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# A criterion for the differential flatness of a nonlinear control system

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## Abstract

Let's consider a control system described by the implicit equation  $F(x, \dot{x}) = 0$ . If this system is differentially flat, then the following criterion is satisfied : For some integer  $r$ , there exists a function  $\varphi(y_0, y_1, \dots, y_r)$  satisfying the following conditions: (1) The map  $(y_0, \dots, y_{r+1}) \mapsto (\varphi(y_0, y_1, \dots, y_r), \frac{\partial \varphi}{\partial y_0} y_1 + \frac{\partial \varphi}{\partial y_1} y_2 + \dots + \frac{\partial \varphi}{\partial y_r} y_{r+1})$  is a submersion on the variety  $F(x, p) = 0$ . (2) The map  $y_0 \mapsto x_0 = \varphi(y_0, 0, \dots, 0)$  is a diffeomorphism on the equilibrium variety  $F(x, 0) = 0$ .

Inversely, if a control system satisfies this flatness criterion, then it is locally controllable at equilibrium points.

## 1 Introduction

The purpose of this note is to propose a new approach to the study of the differential flatness of a control system. For an overview of flatness theory, see for example [1]. For an overview of existing differential flatness criteria, see [3], section 3.1.2, and [2].

We consider a control system defined by the implicit equation  $F(x, \dot{x}) = 0$ , where  $F(x, p)$  is a smooth function defined on  $\mathbb{R}^n \times \mathbb{R}^m$ , with  $\text{rank} \frac{\partial F}{\partial p} = n - m$  ( $m > 0$ ). We assume that for each  $x$  there exists at least one  $p$  satisfying the equation, and more generally that the manifold  $F(x, p)$  can be parameterized using a smooth function  $u \mapsto f(x, u)$  satisfying  $F(x, f(x, u)) = 0$ . This condition is trivially satisfied if the system  $F(x, \dot{x}) = 0$  is derived from some explicit description of the control system such as  $\dot{x} = f(x, u)$ .

## 2 Finding parameter functions

We say that a function  $\varphi(y_0, \dots, y_r)$  defined on  $(\mathbb{R}^m)^{r+1}$  with values in  $\mathbb{R}^n$  is a parameter function for the system  $F$  if we have for all smooth functions  $y(t)$  defined on  $\mathbb{R}$  with values in  $\mathbb{R}^m$ , and writing  $x(t) = \varphi(y(t), \dot{y}(t), \dots, y^{(r)}(t))$ , the equality  $F(x(t), \frac{d}{dt}x(t)) = 0$ . We can also write this condition as :

$$F(\varphi(y(t), \dot{y}(t), \dots, y^{(r)}(t)), \frac{\partial \varphi}{\partial y_0} \dot{y}(t) + \dots + \frac{\partial \varphi}{\partial y_r} y^{(r+1)}(t)) = 0 \quad (1)$$

Considering that for  $t$  fixed, the function  $y$  can be chosen so that  $y(t), \dot{y}(t), \dots, y^{(r+1)}(t)$  take any independent values, we see that we have for any vectors  $y_0, \dots, y_{r+1}$  in  $\mathbb{R}^m$  the equality

$$F(\varphi(y_0, y_1, \dots, y_r), \frac{\partial \varphi}{\partial y_0} y_1 + \frac{\partial \varphi}{\partial y_1} y_2 \dots + \frac{\partial \varphi}{\partial y_r} y_{r+1}) = 0 \quad (2)$$

Inversely, it is clear that if a function  $\varphi$  satisfies this partial differential equation, then equation (1) is satisfied for all smooth functions  $y$ .

The question of the existence and computation of parameter functions is then equivalent to the question of existence and computation of solutions to this implicit first order partial differential equation. A wide variety of techniques are available to study this question : Cartan Kähler theory, method of characteristics and Charpit equations if  $F$  is scalar, etc ..

The main difficulty here is that  $\varphi$  has to be a function of  $y_0, \dots, y_r$  only and must not depend of  $y_{r+1}$ , which appears as a free parameter in the equation. This requires that the ruled manifold criterion described in [4] be satisfied.

### 3 Differentially flat systems

Let's recall (cf [1]) that a system  $F(x, \dot{x})$  is called differentially flat if there exists a parameter function  $\varphi$  satisfying equation (1) and another function  $\psi(x_0, \dots, x_s)$  so that a solution  $x(t)$  of the system can be written as  $x(t) = \varphi(y, \dot{y} \dots y^{(r)})$  if and only if  $y(t) = \psi(x, \dot{x}, \dots x^{(s)})$ . This requirement is stronger than the existence of a parameter function. Let's note  $\mathcal{L}_\tau$  for the derivation operator  $\mathcal{L}_\tau \varphi(y) = \sum_{i=0}^r \frac{\partial \varphi}{\partial y_i} y_{i+1}$ . If  $\varphi$  is associated to a flat system, we have :

**Proposition 3.1** *The smooth map  $\Phi : (y_0, \dots, y_{r+1}) \mapsto (\varphi(y_0 \dots y_r), \mathcal{L}_\tau \varphi(y_0 \dots y_{r+1}))$  is a submersion on the manifold  $F(x, p) = 0$ .*

Remark : this implies that  $d\varphi$  is surjective.

Proof : We have to prove that  $\Phi$  and  $d\Phi$  are surjective.

