Kalman filtering with a class of geometric state equality constraints
Paul Chauchat, A Barrau, S Bonnabel

To cite this version:
Paul Chauchat, A Barrau, S Bonnabel. Kalman filtering with a class of geometric state equality constraints. 2017. hal-01580569v2

HAL Id: hal-01580569
https://hal-mines-paristech.archives-ouvertes.fr/hal-01580569v2
Submitted on 1 Sep 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Kalman filtering with a class of geometric state equality constraints

P. Chauchat, A. Barrau, S. Bonnabel

Abstract—In this paper we consider a noise free class of dynamics encompassing left- and right-invariant on a Lie group with noisy partial state measurements. We assume in addition that the covariance matrix of the state is initially rank-deficient. This, combined with the absence of process noise, keeps the system state within a (time-dependent) subset of the state space at all times. We prove mathematically that the invariant extended Kalman filter (IEKF) perfectly respects this kind of state constraints, contrarily to the standard EKF, or the unscented Kalman filter. This is a strong indication that the IEKF is particularly well suited to navigation when motion sensors are highly precise. The theory is applied to a non-holonomic car example on $SE(2)$, and to an attitude estimation example on $SO(3)$.

I. INTRODUCTION

The Kalman filter (KF) and its extended version (EKF) have appeared in the 1960s, and played a big role in the guidance of spacecrafts during the space age. It has been the state of the art for industrial applications since the 1960s, notably for navigation. However, due to the nonlinear nature of the navigation equations, and in particular to the fact the orientation of the aircraft (i.e., the attitude) does not live in a vector space, the EKF may have some shortcomings. This has motivated the development of alternative filters, especially for attitude estimation, see e.g., [12], [19], [6], [15], [13], [10], [16], [20], [7].

The Invariant Extended Kalman Filter (IEKF) is a relatively recent variant of the EKF meant to account for the nonlinearities of the state space when devising EKFs on Lie groups, see [9], [8], [14], [2]. As such, it can be viewed as a variant of the multiplicative extended Kalman filter (MEKF) [11] for attitude estimation, and as an extension to it for more general state spaces. The two main arguments that advocate its use over other EKF variants are 1- its (local) guaranteed convergence properties [4], and 2- the fact it solves the well-studied inconsistency issues of the EKF for simultaneous localization and mapping (SLAM) and competes with state of the art SLAM algorithms, see [1].

The object of the present paper is to derive a novel general theory that advocates the use of the IEKF for attitude estimation, and more generally state estimation in navigation, when using very precise motion sensors. Indeed, the IEKF has been shown to literally outperform the EKF when the process noise is very small in simulations [4], and patented industrial results have shown great improvement over existing method for the so-called inertial measurement unit (IMU) alignment problem [5]. To be able to derive mathematical result, we need to consider the limit case where the motion sensors are ideal, that is, noise free, and when we have as well some prior deterministic information on the state (that is, the initial covariance matrix is rank deficient). Even if the mathematical guarantees obtained below do not strictly apply to the case of highly precise - but not perfect - sensors, they provide a strong indication that the IEKF is particularly suited for this setting, as previously observed [4], [5]. However, the impact of a rank-deficient initial covariance matrix was only studied in the very particular case of a non-holonomic car with GPS measurements [3].

In the present paper we consider mixed invariant dynamics on a Lie group. We assume the motion is noise free (i.e., no process noise) and the measurements are noisy. We furthermore assume the covariance matrix of the initial state to be rank deficient. Combined with noise-free dynamics this implies that at all times 1- the covariance matrix is rank deficient and 2- the state can only “reach” a well-characterized submanifold of the state space, that is, the “physical” state space is constrained at all times. Unfortunately, due to the fact it is based on approximations, the EKF (and the other Kalman variants such as the unscented Kalman filter) fail to capture those constraints. On the other hand, the IEKF embraces the Lie group structure of the state space, and is shown to perfectly ensure properties 1 and 2 above. Thus, the present paper allows to theorize and generalize our previous results on a non-holonomic car example in [3], where properties 1 and 2 have been shown “manually” on the studied example, and then leveraged to derive some global convergence properties of the filter for that example.

