Linear observation systems on groups (I)

Axel Barrau, Silvere Bonnabel

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Abstract

The present paper is an accessible digest, along with extensions, of previous work by the authors. We propose an unifying and versatile framework for a class of discrete time systems whose state is an element of a group, that we call linear observation systems on groups. Those systems strictly mimic linear systems in the sense that + is replaced with group multiplication, and linear maps with automorphisms. Generalized linear observers on groups, which are the group counterpart of linear observers (and known as invariant observers on groups), are shown to share some important properties with linear observers, namely the fact the estimation error equation is autonomous. We then prove that, linear observation systems are in fact the only ones such that the error equation is autonomous, and relate them to group-affine systems we have previously introduced in continuous time. We also introduce a family of groups called $SE_K(D)$, and leverage it to prove many non-linear discrete-time systems of navigation and robotics (including Simultaneous Localization And Mapping) are in fact linear observation systems on $SE_K(D)$.

1 Introduction

Symmetry-preserving (also called invariant) observers, see [9], or more generally non-linear observers on Lie groups, see e.g. [7, 19, 23, 5, 4, 16, 6, 13, 15, 20] have drawn attention over the past decade, both for their theoretical convergence properties, and for their simplicity and robustness in practical applications, essentially attitude estimation for guidance and control of unmanned aerial vehicles (UAV).

The present article introduces a simple mathematical framework that unifies previous work by the authors. Indeed, in [7] it is advocated that invariant systems on Lie groups with equivariant output maps yield autonomous error equations

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(*Nota Bene*: "autonomous" meaning here the differential equation that governs the evolution of the estimation error is a function of the error itself and the inputs only, that is, it does not explicitly depends on the actual trajectory followed by the state as should be expected for a non-linear system. Yet, through the inputs, it may explicitly depend on time t). This fact was also noticed and exploited in various other works, see e.g., [17, 24] amongst others. In [2], the class of systems such that the invariant error between two trajectories of the system is autonomous was extended, and wholly characterized mathematically. Notably, it encompasses the three classes typically considered of left-invariant, right-invariant, and mixed-invariant systems, known to yield autonomous errors, see [16].

In the present article we consider the discrete-time case, as opposed to our previous works, notably [2], and we propose a definition of linear observation systems on groups. We provide novel characterizations of this class, one of them - the one we use to introduce those systems - being surprisingly simple, as essentially we have a linear system where addition is replaced with group law. We then consider generalized linear observers on groups, which are known as the invariant observers in continuous time, see e.g. [7], or more generally observers on Lie groups in previous literature, and which are the counterpart of linear observers, and are shown to share some important properties with them, namely the fact the error equation is autonomous.

We also introduce a novel family of groups called $SE_K(D)$, and prove many systems of engineering pertaining to navigation and localization are in fact linear observation systems on $SE_K(D)$. Those systems include 0- attitude estimation with unbiased gyrometers, 1- navigation with GPS and an inertial measurement unit (IMU), 2- navigation in a static map of known features, 3- simultaneous localization and mapping (SLAM) for a wheeled robot, 4- SLAM for an unmanned aerial vehicle (UAV) equipped with an IMU, and even 5- SLAM with a mixture of static features and features moving at constant speed (SLAM with dynamical objects tracking). For each of those systems, owing to the fact they are linear systems on the group, we readily obtain large classes of observers with autonomous error dynamics, and under observability conditions we obtain local convergence around any trajectory (which is generally a difficult property to obtain in the non-linear case). 0- is so well-known that we do not detail it. The fact that 1- lends itself to the observers on groups with autonomous errors framework was noticed and proved in [2] in continuous time, but never in discrete time (and the transposition is not straightforward). The fact that 3- lends itself to our framework was noticed in [8] and exploited in [3] to derive an EKF with consistency properties. Regarding 4- and 5-, it had never been noticed before to our best knowledge.

The main contributions are twofold. First, we introduce a framework that unifies prior work by the authors. The choice of discrete time, as opposed to most of our prior work, tends to simplify the theory (notably we do not suppose the group is a Lie group) and as such the paper provides a very tutorial and accessible introduction to the subject (note also that, the observation structure through group actions we introduce is very convenient, although it is essentially a reformulation of the notion of compatible outputs of [9]). The seven characterizations of linear observation systems on groups we provide is novel and extends those already discovered in [2] in continuous time. Then, as underlined before, the paper shows that quite a number of discrete-time systems pertaining to the field of navigation and localization are in fact linear systems on groups, a fact previously unknown for some of them, and this using a single family $SE_K(D)$.

