt \leq \liminf_{n\to\infty}\int_{t_n}^{t_n+T}\int_{\mathbb{T}} a(x)|u_{nx}|^2 dxdt = 0.$$

Therefore v solves

$$v_t - v_{xxt} + v_x + vv_x = 0, \qquad x \in \mathbb{T},\ t \in (0,T), \tag{2.17}$$
$$v_x = 0, \qquad x \in \omega,\ t \in (0,T), \tag{2.18}$$
$$v \in C([0,T]; H^s(\mathbb{T})) \qquad \text{for } s < 1. \tag{2.19}$$

(2.17) and (2.18) yield $v_t = v_x = 0$ in $\omega \times (0,T)$, hence

$$v(x,t) = C \qquad \text{for } (x,t) \in \omega \times (0,T)$$

for some constant $C \in \mathbb{R}$. If the UCP Conjecture is still valid when v is constant on the band $\omega \times (0,T)$, then $v \equiv C$. As $d[u(t)]/dt = 0$, it follows that $C = [u_0]$ and that as $t \to \infty$, $u(t) \to [u_0]$ weakly in $H^1(\mathbb{T})$ and strongly in $H^s(\mathbb{T})$ for any $s < 1$. ■

Remark 1 *Similar results, but with convergences towards 0, hold for the system*

$$\begin{aligned} &u_t - u_{xxt} + u_x + uu_x - (a(x)u_x)_x = 0, \\ &u(0,t) = u(2\pi,t) = 0, \\ &u(x,0) = u_0(x) \end{aligned}$$

provided that $a(x) > 0$ for $x \in (0,\varepsilon) \cup (2\pi - \varepsilon, 0)$ for some $\varepsilon > 0$ and the UCP Conjecture holds.

2.3 Boundary stabilization of BBM

In this section, we are concerned with the boundary stabilization of the BBM equation. We consider the following system

$$u_t - u_{xxt} + u_x + uu_x = 0, \qquad x \in (0,L),\ t \geq 0, \tag{2.20}$$
$$u_{tx}(0,t) = \alpha u(0,t) + \frac{1}{3}u^2(0,t), \tag{2.21}$$
$$u_{tx}(L,t) = \beta u(L,t) + \frac{1}{3}u^2(L,t), \tag{2.22}$$
$$u(x,0) = u_0(x), \tag{2.23}$$

where α and β are some real constants chosen so that

$$\alpha > \frac{1}{2} \quad \text{and} \quad \beta < \frac{1}{2}.$$

Scaling in (2.20) by u yields (at least formally)

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_0^L (|u|^2 + |u_x|^2)dx &= [uu_{tx}]_0^L - \left[\frac{u^2}{2} + \frac{u^3}{3}\right]_0^L \\ &= (\beta - \frac{1}{2})|u(L,t)|^2 + (\frac{1}{2} - \alpha)|u(0,t)|^2, \end{aligned} \tag{2.24}$$

hence $||u(t)||_{H^1}$ is nonincreasing if α and β fulfill (2.3). The wellposedness of (2.20)-(2.23) and the asymptotic behavior are described in the following result.

Theorem 4 *Let $s \in (1/2, 5/2)$ and $u_0 \in H^s(0,L)$. Then there exist a time $T > 0$ and a unique solution $u \in C([0,T], H^s(0,L))$ of (2.20)-(2.23). Furthermore, if $s = 1$, then T may be taken arbitrarily large, and if the UCP Conjecture holds, as $t \to \infty$*

$$u(t) \to 0 \quad \text{weakly in } H^1(0,L) \tag{2.25}$$
$$u(t) \to 0 \quad \text{strongly in } H^s(0,L) \text{ for all } s < 1. \tag{2.26}$$

Proof. Let us begin with the well-posedness part. Pick any $u_0 \in H^s(0,L)$ with $s > \frac{1}{2}$. Let $v = u_t$. Then v solves the elliptic problem

$$(1-\partial_x^2)v = f, \quad x \in (0,L), \tag{2.27}$$
$$v_x(0) = a,\ v_x(L) = b \tag{2.28}$$

with $f := -u_x - uu_x$, $a := \alpha u(0,t) + \frac{1}{3}u^2(0,t)$, $b := \beta u(L,t) + \frac{1}{3}u^2(L,t)$. Note that the solution v of (2.27)-(2.28) may be written as

$$v = w + g$$

where $g(x) = ax + \frac{b-a}{2L}x^2$ and $w = (1-\partial_x^2)_N^{-1}(f - (1-\partial_x^2)g)$ solves

$$(1-\partial_x^2)w = f - (1-\partial_x^2)g$$
$$w_x(0) = w_x(L) = 0.$$

Thus

$$u_t = v = -(1-\partial_x^2)_N^{-1}(u_x + uu_x) + (1 - (1-\partial_x^2)_N^{-1}(1-\partial_x^2))g. \tag{2.29}$$

We seek u as a fixed point of the integral equation

$$\begin{aligned} u(t) = \Gamma(u)(t) \quad := \quad & u_0 + \int_0^t \Big\{ -(1-\partial_x^2)_N^{-1}(u_x + uu_x)(\tau) \\ & +(1-(1-\partial_x^2)_N^{-1}(1-\partial_x^2))\Big[[\alpha u(0,\tau) + \frac{1}{3}u^2(0,\tau)]x \\ & +(2L)^{-1}[\beta u(L,\tau) + \frac{1}{3}u^2(l,\tau) - \alpha u(0,\tau) - \frac{1}{3}u^2(0,\tau)]x^2\Big]\Big\} d\tau. \end{aligned} \tag{2.30}$$