There has been prior authoritative work on Kalman filtering with state constraints, see [18], [17]. Indeed, in case of state constraints various methods exist to make the KF or the EKF respect them, ranging from the addition of perfect measurements to the addition of a projection step and probability density function truncation. But to our best knowledge, nothing has been done in this direction when the state constraints derive from symmetries. To this respect one could object it suffices to devise a state constrained EKF instead of using an IEKF. This might work, but then the obtained filter cannot be used with very small process noise, whereas the IEKF is still shown to outperform the EKF then [4] (but there is no more state constraint to preserve).

The paper is organized as follows. Section II presents the implications of deterministic dynamics in the linear case, and how the KF naturally encodes this information. Section III
describes the counterparts for systems devised on Lie groups and concisely summarizes the general derivation of the IEKF equations. Section IV contains the main theorem of this work stating the Invariant EKF naturally encodes the considered constraints. Finally, the implications of the theorem are illustrated on two examples, the non-holonomic car from [3] and an attitude estimation example.

II. LINEAR KALMAN FILTERING WITH INITIAL STATE CONSTRAINT

We recall in this section a known property of linear Kalman filtering regarding its ability to handle partially deterministic information if no process noise is added. Although theoretical, this limit case is pivotal to filter robustness as will be illustrated on a simple example.

A. Considered system

Consider a classical continuous-discrete linear system in $\mathbb{R}^p$:

$$\begin{align}
X_t &= A_t X_t, \\
Y_t &= C_t X_n + V_n,
\end{align}$$

where $X_t \in \mathbb{R}^p$ is the state of the system at time $t$, $(t_n)_{n \geq 0}$ the sequence of observation times, $(Y_n)_{n \geq 0}$ the corresponding observations and $V_n$ a Gaussian noise polluting observation $n$. $A_t$ and $C_n$ are the matrices defining the dynamics of the system and the function of the system observed through $Y_t$, respectively. Note that process noise has been deliberately removed in Equation (1).

We also make the additional assumption that the initial distribution of the state lies in an affine subspace $V_0$ of $\mathbb{R}^p$:

$$\exists a \in \mathbb{R}^p, X_0 \in a + V_0.$$  (3)

It can then be immediately deduced from (3) that $X_t$ lives in a known affine subspace at any time $t$:

$$X_t \in F_t a + F_t V_0,$$  (4)

where $F_t$ is the matrix solution of the equation $F_0 = I_p$, $\frac{d}{dt} F_t = A_t F_t$ and the vector space defined as $F_t V_0 = \{F_t x, x \in V_0\}$. We will see now that the constraint (4) is automatically met, by the estimate of a Kalman filter.

Remark 1: In the framework of classical Kalman theory, system (1)-(2) is pathological, inasmuch as the state propagation is deterministic. This limit case is intended to give insight into the way a filter handles initial hard constraints such as (3) when the process noise is not sufficient to balance the ill-conditioning of the initial covariance matrix. Section V will show this situation can be extremely troublesome for non-linear systems.

B. The Linear Kalman Filter preserves the additional information

Initial linear information of the form (3) on a linear system is flawlessly captured by a Kalman filter (KF), as illustrated by the following proposition.

Proposition 1: Let $\hat{X}_0$ and $P_0$ be respectively the initial estimate and covariance matrix of a Kalman Filter tracking System (1)-(2), and assume they are consistent with condition (3) in the following sense:

$$\hat{X}_0 \in a + V_0, \quad H_0 P_0 H_0^T = 0_{p \times p},$$  (5)

where matrix $H_0$ is the orthogonal projection over the orthogonal complement of subspace $V_0$. Then, the state $\hat{X}_t$ and covariance matrix $P_t$ returned at any time by the Kalman filter are consistent with (4), i.e., we have:

$$\hat{X}_t \in a + V_t, \quad H_t P_t H_t^T = 0_{p \times p}.$$  (6)

Proof: Between two measurements, the Ricatti equation $\frac{d}{dt} P_t = A_t P_t + P_t A_t^T$ implies $H_t P_t H_t^T = 0$. Before an update, as $P_0$ is symmetric, $H_0 P_0 = 0$ necessarily, and thus $H_0 K_0 = 0$. In turn, this implies $H_0 P_t H_0^T = 0$ and that $x_0^+$ remains in the subspace, as the update writes $P_t^+ = (I - K_n C_n) P_t$ and $\hat{x}_t = \hat{x}_t + K_n(Y_n - C_n \hat{x}_t)$.