Historically, several approaches to filtering or observer design for systems possessing a geometric structure have been developed in previous literature. For stochastic processes on Riemannian manifolds [14] some results have been derived, see e.g., [21]. The specific situation where the process evolves in a vector space but the observations belong to a manifold has also been considered, see e.g. [12] and more recently [22]. For systems on Lie groups powerful tools to study the filtering equations - such as harmonic analysis - have been used, notably in the case of bilinear systems [25] and estimation of the initial condition of a Brownian motion in [11].

Note that, this online preprint contains an appendix where various complementary interesting points are covered.

2 Linear systems and observers

Given a partially observed dynamical system, an observer is another dynamical system fed with observations coming from the first system and designed to provide an estimation of the state, without ever differentiating the measured signals (which are always noisy in practice). In general, building observers with theoretical properties is difficult due to the multiplicity of possible trajectories of the true system. But this task becomes much easier in one special case: if the considered system is linear, using linear observers, also known as Luenberger observers [18].

Definition 1 (Linear system)**.** *For all $n \in \mathbb{N}$, let $F_n \in \mathbb{R}^{N \times N}$, $H_n \in \mathbb{R}^{P \times N}$, and $a_n \in \mathbb{R}^N$. A linear system with state $x \in \mathbb{R}^N$ is defined through the equations:*

$$x_{n+1} = F_n x_n + a_n \tag{1}$$

$$y_n = H_n x_n \tag{2}$$

where $y_n \in \mathbb{R}^P$ is the observed output.

In this case, a specific class of observers has sound theoretical properties:

Definition 2 (Linear observer)**.** *A linear observer of the system* (1)*,* (2) *is an observer of the form:*

$$\hat{x}_{n+1|n} = F_n \hat{x}_{n|n} + a_n, \tag{3}$$

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} + L_n \left(y_{n+1} - H_{n+1} \hat{x}_{n+1|n} \right) \tag{4}$$

with L_n a tunable matrix called the gain.

The rationale is as follows: first the estimate is propagated using the model equations [propagation step]. Then, it is corrected using a correction term that is proportional to the discrepancy between the measured output and the predicted one [udpate step], and which "nudges" the estimate towards the true state. This discrepancy between measured output and predicted (or estimated) one is called the "innovation". The property making the study of linear observers for linear systems particularly easy, is related to the evolution of the estimation error e defined as:

$$e = x - \hat{x}.$$

During the propagation step we have indeed:

$$e_{n+1|n} = x_{n+1} - \hat{x}_{n+1|n} = F_n x_n - F_n \hat{x}_{n|n} = F_n e_{n|n}, \tag{5}$$

and during the update:

$$\begin{aligned} &e_{n+1|n+1} = x_{n+1} - \hat{x}_{n+1|n+1} \\ &= x_{n+1} - \hat{x}_{n+1|n} - L_n \left[H_{n+1} x_{n+1} - H_{n+1} \hat{x}_{n+1|n} \right] \\ &= \left(I - L_n H_{n+1} \right) e_{n+1|n} = \left(I - L_n H_{n+1} \right) F_n e_{n|n}. \end{aligned} \tag{6}$$

At both propagation and update steps, the error variable follows an autonomous equation, i.e., independent from the trajectory x_n followed by the system, (or equivalently on the actual value of $\hat{x}_n$). This means matrices L_n can be tuned without considering any specific trajectory of the system (and the corresponding tuning will be satisfactory for all trajectories). The remainder of this paper shows this property generalizes to a much broader class of systems.

Remark 1. *The equation governing the error evolution* (6) *explicitly depends on the time step n. As such the term "autonomous" is abusive, but it is used here to insist on the absence of explicit dependency on the estimated state's trajectory. Indeed, if F_n and H_{n+1} were non-linear maps, $e_{n+1|n+1}$ would depend on both x_n and $\hat{x}_n$ and would by no means be a function of $e_{n|n}$ only.*

3 Linear observation systems on groups

A natural question that arises when dealing with nonlinear systems is wether an analog to equations (5) and (6) can be found. Of course, the answer is no as linearity is needed when factorizing matrices F and H. Linearity is the key to autonomous error variables because it factorizes vector additions and subtractions. As there is apparently no way to obtain the same property with a nonlinear system, the idea we develop here is finding systems which factorize under a different operation. This means that, in the sequel, we are going to work with a general operation which can be, or not be, the vector addition. The only hypothesis we impose on the operation is being a *group law*, which is necessary to build error variables.