Note that $\mathcal{D}((1-\partial_x^2)_N^{\frac{s}{2}}) = H^s(0,L)$ for $-1/2 < s < 3/2$. Let $R > 0$ be given and let B_R denote the closed ball in $C([0,T], H^s(0,L))$ of center 0 and radius R. For $1/2 < s < 5/2$ and $u \in B_R, v \in B_R$, we have that

$$\begin{aligned} ||\Gamma(u)(t) - \Gamma(v)(t)||_{H^s(0,L)} &\leq C\int_0^t (1+R)||u(\tau) - v(\tau)||_{H^s(0,L)}d\tau \\ &\leq CT(1+R)||u-v||_{C([0,T],H^s(0,L))} \end{aligned}$$

and

$$||\Gamma(u)(t)||_{H^s(0,L)} \leq ||u_0||_{H^s(0,L)} + CT(1+R)||u||_{C([0,T],H^s(0,L))}.$$

Pick $R = 2||u_0||_{H^s(0,L)}$ and $T = (2C(1+R))^{-1}$. Then Γ is a contraction in B_R, hence it admits a unique fixed point which solves (2.30), or (2.29) and (2.23).

Assume now that $s = 1$. It follows from (2.29) that $u_t \in C([0,T], H^2(0,L))$, hence we can scale by u in (2.20) to derive (2.24). Thus $||u(t)||_{H^1(0,L)}$ is nonincreasing, and T may be taken as large as desired. Let us turn our attention to the asymptotic behavior. Let $(t_n)_{n\geq 0}$ be any sequence with $t_n \to \infty$ as $n \to \infty$. Extracting a subsequence if needed, we can assume that $t_{n+1} - t_n \geq T$ for each n and that, for some $v_0 \in H^1(0,L)$, $u(t_n) \to v_0$ weakly in $H^1(0,L)$ (hence strongly in $H^s(0,L)$ for $s < 1$) as $n \to \infty$. The continuity of the flow map (which follows at once from the fact that the map Γ is a contraction) yields

$$u(t_n + \cdot) \to v(\cdot) \qquad \text{in } \ C([0,T], H^s(0,L)) \quad \text{for } s < 1,$$

where v denotes the solution of (2.20)-(2.23) issued from v_0 at $t = 0$. Integrating (2.24) on $[t_n, t_{n+1}]$ yields

$$\frac{1}{2}||u(t_{n+1})||^2_{H^1(0,L)} - \frac{1}{2}||u(t_n)||^2_{H^1(0,L)} + (\frac{1}{2} - \beta)\int_{t_n}^{t_{n+1}} |u(L,t)|^2 dt + (\alpha - \frac{1}{2})\int_{t_n}^{t_{n+1}} |u(0,t)|^2 dt.$$

Letting $n \to \infty$ yields

$$(\frac{1}{2} - \beta) \int_0^T |v(L,t)|^2 dt + (\alpha - \frac{1}{2}) \int_0^T |v(0,t)|^2 dt = 0. \tag{2.31}$$

Extending $v(x,t)$ by 0 for $x \in \mathbb{R} \setminus (0,L)$ and $t \in (0,T)$, we infer from (2.20), (2.21), (2.22) (for v) and (2.31) that

$$v_t - v_{txx} + v_x + vv_x = 0 \qquad \text{for } x \in \mathbb{R},\ t \in (0,T)$$

with

$$v(x,t) = 0 \qquad \text{for } x \in \mathbb{R} \setminus (0,L),\ t \in (0,T).$$

Since $v \in C([0,T], H^s(\mathbb{R}))$ with $1/2 < s < 1$, we infer from the UCP Conjecture (if true) that $v \equiv 0$, hence $v_0 = 0$. ■

References

[1] T.B. Benjamin, J.L. Bona, J.J. Mahoney, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. R. Soc. Lond. **272** (1972) 47–78.

[2] J.L. Bona, M. Chen, J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory*, J. Nonlinear Sci. **12** (2002), 282-318.

[3] J.L. Bona, M. Chen, J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II. Nonlinear theory*, Nonlinearity, **17** (2004), 925–952.

[4] J.L. Bona, N. Tzvetkov, *Sharp well-posedness for the BBM equation*, Discrete Contin. Dyn. Syst. **23** (4) (2009) 1241–1252.

[5] S. Micu, *On the controllability of the linearized Benjamin-Bona-Mahony equation*, SIAM J. Control Optim., **39** (6) (2001) 1677–1696.

[6] S. Micu, J.H. Ortega, L. Rosier, B.-Y. Zhang, *Control and stabilization of a family of Boussinesq systems*, Discrete Contin. Dyn. Syst. **24** (2009), no. 2, 273–313.

[7] S. Micu, A.F. Pazoto, *Stabilization of a Boussinesq system with localized damping*, Journal d'Analyse Mathématique, to appear.

[8] L. Rosier, *Exact controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var. **2** (1997) 33–55.

[9] D. Roumégoux, *A symplectic non-squeezing theorem for BBM equation*, Dyn. Partial Differ. Equ. **7** (4) (2010) 289–305.

[10] L. Rosier, B.-Y. Zhang, *Control and stabilization of the Korteweg-de Vries equation: recent progresses*, J. Syst. Sci. Complex. **22** (2009), no. 4, 647–682.

[11] L. Rosier, B.-Y. Zhang, *Unique continuation property and control for the Benjamin-Bona-Mahony equation*, J. Differential Equations **254** (2013) 141–178.

[12] M. Yamamoto, *One unique continuation for a linearized Benjamin-Bona-Mahony equation*, J. Inverse Ill-Probl. **11** (5) (2003) 537–543.