A criterion for the differential flatness of a nonlinear control system

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Abstract

Let's consider a control system described by the implicit equation $F(x, \dot{x}) = 0$. If this system is differentially flat, then the following criterion is satisfied : For some integer r, there exists a function $\varphi(y_0, y_1, .., y_r)$ satisfying the following conditions: (1) The map $(y_0, .., y_{r+1}) \mapsto (\varphi(y_0, y_1, .., y_r), \frac{\partial\varphi}{\partial y_0} y_1 + \frac{\partial\varphi}{\partial y_1} y_2 + .. + \frac{\partial\varphi}{\partial y_r} y_{r+1})$ is a submersion on the variety $F(x, p) = 0$. (2) The map $y_0 \mapsto x_0 = \varphi(y_0, 0, .., 0)$ is a diffeomorphism on the equilibrium variety $F(x, 0) = 0$.

Inversely, if a control system satifies this flatness criterion, then it is locally controllable at equilibrium points.

1 Introduction

The purpose of this note is to propose a new approach to the study of the differential flatness of a control system. For an overview of flatness theory, see for example [1]. For an overview of existing differential flatness criteria, see [3], section 3.1.2, and [2].

We consider a control system defined by the implicit equation $F(x, \dot{x}) = 0$, where $F(x, p)$ is a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^n$, with rank $\frac{\partial F}{\partial p} = n - m$ $(m > 0)$. We assume that for each x there exists at least one p satisfying the equation, and more generally that the manifold $F(x, p)$ can be parameterized using a smooth function $u \mapsto f(x, u)$ satisfying $F(x, f(x, u)) = 0$. This condition is trivially satisfied if the system $F(x, \dot{x}) = 0$ is derived from some explicit description of the control system such as $\dot{x} = f(x, u)$.

2 Finding parameter functions

We say that a function $\varphi(y_0, ..y_r)$ defined on $(\mathbb{R}^m)^{r+1}$ with values in $\mathbb{R}^n$ is a parameter function for the system F if we have for all smooth functions $y(t)$ defined on $\mathbb{R}$ with values in $\mathbb{R}^m$, and writing $x(t) = \varphi(y(t), \dot{y}(t), .., y^{(r)}(t))$, the equality $F(x(t), \frac{d}{dt}x(t)) = 0$. We can also write this condition as :

$$F(\varphi(y(t), \dot{y}(t)..y^{(r)}(t)), \frac{\partial\varphi}{\partial y_0}\dot{y}(t) + .. + \frac{\partial\varphi}{\partial y_r} y^{(r+1)}(t)) = 0 \quad (1)$$

Considering that for t fixed, the function y can be chosen so that $y(t), \dot{y}(t), .., y^{(r+1)}(t)$ take any independent values, we see that we have for any vectors $y_0, .., y_{r+1}$ in $\mathbb{R}^m$ the equality

$$F(\varphi(y_0, y_1, .., y_r), \frac{\partial \varphi}{\partial y_0} y_1 + \frac{\partial \varphi}{\partial y_1} y_2 .. + \frac{\partial \varphi}{\partial y_r} y_{r+1}) = 0 \tag{2}$$

Inversely, it is clear that if a function φ satisfies this partial differential equation, then equation (1) is satisfied for all smooth functions y.

The question of the existence and computation of parameter functions is then equivalent to the question of existence and computation of solutions to this implicit first order partial differential equation. A wide variety of techniques are available to study this question : Cartan Kähler theory, method of characteristics and Charpit equations if F is scalar, etc ..

The main difficulty here is that φ has to be a function of $y_0, ..y_r$ only and must not depend of y_{r+1}, which appears as a free parameter in the equation. This requires that the ruled manifold criterion described in [4] be satisfied.

3 Differentially flat systems

Let's recall (cf [1]) that a system $F(x, \dot{x})$ is called differentially flat if there exists a parameter function φ satisfying equation (1) and another function $\psi(x_0, .., x_s)$ so that a solution $x(t)$ of the system can be written as $x(t) = \varphi(y, \dot{y}...y^{(r)})$ if and only if $y(t) = \psi(x, \dot{x}, ..x^{(s)})$. This requirement is stronger than the existence of a parameter function. Let's note $\mathcal{L}_\tau$ for the derivation operator $\mathcal{L}_\tau \varphi(y) = \sum_{i=0}^{r} \frac{\partial \varphi}{\partial y_i} y_{i+1}$. If φ is associated to a flat system, we have :

Proposition 3.1 *The smooth map $\Phi : (y_0, .., y_{r+1}) \mapsto (\varphi(y_0..y_r), \mathcal{L}_\tau \varphi(y_0..y_{r+1}))$ is a submersion on the manifold $F(x, p) = 0$.*

Remark : this implies that $d\varphi$ is surjective.
Proof : We have to prove that Φ and $d\Phi$ are surjective.