3.1 Groups and group laws

Definition 3 (Group)**.** *A group is a set G endowed with a group composition law, i.e. a map $G \times G \to G$ denoted $a \cdot b$, and referred to as "dot", that verifies:*

- *Associativity:*
$$\forall x, y, z \in G, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$
- *Neutral element: There exists an element $Id \in G$ such that*
$$\forall x \in G, \quad x \cdot Id = Id \cdot x = x.$$
- *Inversion: For any $x \in G$ there exists an element $x^{-1} \in G$ such that*
$$x \cdot x^{-1} = x^{-1} \cdot x = Id.$$

Note that the group law is usually referred to as the "product" on G, and arguments of the operation are told to be "multiplied". This terminology is of course inspired by classical scalar or matrix multiplication.

The neutral element is the equivalent of zero in a vector space: it does not affect other elements through the operation. It is the value we will want our error variable to reach. The inverse is the equivalent of $-x$ when x is a vector: it is the quantity bringing x to zero through the operation. Of course, vector spaces are groups, their group law being addition, their neutral element being zero and their inversion being $x \to -x$.

Remark 2. *Def. 3 automatically implies some classical properties that are reminiscent of matrix calculus, such as*
$$Id^{-1} = Id$$
and
$$(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}.$$

Given a group law, Def. 3 gives all the tools we need to transpose the definition of the error variable $e = x - \hat{x}$:

Definition 4 (Left-invariant error variable)**.** *Let x and $\hat{x}$ be two elements of a group G. We define the left-invariant error e from $\hat{x}$ to x as:*
$$e = \hat{x}^{-1} \cdot x$$

If the dot is addition, we obtain $e = -\hat{x} + x = x - \hat{x}$, the classical error variable. In the general case, the true state is related to the estimate through the error variable as $x = Id \cdot x = \hat{x} \cdot \hat{x}^{-1} \cdot x = \hat{x} \cdot e$ and we see again that Id plays the role of the zero: the latter expression with $e = Id$ gives $x = \hat{x}$.

Now, we define two notions, one being the generalization of linear dynamics, and the other of linear observations, to state spaces having a group structure.

3.2 Group automorphisms and group actions

To generalize linearity to any group law in view of designing observers, remember that the useful property of linearity was factorizing the addition. The counterpart of this property for a general group law is as follows:

Definition 5 (Group automorphism)**.** *An automorphism of a group G is a function $\phi : G \to G$ such that:*

$$\forall a, b \in G, \phi(a \cdot b) = \phi(a) \cdot \phi(b)$$

It can be easily shown that this assumption implies $\phi(Id) = Id$ and $\forall x \in G, \phi(x^{-1}) = \phi(x)^{-1}$. The set of all automorphisms of G is usually denoted by $Aut(G)$.

If the dot is the vector addition, the hypothesis becomes simply $\phi(x + y) = \phi(x)+\phi(y)$. This is obviously verified by linear mappings and is the property used in Eq. (5) to factorize F. In the sequel, this hypothesis will make the computations of Eq.(5) work for any operation. But for the moment, let us consider a second class of maps which can be considered as a generalization of linear maps, and will prove to be suited to the generalization of linear observations.

Definition 6 (Left group action)**.** *Let G be a group and Y a set with no specific structure. A left group action of G on Y is an operation we will denote by a star $\star$:*

$$\begin{aligned} G \times Y &\to Y \\ (x, y) &\mapsto x \star y \end{aligned}$$

and which satisfies:

$$\forall a, b \in G, \forall y \in Y, \quad a \star (b \star y) = (a \cdot b) \star y, \tag{7}$$

which is a sort of associativity property. In this paper, the argument $y \in Y$ will be referred to as a "target".

We first illustrate group actions of a matrix group because this makes the assumptions and notations clear.