Conditions (5) and (6) are easily interpreted: initial error covariance over a direction orthogonal to $V_0$ (resp. $V_t$) is zero at time 0 (resp. $t$). Thus, Proposition 1 implies that at all times the estimate of the Kalman filter remains in the subspace the state lives in (if initialized in $V_0$), and $P_t$ keeps reflecting the absence of dispersion of the probability distribution of the state orthogonally to this subspace.

C. Non-linear case

In the non-linear case, the initial subset in which the state lives may not be a vector space. But even if it is, it is distorted by the dynamics. Therefore, linearizations do not lead to updates that remain in that space. In turn, this leads the EKF to degraded performance, even in the presence of small process noise, as illustrated in the simulations of [3] and [4].

The aim of the present paper is to show that although the property above, along with Proposition 1, seems to be reserved for linear systems, it has in fact a counterpart for a class of non-linear dynamics and carries over to the Invariant EKF. This was already proved “manually” for a particular example in [3]. The results of the latter paper will prove to be a particular case of the general theory developed herein.

III. INVARIANT KALMAN FILTERING WITH GEOMETRIC CONSTRAINTS

In this section, the general results will systematically be illustrated by an attitude estimation example, to help the reader grasp the theoretical ideas and concepts.

A. A short primer on matrix Lie groups

A matrix Lie group $G$ is a subset of square invertible $N \times N$ matrices $\mathcal{M}_N(\mathbb{R})$ verifying the following properties:

$$\text{Id} \in G, \quad \forall g \in G, g^{-1} \in G, \quad \forall a, b \in G, ab \in G.$$  (7)

If $\gamma(t)$ is a curve over $G$ with $\gamma(0) = \text{Id}$, then its derivative at $t = 0$ necessarily lies in a subset $g$ of $\mathcal{M}_N(\mathbb{R})$. $g$ is a vector space, called the Lie algebra of $G$ and has same dimension as $G$. Thanks to a linear map from $\mathbb{R}^{\dim g} \rightarrow g$ denoted by $\xi \rightarrow \xi^*$, one can advantageously identify $g$ to $\mathbb{R}^q$ where $q = \dim G$. Besides, the vector space $g$ can be mapped to the matrix Lie group $G$ through the classical matrix exponential $\exp_m$. Thus, $\mathbb{R}^q$ can be mapped to $G$ through the Lie exponential
map defined by \( \exp(\xi) := \exp_{a}(\xi) \) for \( \xi \in \mathbb{R}^{q} \). We have thus \( \exp(\xi) = I + \xi + O(\xi^{2}) \in \mathcal{M}_{N}(\mathbb{R}) \).

B. Dynamics on a Lie group with equivariant state equality constraints

1) Considered nonlinear system: We first introduce a subclass of systems of [4]. Consider the following deterministic dynamics on a matrix Lie group \( G \subset \mathbb{R}^{N \times N} \):

\[
\frac{d}{dt} X_{t} = X_{t} u_{t} + v_{t} X_{t},
\]

where \( X_{t} \in G \) is the state space, \( u_{t} \) and \( v_{t} \) are processes taking values in the Lie algebra \( g \).

Note that, although the system might “look” linear, (7) are non-linear dynamics. Indeed, linearity may not even make sense in this context, as \( G \) is usually not a vector space.

2) Geometric constraints: Suppose that initially the state is known to satisfy \( k \) equivariant constraints of the form

\[
\chi_{0} b^{i} = c^{i},
\]

for some \( (b^{i}, c^{i})_{1 \leq i \leq k} \subset \mathbb{R}^{N} \). That is, the system verifies a multiplicative counterpart of (3), as (8) is equivalent to

\[
\chi_{0} \in a \cdot G_{0} = \{ a x, x \in G_{0} \},
\]

where \( a \) is an element of \( G \) verifying \( a \cdot b^{i} = c^{i} \) and \( G_{0} \) is the stabilizer subgroup of \( b^{i} \) with respect to \( b^{i} \), i.e., \( G_{0} \in \{ x, x b = b \} \). Now, we derive in Proposition 2 the multiplicative counterpart of condition (4) of the linear case.