Let's consider an element (x_0, p_0) of the manifold and the associated parameter u_0 so that $p_0 = f(x_0, u_0)$. Let's now choose the constant function $u(t)$ defined by the formula $u(t) = u_0$. Then the unique solution of the firts order ODE $\dot{x} = f(x, u)$ with $\dot{x}(0) = p_0$, $x(0) = x_0$ satisfies $F(x, \dot{x}) = 0$ for all values of t and is a smooth function of x_0 and p_0. Using the function ψ, we get some function $y(t)$ which is also a smooth function of x_0 and p_0 so that $x(t) = \varphi(y, \dot{y}, ..y^{(r)})$. Writing $y(0) = y_0, .., y^{(r+1)}(0) = y_{r+1}$ can then write that

$$(x_0, p_0) = (\varphi(y_0, .., y_r), \mathcal{L}_\tau \varphi(y_0..y_{r+1}) = \Phi(y_0, ..y_{r+1})$$

This shows that Φ is surjective. Taking the differential of this equation with respect to (x_0, p_0), we get

$$Id = d\Phi \circ (..)$$

which shows that $d\Phi$ is surjective $\square$

Proposition 3.2 *The restriction of Φ to the elements of the form $(y_0, 0..0)$ is a diffeomorphism on the equilibrium manifold $F(x_0, 0) = 0$*

Proof : It is well known that Lie-Bäcklund equivalence preserves equilibrium points (cf [1], Th 5.2) $\square$

4 Controllability at equilibrium points

We now consider in this section a control system described by an equation $F(x, \dot{x}) = 0$ such that the equation (2) has a solution φ satisfying propositions 3.1 and 3.2.

Proposition 4.1 *The system is locally controllable at equilibrium points.*

Proof : We consider some equilibrium point x_0, so that $F(x_0, 0) = 0$. Using 3.1, we see that $(x_0, 0)$ is in the image of Φ and that $d\varphi$ is surjective at this point.

We now show that Kalman criterion is satisfied. Let's note $A = \frac{\partial f}{\partial x}$ and $B = Im\frac{\partial f}{\partial u}$ Let's take the derivative of the equation $F(x, f(x, u)) = 0$ We get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} A = 0$$

and, considering that for x fixed, the map $u \mapsto f(x, u)$ is assumed to be a diffeomorphism on the manifold $F(x, p) = 0$,

$$B = Ker\frac{\partial F}{\partial p}$$

We can write the equation (2) as

$$F(\varphi, \mathcal{L}_\tau \varphi) = 0$$

Taking differentials with respect to $y_0..y_{r+1}$, we get the equalities :

$$\frac{\partial F}{\partial x}\frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p}\frac{\partial}{\partial y_i}\mathcal{L}_\tau \varphi = 0$$

using the commutation relation $[\frac{\partial}{\partial y_i}, \mathcal{L}_\tau] = \frac{\partial}{\partial y_{i-1}}$ for $1 \leq i \leq r+1$ and $[\frac{\partial}{\partial y_0}, \mathcal{L}_\tau] = 0$, we get

- Differential with respect to y_0 :

$$\frac{\partial F}{\partial x}\frac{\partial \varphi}{\partial y_0} + \frac{\partial F}{\partial p}\mathcal{L}_\tau \frac{\partial \varphi}{\partial y_0} = 0$$

- Differential with respect to y_i for $i \in 1..r$:

$$\frac{\partial F}{\partial x}\frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p}\mathcal{L}_\tau \frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p}\frac{\partial \varphi}{\partial y_{i-1}} = 0$$

- Differential with respect to y_{r+1} (considering that φ is a function of $y_0..y_r$ only) :

$$\frac{\partial F}{\partial p}\frac{\partial \varphi}{\partial y_r} = 0$$

Considering that $(x_0, 0)$ is an equilibrium point of the system, we now use 3.2 to get $y_1 = y_2 = .. = y_r = y_{r+1} = 0$. The equations become :

- Differential with respect to y_0 :

$$\frac{\partial F}{\partial x}\frac{\partial \varphi}{\partial y_0} = 0$$

- Differential with respect to y_i for $i \in 1..r$:

$$\frac{\partial F}{\partial x}\frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p}\frac{\partial \varphi}{\partial y_{i-1}} = 0$$

- Differential with respect to y_{r+1} :

$$\frac{\partial F}{\partial p}\frac{\partial \varphi}{\partial y_r} = 0$$

Writing $\frac{\partial F}{\partial x} = -\frac{\partial F}{\partial p}A$, The equations become

$$\frac{\partial F}{\partial p}(A\frac{\partial \varphi}{\partial y_0}) = 0$$

$$\frac{\partial F}{\partial p}(-A\frac{\partial \varphi}{\partial y_i} + \frac{\partial \varphi}{\partial y_{i-1}}) = 0$$

$$\frac{\partial F}{\partial p}\frac{\partial \varphi}{\partial y_r} = 0$$

this implies inductively the inclusions

$$Im\frac{\partial \varphi}{\partial y_r} \subset B$$

$$Im\frac{\partial \varphi}{\partial y_{r-1}} \subset AB + B$$

..

$$Im\frac{\partial \varphi}{\partial y_0} \subset A^rB + A^{r-1}B + .. + B$$

and we see that $A^rB + A^{r-1}B + .. + B$ has to contain the range of all the matrices $\frac{\partial \varphi}{\partial y_i}$, which generate $\mathbb{R}^n$ $\square$

References

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