Example 1 (Group of invertible matrices)**.** *Let G be the group of invertible $N \times N$ matrices, with matrix multiplication as the group law (the "dot"). Then, defining $Y = \mathbb{R}^N$ and $\star$ as the classical matrix-vector product $M \star y = My$, property (7) is easily checked:*

$$(M_1 \cdot M_2) \star y = (M_1M_2)y = M_1(M_2y) = M_1 \star (M_2 \star y).$$

We see that (7) means that a group action can be manipulated essentially like a matrix-vector product.

Let us come back to our motivating example, which is vector spaces endowed with addition.

Example 2. *Let G be $\mathbb{R}^N$ be endowed with vector addition as usual, Y be a second vector space $\mathbb{R}^P$ and H a marix of size $N \times P$. Then the operation $\star : (G, Y) \to Y$ defined by $a \star y = Ha + y$ is an action of $\mathbb{R}^N$ on $\mathbb{R}^P$, as*

$$(a + b) \star y = H(a + b) + y = Ha + Hb + y = Ha + b \star y = a \star (b \star y).$$

In particular, linear mappings of the form $y = Hx$ can be written as $y = x \star b$ with $\star$ the action of $\mathbb{R}^N$ on $\mathbb{R}^P$ of Example 2 and $b = 0_{P,1}$, making group actions a good candidate for generalizing linear observations (note that this assumption is closely related to the compatibility of the output map defined in [9]).

We have now all the concepts we need to build nonlinear systems with autonomous error variables. The next subsection gives the shape of "generalized" linear systems.

Remark 3. *For readability, all over this paper, dots $\cdot$ will always refer to group laws, stars $\star$ will always refer to group actions, plus $+$ will always refer to classical matrix, vector or scalar addition, blank will always refer to classical matrix-matrix, matrix-vector, scalar-matrix, scalar-vector or scalar-scalar multiplication and ring $\circ$ will always refer to operator composition.*

3.3 Linear observation systems on groups

For linear dynamics the propagated state is equal to "linear mapping of the state + vector". The counterpart we expect here is thus : "automorphism of the state dot group element". Linear observation have the form "linear mapping + vector", which is a specific case of group action (see Example 2). The natural generalization of linear systems we propose is thus:

Definition 7 (Left linear observation system)**.** *Let G be a group, and for all $n \in \mathbb{N}$, let $a_n \in G$. A linear observation system with state $x \in G$ is defined through the equations:*

$$x_{n+1} = \phi_n(x_n) \cdot a_n \tag{8}$$

$$y_n = x_n \star b_n \tag{9}$$

with ϕ_n a group automorphism (possibly varying with n) and star $\star$ a group action of x_n on an element b_n of a set Y. Note that the group action can also be different for different time steps, but denoting it all the same by $\star$ is no source of ambiguity.

Just like linear systems, linear observation systems have autonomous error propagation. This is what we check now. Let $(x_n)_{n \geq 0}$ and $(\hat{x}_n)_{n \geq 0}$ be two sequences verifying Eq. (8). We define the left-invariant error between $\hat{x}_n$ and x_n

as $e_n = \hat{x}_n^{-1} \cdot x_n$ and have then:

$$\begin{aligned} e_{n+1} &= \hat{x}_{n+1}^{-1} \cdot x_{n+1} \\ &= [\phi_n(\hat{x}_n) \cdot a_n]^{-1} \cdot [\phi_n(x_n) \cdot a_n] \\ &= a_n^{-1} \cdot \phi_n(\hat{x}_n)^{-1} \cdot \phi(x_n) \cdot a_n \\ &= a_n^{-1} \cdot \phi_n(\hat{x}_n^{-1}) \cdot \phi(x_n) \cdot a_n \\ &= a_n^{-1} \cdot \phi_n(\hat{x}_n^{-1} x_n) \cdot a_n \\ &= a_n^{-1} \cdot \phi_n(e_n) \cdot a_n \\ &= I_{a_n^{-1}}(\phi_n(e_n)) \end{aligned} \tag{10}$$

where we let I_g denote the map (called the inner automorphism):

$$I_g : x \mapsto g \cdot x \cdot g^{-1}.$$

We see that $\hat{x}_n$ and x_n collapse to e_n at the last line and disappear from the equation, as in the linear case. As well, *innovations*, that is, the discrepancy between the measured output and the estimated (or predicted) one, are only dependent on the error variable, if defined as:

$$z_n = \hat{x}_n^{-1} \star y_n = \hat{x}_n^{-1} \star (x_n b) = (\hat{x}_n^{-1} \cdot x_n) \star b = e_n \star b_n \tag{11}$$