Proposition 2: Consider the dynamics (7) with initial condition (8). For all \( 1 \leq i \leq k \), let \( b_{t}^{i} \) and \( c_{t}^{i} \) be defined by the differential equations

\[
\begin{align*}
\dot{b}_{t}^{i} &= b^{i} - u_{t} b_{t}^{i} , \\
\dot{c}_{t}^{i} &= c^{i} + v_{t} c_{t}^{i} .
\end{align*}
\]

Then at all times we have necessarily

\[
\forall t \geq 0, \, \chi_{t} \in \{ \chi \in G \mid v_{t}, \chi b^{i} = c^{i} \} .
\]

Proof: It is straightforward to see that, for all \( t \),

\[
\frac{d}{dt}(\chi_{t} b_{t}^{i} - c_{t}^{i}) = u_{t}(\chi_{t} b_{t}^{i} - c_{t}^{i}).
\]

This linear system being equal to zero at \( t = 0 \) and linear, it is identically zero, which leads to \( \forall t, \chi_{t} b_{t}^{i} = c_{t}^{i} \).

C. Illustration in terms of attitude estimation

As announced, let us present now a specific case of (7) modeling an attitude estimation problem. The state is represented by a rotation matrix \( R_{t} \in SO(3) \), which maps the coordinates of a vector expressed in the body frame to those in the static frame. Letting \( \omega \) denote the perfectly measured angular velocity, the dynamics read:

\[
\frac{d}{dt} R_{t} = R_{t}(\omega_{t})_{\times},
\]

with \((a)_{\times}\) the skew symmetric matrix associated to vector \( a \).

A geometric constraint of the form of (8) for this system can mean that, when initialising, the vehicle was able to measure in its frame a known vector with certainty, say the direction of a distant star thanks to a high-definition camera. Denoting by \( s_{fixed} \) and \( s_{0} \) the direction of the star in the fixed and the initial mobile frame respectively, this reads:

\[
R_{0}^{T} s_{fixed} = s_{0} \Leftrightarrow R_{0} s_{0} = s_{fixed}
\]

Thus, Proposition 2 states that the true system always knows the true direction of the star, i.e., satisfies, for \( s_{t} \), such that \( \frac{d}{dt} s_{t} = -(\omega_{t})_{\times} s_{t} \),

\[
\forall t, \, R_{t}^{T} s_{fixed} = s_{t} .
\]

D. Invariant filtering on a Lie group

Consider a system with equivariant constraints such as in Section III-B, and left-equivariant noisy output \( Y^{L} \):

\[
Y_{n}^{L} = \chi_{n} \cdot d + V_{n},
\]

where \( d \in \mathbb{R}^{p} \) and the \( V_{n} \)’s are Gaussian independent noises. Of course the output can consist of several measurements of this type at each \( t_{n} \) considering various vectors \( d_{1}, d_{2}, \cdots \).

1) Review of the L-IEKF equations for this system: The systems described in this section are suitable for the design of a Left-Invariant EKF (LIEKF), see e.g., [4]. It is defined through the usual propagation and update sequence. Assume discrete observations at times \( t_{n}, n \geq 0 \), then it writes:

\[
\begin{align*}
\frac{d}{dt} \hat{z}_{t} &= \hat{z}_{t} u_{t} + v_{t} \hat{z}_{t} , & \forall n-1 \leq t < t_{n} \quad \text{Propagation} \ (16) \\
\hat{z}_{n}^{+} &= \hat{z}_{n} \exp \left[ K_{n} (\hat{z}_{n}^{+} Y_{n} - d) \right] \quad \text{LIEKF Update} \ (17)
\end{align*}
\]

where the function \( K_{n} : \mathbb{R}^{N} \rightarrow \mathbb{R}^{q} \) is defined through linearizations as in the conventional EKF theory, using the following left-invariant state estimation error:

\[
\eta_{n} = \chi_{n}^{-1} \hat{z}_{n} .
\]

Such a non-linear error \( \eta_{n} \) can be associated to a vector \( \zeta_{n} \in \mathbb{R}^{q} \) such that \( \eta_{n} = \exp \zeta_{n} \). This form, along with the extensive use of the first-order linearization \( \eta_{n} \approx I d + \zeta_{n} \), allow us to derive the equations defining the observer’s covariance, its propagation, and the gain used in (17). Indeed the linearizations of the left-invariant error system associated to (16), and (17) write:

\[
\begin{align*}
\frac{d}{dt} \zeta_{n} &= A_{f} \zeta_{n} \quad \text{(19)} \\
\zeta_{n}^{+} &= \zeta_{n} + K_{n} (C_{n} \zeta_{n} + V_{n}) \quad \text{(20)}
\end{align*}
\]

where \( V_{n} = \hat{z}_{n}^{-1} \) represents the observation noise, \( A_{f} \) is the map defined by \( (A_{f} \zeta)_{\times} = \zeta_{\times} - u_{t} \zeta_{\times} \) (see indeed Theorem 2 of [4]) and \( C_{n} \) is the matrix defined by \( C_{n} \zeta = -\zeta_{\times} d \). Linearizations leading to (19)-(20) are detailed in the proof below.

