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A criterion for the differential flatness of a nonlinear control system

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Abstract

Let's consider a control system described by the implicit equation $F(x, \dot{x}) = 0$. If this system is differentially flat, then the following criterion is satisfied : For some integer r , there exists a function $\varphi(y_0, y_1, \dots, y_r)$ satisfying the following conditions: (1) The map $(y_0, \dots, y_{r+1}) \mapsto (\varphi(y_0, y_1, \dots, y_r), \frac{\partial \varphi}{\partial y_0} y_1 + \frac{\partial \varphi}{\partial y_1} y_2 + \dots + \frac{\partial \varphi}{\partial y_r} y_{r+1})$ is a submersion on the variety $F(x, p) = 0$. (2) The map $y_0 \mapsto x_0 = \varphi(y_0, 0, \dots, 0)$ is a diffeomorphism on the equilibrium variety $F(x, 0) = 0$.

Inversely, if a control system satisfies this flatness criterion, then it is locally controllable at equilibrium points.

1 Introduction

The purpose of this note is to propose a new approach to the study of the differential flatness of a control system. For an overview of flatness theory, see for example [1]. For an overview of existing differential flatness criteria, see [3], section 3.1.2, and [2].

We consider a control system defined by the implicit equation $F(x, \dot{x}) = 0$, where $F(x, p)$ is a smooth function defined on $\mathbb{R}^n \times \mathbb{R}^m$, with $\text{rank} \frac{\partial F}{\partial p} = n - m$ ($m > 0$). We assume that for each x there exists at least one p satisfying the equation, and more generally that the manifold $F(x, p)$ can be parameterized using a smooth function $u \mapsto f(x, u)$ satisfying $F(x, f(x, u)) = 0$. This condition is trivially satisfied if the system $F(x, \dot{x}) = 0$ is derived from some explicit description of the control system such as $\dot{x} = f(x, u)$.

2 Finding parameter functions

We say that a function $\varphi(y_0, \dots, y_r)$ defined on $(\mathbb{R}^m)^{r+1}$ with values in \mathbb{R}^n is a parameter function for the system F if we have for all smooth functions $y(t)$ defined on \mathbb{R} with values in \mathbb{R}^m , and writing $x(t) = \varphi(y(t), \dot{y}(t), \dots, y^{(r)}(t))$, the equality $F(x(t), \frac{d}{dt}x(t)) = 0$. We can also write this condition as :

$$F(\varphi(y(t), \dot{y}(t), \dots, y^{(r)}(t)), \frac{\partial \varphi}{\partial y_0} \dot{y}(t) + \dots + \frac{\partial \varphi}{\partial y_r} y^{(r+1)}(t)) = 0 \quad (1)$$

Considering that for t fixed, the function y can be chosen so that $y(t), \dot{y}(t), \dots, y^{(r+1)}(t)$ take any independent values, we see that we have for any vectors y_0, \dots, y_{r+1} in \mathbb{R}^m the equality

$$F(\varphi(y_0, y_1, \dots, y_r), \frac{\partial \varphi}{\partial y_0} y_1 + \frac{\partial \varphi}{\partial y_1} y_2 \dots + \frac{\partial \varphi}{\partial y_r} y_{r+1}) = 0 \quad (2)$$

Inversely, it is clear that if a function φ satisfies this partial differential equation, then equation (1) is satisfied for all smooth functions y .

The question of the existence and computation of parameter functions is then equivalent to the question of existence and computation of solutions to this implicit first order partial differential equation. A wide variety of techniques are available to study this question : Cartan Kähler theory, method of characteristics and Charpit equations if F is scalar, etc ..

The main difficulty here is that φ has to be a function of y_0, \dots, y_r only and must not depend of y_{r+1} , which appears as a free parameter in the equation. This requires that the ruled manifold criterion described in [4] be satisfied.

3 Differentially flat systems

Let's recall (cf [1]) that a system $F(x, \dot{x})$ is called differentially flat if there exists a parameter function φ satisfying equation (1) and another function $\psi(x_0, \dots, x_s)$ so that a solution $x(t)$ of the system can be written as $x(t) = \varphi(y, \dot{y} \dots y^{(r)})$ if and only if $y(t) = \psi(x, \dot{x}, \dots x^{(s)})$. This requirement is stronger than the existence of a parameter function. Let's note \mathcal{L}_τ for the derivation operator $\mathcal{L}_\tau \varphi(y) = \sum_{i=0}^r \frac{\partial \varphi}{\partial y_i} y_{i+1}$. If φ is associated to a flat system, we have :

Proposition 3.1 *The smooth map $\Phi : (y_0, \dots, y_{r+1}) \mapsto (\varphi(y_0 \dots y_r), \mathcal{L}_\tau \varphi(y_0 \dots y_{r+1}))$ is a submersion on the manifold $F(x, p) = 0$.*

Remark : this implies that $d\varphi$ is surjective.

Proof : We have to prove that Φ and $d\Phi$ are surjective.