Remark 4. *At first glance, this definition could not look consistent with the linear case and we would rather expect something like a difference between $x_n \star b_n$ and $\hat{x}_n \star b_n$. But this is only an impression due to the group action formalism. Indeed, a linear observation $x \to Hx$ is simply the action $x \star b$ as in Example 2 applied to target $b = 0_{P,1}$. In the group actions formalism we would write: $Hx = x \star 0_{P,1}$. Then we have: $z_n = \hat{x}_n^{-1} \star (x_n \star 0_{P,1}) = (-\hat{x}_n) \star [Hx_n] = -H\hat{x}_n + Hx_n = H(x_n - \hat{x}_n)$, and z_n boils down to the classical innovation!*

When the elements of G are matrices, a good candidate group action to define observations (9) is matrix-vector product of Example 1. Note that, matrix-vector products are the observation model we retained to prove some stability properties in continuous time in of the invariant extended Kalman filter (IEKF) in [2]. Another case of interest is when the observations are a group element, and this is still compatible with the framework introduced in the present paper as shown by the following result.

Proposition 1 (Full state observations)**.** *The case where the observations are defined through a morphism from G to another group H (a morphism is a function verifying the same property as an automorphism, but taking values in a group other than G) is also a specific case of* (11)*. In particular this applies to the case where the full state is observed, that is, $Y = G$ and $y_n = x_n$.*

Proof. Consider an observation $y_n = \psi(x_n)$, with ψ a morphism from G to H. Then the operation $G, H \to H,\ (x, b) \mapsto \psi(x) \cdot b$ defines an action of G on H indeed and can thus be denoted by a star: $x \star b = \psi(x) \cdot b$. Then, setting $b = Id$ (neutral element of H) we obtain $y_n = \psi(x) = \psi(x) \cdot Id = \psi(x) \cdot b = x \star b$. In particular, an observation of the full state still matches with Def. (9). $\square$

4 Linear (pre-)observers on groups

4.1 Definition and properties

In the previous section we defined a class of systems that are akin to linear systems. The corresponding properties will be leveraged now to build non-linear observers sharing some of the properties of linear observers:

Definition 8 (Left linear pre-observer)**.** *For the linear observation system* (8), (9) *a generalized pre-observer on the group G is defined by a sequence of estimates* $(\hat{x}_n)$ *of the following form:*

$$\hat{x}_{n+1|n} = \phi_n(\hat{x}_{n|n}) \cdot a_n \tag{12}$$

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} \cdot L_n\left(\hat{x}_{n+1|n}^{-1} \star y_n\right), \tag{13}$$

with $L_n()$ any operator from Y to G, possibly depending on n.

We already know that the error variable is autonomous during the propagation step, see (10). We also know that the innovation $\hat{x}_{n+1|n}^{-1} \star y_n$ is equal to $e_n \star b_n$, see (11), and is thus only a function of the error variable. Let us compute the error variable after the update:

$$\begin{aligned} e_{n+1|n+1} &= \hat{x}_{n+1|n+1}^{-1} \cdot x_{n+1} \\ &= \left[\hat{x}_{n+1|n} \cdot L_n(e_{n+1|n} \star b_n)\right]^{-1} \cdot x_{n+1} \\ &= L_n\left(e_{n+1|n} \star b_n\right)^{-1} \cdot \hat{x}_{n+1|x_n}^{-1} \cdot x_{n+1} \\ &= L_n\left(e_{n+1|n} \star b_n\right)^{-1} \cdot e_{n+1|n} \end{aligned} \tag{14}$$

We see the update step yields an autonomous error equation too. We have thus proved the following result:

Proposition 2. *Consider a linear observation system* (15)-(16) *as introduced in Definition 7, and a linear pre-observer* (12)-(13) *as introduced in Definition 8. Then, the error evolution (including propagation step and update step) is autonomous. Namely it writes:*

$$e_{n+1|n} = a_n^{-1} \cdot \phi_n(e_{n|n}) \cdot a_n \tag{15}$$

$$e_{n+1|n+1} = L_n\left(e_{n+1|n} \star b_n\right)^{-1} \cdot e_{n+1|n} \tag{16}$$

Of course, this boils down to (5), (6), i.e., linear observers for linear systems, if the group is a vector space with addition as group law, if morphisms ϕ_n are linear mappings and if the group action on Y is $x \star b = Hx + b$ of Example 2 with target $b = 0$. We obtained a generalization of linear pre-observers, i.e., observers ensuring autonomous error evolution. Just as with pre-observers, a function $L_n()$ bringing the error to zero still has to be found, and it is not as simple as in the linear case.