The state error covariance \( P_{t} \) output by the IEKF is an approximation to \( E(\zeta^{T}_{n} \zeta^{\dagger}_{n}) \). Thus Equations (19) and (20) lead to the following Kalman gain and covariance updates, where \( \hat{N} \) denotes the covariance matrix of the observation noise \( \hat{V}_{n} \):

\[
\begin{align*}
P_{t} &= A_{f} P_{t} + P_{t} A_{f}^{T} \quad \text{(21)} \\
S_{n} &= C_{n} C_{n}^{T} + \hat{N}_{n} \quad \text{(22)} \\
P_{n}^{+} &= (I - K_{n} C_{n}) P_{n} \quad \text{(23)}
\end{align*}
\]
IV. Mathematical results

In this section, we show our main result, stating that the equivariant equality constraints defined by Equation (8) are propagated by the LIEKF the same way as in Proposition 2, without having to incorporate them in the filter as hard constraints like (artificial) perfect measurements [17].

**Theorem 1**: Consider the LIEKF described by (16) and (17), associated to the dynamics (7), with initial equality constraint (8). It implies the constraint (11) at all times. Now, note that \( \xi_{t} b_{t} = c_{t} \), for \( t \leq t_{n+1} \), and \( H_{n-1} P_{n} H_{n-1}^{T} = 0 \) (25).

Proof: Let \( T_{n} \) denote the condition \( H_{n} P_{n} H_{n}^{T} = 0 \). To prove the theorem, it is enough to show the following four implications:

1. \( \eta_{n} b_{n} = b_{n} \Rightarrow \eta_{t} b_{t} = b_{t} \), for \( t_{n} \leq t < t_{n+1} \)
2. \( T_{n} \Rightarrow T_{n} \), for \( t_{n} \leq t < t_{n+1} \)
3. \( T_{n} \Rightarrow T_{n+1} \) for \( n > 0 \)
4. \( \eta_{n} b_{n} = b_{n} \) \( \wedge T_{n} \Rightarrow \eta_{n} b_{n} = b_{n} \)

Proof of (i): As the considered system has deterministic dynamics, this directly comes from Proposition 2 and (16).

Proof of (ii): We have \( T_{n} = [\eta]^{-1} \). Linearizing, it follows that \( \xi_{n} b_{n} = c_{n} \), and \( H_{n} \phi_{n} \xi_{n} \), for all \( \eta \). Recap all that \( \frac{d}{dt} P_{t} = A_{t} P_{t} + P_{t} A \). As \( P_{t} \) is symmetric, we can write \( P_{t} = Q_{t} Q_{t}^{T} \), where \( \frac{d}{dt} Q_{t} = A_{t} Q_{t} \). We will then prove that \( H_{t} Q_{t} \) is identical zero, which implies \( H_{t} P_{t} H_{t}^{T} = 0 \) as wanted. We have:

\[
\frac{d}{dt} H_{t} Q_{t} = (H_{t} + H_{t} A_{t}) Q_{t}
\]

From the definitions of \( H_{t} \) and \( A_{t} \), we get that for all \( \xi \), \( H_{t} \xi = -\xi \xi u_{t} - H_{t} A_{t} \xi = (\xi u_{t} - u_{t} \xi \xi) b_{t} \). Finally, \( (H_{t} + H_{t} A_{t}) \xi = -u_{t} \xi \xi b_{t} = -u_{t} H_{t} \xi \)

Replacing \( \xi \) by the columns of \( Q_{t} \), we get \( \frac{d}{dt} [H_{t} Q_{t}] = -u_{t} [H_{t} Q_{t}] \), a linear system initialized at 0. Thus, it is identical zero, which proves \( T_{n+1} \Rightarrow T_{n} \) for \( t_{n} \leq t < t_{n+1} \).