Let's consider an element (x_0, p_0) of the manifold and the associated parameter u_0 so that $p_0 = f(x_0, u_0)$. Let's now choose the constant function $u(t)$ defined by the formula $u(t) = u_0$. Then the unique solution of the first order ODE $\dot{x} = f(x, u)$ with $\dot{x}(0) = p_0, x(0) = x_0$ satisfies $F(x, \dot{x}) = 0$ for all values of t and is a smooth function of x_0 and p_0 . Using the function ψ , we get some function $y(t)$ which is also a smooth function of x_0 and p_0 so that $x(t) = \varphi(y, \dot{y}, \dots y^{(r)})$. Writing $y(0) = y_0, \dots, y^{(r+1)}(0) = y_{r+1}$ can then write that

$$(x_0, p_0) = (\varphi(y_0, \dots, y_r), \mathcal{L}_\tau \varphi(y_0 \dots y_{r+1})) = \Phi(y_0, \dots, y_{r+1})$$

This shows that Φ is surjective. Taking the differential of this equation with respect to (x_0, p_0) , we get

$$Id = d\Phi \circ (..)$$

which shows that $d\Phi$ is surjective \square

Proposition 3.2 *The restriction of Φ to the elements of the form $(y_0, 0 \dots 0)$ is a diffeomorphism on the equilibrium manifold $F(x_0, 0) = 0$*

Proof : It is well known that Lie-Bäcklund equivalence preserves equilibrium points (cf [1], Th 5.2) \square

4 Controllability at equilibrium points

We now consider in this section a control system described by an equation $F(x, \dot{x}) = 0$ such that the equation (2) has a solution φ satisfying propositions 3.1 and 3.2.

Proposition 4.1 *The system is locally controllable at equilibrium points.*

Proof : We consider some equilibrium point x_0 , so that $F(x_0, 0) = 0$. Using 3.1, we see that $(x_0, 0)$ is in the image of Φ and that $d\varphi$ is surjective at this point.

We now show that Kalman criterion is satisfied. Let's note $A = \frac{\partial f}{\partial x}$ and $B = \text{Im} \frac{\partial f}{\partial u}$. Let's take the derivative of the equation $F(x, f(x, u)) = 0$. We get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial p} A = 0$$

and, considering that for x fixed, the map $u \mapsto f(x, u)$ is assumed to be a diffeomorphism on the manifold $F(x, p) = 0$,

$$B = \text{Ker} \frac{\partial F}{\partial p}$$

We can write the equation (2) as

$$F(\varphi, \mathcal{L}_\tau \varphi) = 0$$

Taking differentials with respect to $y_0 \dots y_{r+1}$, we get the equalities :

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p} \frac{\partial}{\partial y_i} \mathcal{L}_\tau \varphi = 0$$

using the commutation relation $[\frac{\partial}{\partial y_i}, \mathcal{L}_\tau] = \frac{\partial}{\partial y_{i-1}}$ for $1 \leq i \leq r+1$ and $[\frac{\partial}{\partial y_0}, \mathcal{L}_\tau] = 0$, we get

- Differential with respect to y_0 :

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_0} + \frac{\partial F}{\partial p} \mathcal{L}_\tau \frac{\partial \varphi}{\partial y_0} = 0$$

- Differential with respect to y_i for $i \in 1 \dots r$:

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p} \mathcal{L}_\tau \frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_{i-1}} = 0$$

- Differential with respect to y_{r+1} (considering that φ is a function of $y_0 \dots y_r$ only) :

$$\frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_r} = 0$$

Considering that $(x_0, 0)$ is an equilibrium point of the system, we now use 3.2 to get $y_1 = y_2 = \dots = y_r = y_{r+1} = 0$. The equations become :

- Differential with respect to y_0 :

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_0} = 0$$

- Differential with respect to y_i for $i \in 1..r$:

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_i} + \frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_{i-1}} = 0$$

- Differential with respect to y_{r+1} :

$$\frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_r} = 0$$

Writing $\frac{\partial F}{\partial x} = -\frac{\partial F}{\partial p} A$, The equations become

$$\frac{\partial F}{\partial p} \left(A \frac{\partial \varphi}{\partial y_0} \right) = 0$$

$$\frac{\partial F}{\partial p} \left(-A \frac{\partial \varphi}{\partial y_i} + \frac{\partial \varphi}{\partial y_{i-1}} \right) = 0$$

$$\frac{\partial F}{\partial p} \frac{\partial \varphi}{\partial y_r} = 0$$

this implies inductively the inclusions

$$Im \frac{\partial \varphi}{\partial y_r} \subset B$$

$$Im \frac{\partial \varphi}{\partial y_{r-1}} \subset AB + B$$

..

$$Im \frac{\partial \varphi}{\partial y_0} \subset A^r B + A^{r-1} B + .. + B$$

and we see that $A^r B + A^{r-1} B + .. + B$ has to contain the range of all the matrices $\frac{\partial \varphi}{\partial y_i}$, which generate \mathbb{R}^n \square

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