However, the autonomy property, already known in more specific cases, has been key to much of the successes of non-linear observers on Lie groups, since it turns a tuning $L_n()$ successful for a single trajectory into a tuning that is successful for all of them. Furthermore, it allows to readily prove local non-trivial convergence results in the Lie group case as further discussed in Section 4.4.

Remark 5. *Observers of the form* (12), (13) *are not novel. This structure arises naturally on Lie groups, and it is key to most approaches to observer design on groups, such as e.g., [7, 19, 23, 5, 4, 16, 6, 13, 15, 20] . Observers* (12), (13) *are also called left-invariant observers, or invariant observers, [7]. However, this terminology is not particularly desirable in the present setting, since equation* (12) *is neither left nor right invariant, our setting being more general. Left and right multiplications are the most basic kinds of group automorphisms, but the autonomy property is shown to carry over to the general setting.*

4.2 Generalized linear observers for right group actions

When the output is an action with respect to right group composition, the generalized linear observers have a slightly different form.

Definition 9 (Right group action). *Let G be a group and Y a set. A right group action of G on Y is an operation $Y \times G \to Y$ we will denote by a star $\star$ and defined as $(b, x) \mapsto b \star x$ and verifying:*

$$\forall x, y \in G, \forall b \in Y, \quad (b \star x) \star y = b \star (x \cdot y), \tag{17}$$

Considering this second type of observations we obtain a second family of systems:

Definition 10 (Right linear observation system). *A right generalized linear system with state $x \in G$ is a system with dynamics as follows, and observed through right group actions on a set Y:*

$$x_{n+1} = \phi_n(x_n) \cdot a_n \tag{18}$$

$$y_n = b_n \star x_n \tag{19}$$

with ϕ_n a group automorphism.

In Section 3 we chose $e = \hat{x}^{-1} \cdot x$ as the error variable. This error is called left-invariant, since it is unchanged by the transformation $(\hat{x}, x) \mapsto (g \cdot \hat{x}, g \cdot x)$, for arbitrary $g \in G$: $(g \cdot \hat{x})^{-1} \cdot (g \cdot x) = \hat{x}^{-1} \cdot g^{-1} \cdot g \cdot x = \hat{x}^{-1} \cdot x$. Another natural transposition of the linear difference would be $e = x \cdot \hat{x}^{-1}$, and this error turns out to be suited to observations defined through right group actions. In this case, the linear pre-observers read:

$$\hat{x}_{n+1|n} = \phi_n(\hat{x}_{n|n}) \cdot a_n \tag{20}$$

$$\hat{x}_{n+1|n+1} = L_n\left(y_n \star \hat{x}_{n+1|n}^{-1}\right) \cdot \hat{x}_{n+1|n}, \tag{21}$$

Defining the right-invariant errors $e_{n+1|n} = x_n \cdot \hat{x}_{n+1|n}^{-1}$ and $e_{n|n} = x_n \cdot \hat{x}_{n|n}^{-1}$, the errors equations are again autonomous owing to similar computations as (14):

$$e_{n+1|n} = \phi_n(e_{n|n}),$$
$$e_{n+1|n+1} = e_{n+1|n} \cdot L_n \left(y_n \star e_{n+1|n}\right)^{-1}.$$

4.3 Right and left equivalence

The notion of right action is in fact not very useful in practice because of the following results:

Proposition 3. *If $(g, x) \mapsto g \star x$ denotes a left-action, then $(g, x) \mapsto g^{-1} \star x$ is a right-action. For this reason, we can ignore the notion of right-action and write them as $g^{-1} \star x$, with $\star$ always denoting a left action.*

Proposition 4 (Inverse-state linear system)**.** *Let G be a group and x_n the state of a system of the form*

$$x_{n+1} = \phi_n(x_n) \cdot a_n \tag{22}$$
$$y_n = x_n^{-1} \star b_n. \tag{23}$$