Proof of (iii): Equation (17) rewrites in terms of error:

\[
\eta_{n}^{+} = \eta_{n} \exp [K_{n}(\eta_{n}^{-1} d - d + V_{n})]
\]

Linearizing then leads to \( \xi_{n}^{+} = \xi_{n} + K_{n}(\eta_{n}^{-1} d - d + V_{n}) \approx \xi_{n} + K_{n}(\xi_{n} d - d + V_{n}) \), that is, Equation (20). As \( K_{n} = P_{n} C S_{n}^{-1} \) from (22), the image of \( K_{n} \) is included in that of \( P_{n} \), which means that, thanks to \( T_{n} \), we have \( \text{Im} K_{n} \subset \text{ker} H_{n} \). This directly leads to \( H_{n} P_{n}^{+} = H_{n} P_{n} - H_{n} K_{n} CP_{n} = 0 \), i.e., \( T_{n} \Rightarrow T_{n}^{+} \).

**Proof of (iv):** Suppose that \( \eta_{n} b_{n} = b_{n} \). The matrix exponential map is defined by \( \exp M = I d + \sum_{k=1}^{M} \frac{M^{k}}{k!} \). By noting \( z = \eta_{n}^{-1} d - d \), we thus have according to (26):

\[
\eta_{n} b_{n} = \eta_{n}(I d + \sum_{k=1}^{k_{n}} \frac{(K_{n} z)^{k}}{k!})b_{n}
\]

\[
= \eta_{n} b_{n} + \sum_{k=0}^{k_{n}} \frac{(K_{n} z)^{k}}{k!}H_{n}(Ke z) = b_{n}
\]

which concludes the proof.

**Theorem 2**: If the hypotheses of Theorem 1 are satisfied, with output (27) instead of (15), then the RIEKF estimates also verify (25).

Proof: The dynamics (7) is neither right nor left invariant. It is such that \( \chi^{-1} \) satisfies similar dynamics. Using \( \chi^{-1} \in S \) as the state variable, the output (27) becomes left-equivalent as (15). And the RIEKF update for the variable \( \chi^{-1} \) is exactly the LIEKF update. Theorem 1 then applies.

B. Graphical illustration of the theorem and discussion

As it was already known, part of what makes the IEKF work where a filter with linear update such as the EKF fails mostly is where the linearization is done. Indeed, the EKF tries to linearize on a non-linear space by embedding the state in the ambient vector space. Think again of \( SO(3) \) : there is no simple way of expressing a rotation as the sum of another rotation and some matrix. The IEKF however linearizes on the Lie algebra of the system, which is a linear space in its own right, the exponential map being just a translation between the two. When one writes \( \chi = \exp(\xi) \), \( \xi \) is the axis of rotation of \( \chi \), the angle being the vector’s norm, and summing rotation vectors makes perfect sense. It was already the idea behind the MEKF.

The second main argument which makes the proof work is the fact that the image of a Lie sub-algebra by the exponential map is a subgroup of the associated Lie group. This comes from the Baker-Campbell-Hausdorff formula, which states that if \( X, Y \in \mathbb{g} \), then \( e^{X} e^{Y} = e^{Z} \) where \( Z \) is a series of \( X, Y \) and nested Lie bracket terms. \( Z \) thus stays in the sub-algebra.

This is illustrated by Figure 1, which gives a schematic view of the difference among the linear, the MEKF and IEKF updates, for an estimate lying on the subgroup represented by the circle. The IEKF update, through the exponential map, makes the estimate move along the circle. Since the covariance is expressed in the Lie algebra, its alignment stays consistent with the subgroup (11). On the contrary,
the MEKF gives no guarantee that the estimate will stay in the subgroup. In the meantime, the EKF updates along a straight line in the direction of the covariance, so there are no guarantee that the estimate will even remain in $G$, or that the covariance will stay consistent with the curvature of the space.

As already said, the interest is that when process noise is low, the hard constraint becomes useless. However, the state will live near the manifold defined by (11), and so the IEKF stays on the subgroup, while the MEKF and the EKF update respectively leave the subgroup and even the group $G$.

V. Examples

This section presents two examples illustrating the implications of Theorems 1 and 2. The first one shows that the result of [3] now appears as a direct application of Theorem 1. The second one presents the implications of Theorem 2 for the attitude estimation example of Section III-C, and illustrates what happens when noise is turned on.