Let's consider an element  $(x_0, p_0)$  of the manifold and the associated parameter  $u_0$  so that  $p_0 = f(x_0, u_0)$ . Let's now choose the constant function  $u(t)$  defined by the formula  $u(t) = u_0$ . Then the unique solution of the first order ODE  $\dot{x} = f(x, u)$  with  $\dot{x}(0) = p_0$ ,  $x(0) = x_0$  satisfies  $F(x, \dot{x}) = 0$  for all values of  $t$  and is a smooth function of  $x_0$  and  $p_0$ . Using the function  $\psi$ , we get some function  $y(t)$  which is also a smooth function of  $x_0$  and  $p_0$  so that  $x(t) = \varphi(y, \dot{y}, \dots y^{(r)})$ . Writing  $y(0) = y_0, \dots, y^{(r+1)}(0) = y_{r+1}$  can then write that

$$(x_0, p_0) = (\varphi(y_0, \dots, y_r), \mathcal{L}_\tau \varphi(y_0 \dots y_{r+1})) = \Phi(y_0, \dots, y_{r+1})$$

This shows that  $\Phi$  is surjective. Taking the differential of this equation with respect to  $(x_0, p_0)$ , we get

$$Id = d\Phi \circ (..)$$

which shows that  $d\Phi$  is surjective  $\square$

**Proposition 3.2** *The restriction of  $\Phi$  to the elements of the form  $(y_0, 0 \dots 0)$  is a diffeomorphism on the equilibrium manifold  $F(x_0, 0) = 0$*

Proof : It is well known that Lie-Bäcklund equivalence preserves equilibrium points ( cf [1], Th 5.2)  $\square$

## 4 Controllability at equilibrium points

We now consider in this section a control system described by an equation  $F(x, \dot{x}) = 0$  such that the equation (2) has a solution  $\varphi$  satisfying propositions 3.1 and 3.2.

**Proposition 4.1** *The system is locally controllable at equilibrium points.*

Proof : We consider some equilibrium point  $x_0$ , so that  $F(x_0, 0) = 0$ . Using 3.1, we see that  $(x_0, 0)$  is in the image of  $\Phi$  and that  $d\varphi$  is surjective at this point.

We now show that Kalman criterion is satisfied. Let's note  $A = \frac{\partial f}{\partial x}$  and  $B = \text{Im} \frac{\partial f}{\partial u}$ . Let's take the derivative of the equation  $F(x, f(x, u)) = 0$ . We get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} A = 0$$

and, considering that for  $x$  fixed, the map  $u \mapsto f(x, u)$  is assumed to be a diffeomorphism on the manifold  $F(x, p) = 0$ ,

$$B = \text{Ker} \frac{\partial F}{\partial p}$$

We can write the equation (2) as

$$F(\varphi, \mathcal{L}_\tau \varphi) = 0$$

Taking differentials with respect to  $y_0 \dots y_{r+1}$ , we get the equalities :

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p} \frac{\partial}{\partial y_i} \mathcal{L}_\tau \varphi = 0$$

using the commutation relation  $[\frac{\partial}{\partial y_i}, \mathcal{L}_\tau] = \frac{\partial}{\partial y_{i-1}}$  for  $1 \leq i \leq r+1$  and  $[\frac{\partial}{\partial y_0}, \mathcal{L}_\tau] = 0$ , we get

- Differential with respect to  $y_0$  :

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_0} + \frac{\partial F}{\partial p} \mathcal{L}_\tau \frac{\partial \varphi}{\partial y_0} = 0$$

- Differential with respect to  $y_i$  for  $i \in 1 \dots r$  :

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p} \mathcal{L}_\tau \frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_{i-1}} = 0$$

- Differential with respect to  $y_{r+1}$  ( considering that  $\varphi$  is a function of  $y_0 \dots y_r$  only) :

$$\frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_r} = 0$$

Considering that  $(x_0, 0)$  is an equilibrium point of the system, we now use 3.2 to get  $y_1 = y_2 = \dots = y_r = y_{r+1} = 0$ . The equations become :

- Differential with respect to  $y_0$  :

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_0} = 0$$

- Differential with respect to  $y_i$  for  $i \in 1..r$  :

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_{i-1}} = 0$$

- Differential with respect to  $y_{r+1}$  :

$$\frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_r} = 0$$

Writing  $\frac{\partial F}{\partial x} = -\frac{\partial F}{\partial p} A$ , The equations become

$$\frac{\partial F}{\partial p} \left( A \frac{\partial \varphi}{\partial y_0} \right) = 0$$

$$\frac{\partial F}{\partial p} \left( -A \frac{\partial \varphi}{\partial y_i} + \frac{\partial \varphi}{\partial y_{i-1}} \right) = 0$$

$$\frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_r} = 0$$

this implies inductively the inclusions

$$Im \frac{\partial \varphi}{\partial y_r} \subset B$$

$$Im \frac{\partial \varphi}{\partial y_{r-1}} \subset AB + B$$

..

$$Im \frac{\partial \varphi}{\partial y_0} \subset A^r B + A^{r-1} B + .. + B$$

and we see that  $A^r B + A^{r-1} B + .. + B$  has to contain the range of all the matrices  $\frac{\partial \varphi}{\partial y_i}$ , which generate  $\mathbb{R}^n$   $\square$

## References

- [1] J. Levine, *Analysis and Control of Nonlinear Systems : A Flatness based approach*, Springer, Mathematical Engineering, 2009
- [2] J. Levine *On necessary and sufficient conditions for differential flatness* J. AAEECC (2011) 22:47
- [3] Ph. Martin, R.M. Murray, and P. Rouchon. *Flat systems*, In G. Bastin and M. Gevers, editors, Plenary Lectures and Minicourses, Proc ECC 97, Brussels, pages 211-264, 1997
- [4] P. Rouchon, *Necessary Condition and Genericity of Dynamic Feedback Linearization*, J. of Math. Systems, Estim. and Control 5(3),1995, pp. 345-358

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