A. Car position and heading estimation

1) Recall of the results of [3]: Consider the simple case of a non-holonomic car with perfect odometry, unknown heading and noisy position measurements. Suppose the initial position of the car is known. The dynamics are given by:

$$
\frac{d}{dt} \theta = \omega, \quad \frac{d}{dt} x_l = \begin{pmatrix} \cos(\theta) u_l \\ \sin(\theta) u_l \end{pmatrix},
$$

(28)

where $\theta$ is the heading of the car, $x_l$ its position vector, and $\omega, u_l$ are the angular and linear velocities. Noisy position measurements $Y_n = x_n + \nu_n$ are acquired at discrete times $t_n \in \mathbb{N}$, corrupted by white noise $\nu_n$.

It was then proven in [3] that if $R(\theta)$ denotes the rotation matrix of angle $\theta$, and $\hat{\theta}_n, \hat{x}_n$ denote the IEKF estimates, then $R(\hat{\theta}_n)^T x_n = R(\hat{\theta}_n)^T \hat{x}_n = b_t$, where $b_t$ is defined through the differential equation

$$
b_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \frac{d}{dt} b_t = -\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} b_t + \begin{pmatrix} u_l \\ 0 \end{pmatrix}.
$$

(29)

Figure 2, reproduced from [3], displays the trajectories of the true car and both EKF and IEKF estimates, for $\omega_t \equiv 0$. The IEKF car estimate is always traveling on a ray that passes through the origin, while that is not true for the EKF.

2) Translation in the Lie group formalism: The system (28) can be seen as living in the Lie group $SE(2)$ with a state $\chi_t$ verifying

$$
\chi_t = \begin{pmatrix} R(\hat{\theta}_t) & x_t \\ 0_{1,2} & 1 \end{pmatrix}, \quad \dot{\chi}_t = \chi_t U_t, \quad U_t = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

(30)

Moreover, the position measurements and initial known position are respectively given by

$$
Y_0 = \chi_{x_0} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T, \quad \chi_0 (b_0 \, 1)^T = (0 \, 0 \, 1)^T
$$

(31)

Thus, Proposition 2 implies that for $\beta_0 = (b_0 \, 1)^T$ and $\dot{\beta}_t = -U_t \beta_t$, the state satisfies $\chi_t \beta_t = (0 \, 0 \, -1)^T$ for all $t$, which is exactly what is stated in Section V-A.1.

In turn, the result of [3] showing that the LIEKF preserves the latter property is in fact a direct corollary of Theorem 1.

B. Attitude estimation

This example builds upon the work of [2]. Indeed, it did not consider the structure of the initial covariance, and in particular the impact of rank-deficiency. Consider that the system receives at times $t_n$, a measurement $Y_n$ of the gravity
field $g$ and the earth magnetic field $b$ through a triplet of accelerometers and magnetometers, i.e.,
\[ Y_n = (R_n^Tg + V_n^g, R_n^Tb + V_n^b), \]
with $V_n^g, V_n^b$ two centered noises in $\mathbb{R}^3$. It has thus right-equivariant outputs, and the system is suitable for the design of a RIEKF. The corresponding filter’s equations for this problem were already written in [2]. They are similar to (17), with the update consisting of a left multiplication by a term of the form $\exp(K_nz)$, with $z \in \mathbb{R}^N$, that is, a rotation around the axis $K_nz$. Thus, theorem 2 implies that, if initialized correctly, the estimate will correctly estimate the direction of the star at all times whatever the motion. Figure 3 displays the results of numerical experiments. As stated, all the updates consist of rotations sharing the same axis, the direction of the star denoted $c_0$.

**Impact of noise:** We also performed simulations with process noise having standard deviation equal to 0.02 degrees/s. During all the experiment, the angle between $c_0$ and the axis of the updates’ rotations was of course impacted and did not remain null. However, it never exceeded 0.01 degree, and the estimated star direction remained in a sharp cone, as could be expected from the IEKF’s geometrical structure.

**VI. CONCLUSION**

This paper highlighted the shortcomings of Extended Kalman Filtering for high-accuracy navigation problems through the degenerate situation of an infinitely accurate geometric prior. Without resorting to artificial process noise, the EKF fails to propagate this information. These situations are especially important in inertial navigation where they have to be handled carefully.

The novel result we proved shows the algebraic approach to filtering of the IEKF is a robust response to the issue of assimilating precise nonlinear prior information, i.e rank-deficient initial covariance. In future work, this result will be extended to any system defined on a Lie group having group-affine dynamics as defined in [4], and to any information restricting the admissible state space to a subgroup.

**VII. ACKNOWLEDGEMENTS**

This work is supported by the company Safran through the CIFRE convention 2016/1444.

**REFERENCES**