Then x_n^{-1} is the state to a left linear observation system of the form (8)-(9)*.*

Proof. The observation part is straightforward, let us show the propagation part. If x_n verifies $x_{n+1} = \phi_n(x_n) \cdot a_n$ then x_n^{-1} verifies

$$x_{n+1}^{-1} = a_n^{-1} \cdot \phi_n(x_n)^{-1} = a_n^{-1} \cdot \phi_n(x_n^{-1}),$$

(ϕ_n being a automorphism, it commutes with inversion) and finally

$$x_{n+1}^{-1} = a_n^{-1} \cdot \phi_n(x_n^{-1}) \cdot a_n \cdot a_n^{-1} = I_{a_n^{-1}} \circ \phi_n(x_n^{-1}) \cdot a_n^{-1} = \tilde{\phi}_n(x_n^{-1}) \cdot a_n^{-1}$$

(see our notation for I above) where $I_{a_n^{-1}} \circ \phi_n = \tilde{\phi}_n$ is a automorphism as composition of automorphisms. ∎

Thus, it turns out that studying (22), (23) separately is not necessary: all we have to do is working with x_n^{-1} instead of x_n. This readily proves the following result:

Proposition 5. *For the linear observation system* (22)-(23) *the generalized observer*

$$\hat{x}_{n+1|n} = \phi_n(\hat{x}_{n|n}) \cdot a_n \tag{24}$$
$$\hat{x}_{n+1|n+1} = L_n \left(\hat{x}_{n+1|n} \star y_n\right) \cdot \hat{x}_{n+1|n}, \tag{25}$$

is such that the error equation is autonomous.

This confirms the observation action structure (left or right) dictates the error (left or right), on which the observer should be based (to get an autonomous error).

4.4 Complementary results for the Lie group case

In this section, we momentarily assume the state space G is an N-dimensional Lie group. Readers unfamiliar with Lie groups are advised to skip the present section.

The present section builds upon our previous work [2] devoted to continuous time and matrix Lie groups, and yields a discrete time version of one of the main results of [2] that is that the error follows in fact a *linear* propagation equation, resorting to a suitable change of variables. Moreover, making use in the present section of the standard notion of Lie group - Lie algebra homomorphism correspondance we much simplify the analysis of [2].

We call "linearized error variable" an element $\xi_{n|n}$ (resp. $\xi_{n+1|n}$) of the Lie algebra $\mathfrak{g}$ (identified to $\mathbb{R}^N$) whose image through the Lie exponential map is the error variable e of the previous sections, i.e.:

$$e_{n|n} = \exp(\xi_{n|n}), \quad e_{n+1|n} = \exp(\xi_{n+1|n}) \tag{26}$$

A key property of linear observation systems is that the linearized error variable $\xi_{n|n}$ evolves linearly during the propagation step (15), i.e., there exists a matrix F_n such that:

$$\xi_{n+1|n} = F_n \xi_{n|n} \tag{27}$$

To avoid any misunderstanding here, we insist that this is *not* the result of a first-order expansion: Eq. (27) is exact. This is a consequence of the Lie group - Lie algebra homomorphism correspondance, a classical result of the theory of Lie groups.

Theorem 1 (Lie group - Lie algebra correspondance)**.** *Let G be a Lie group and $\phi : G \to G$ an automorphism of G as defined in Section 3.2. Then, there exists a linear map $f : \mathfrak{g} \to \mathfrak{g}$ sugh that:*

$$\phi \circ \exp = \exp \circ f$$

The map f appearing in Theorem 1 being linear, it can be represented under a classical matrix form: $f(\xi) = F\xi$. Now, Eq. (15) reads:

$$e_{n+1|n} = I_{a_n^{-1}} \circ \phi_n(e_{n|n}),$$

It can be easily checked that 1- I_x is an automorphism for any x, and 2- the composition of two automorphisms is an automorphism. Thus, $I_{a_n^{-1}} \circ \phi_n$ is an automorphism and Theorem 1 applies: there exists a matrix F_n such that $a_n^{-1} \cdot \phi_n(\exp(\xi)) \cdot a_n = \exp(F_n \xi)$ for any $\xi \in \mathfrak{g}$, which yields in particular Eq. (27).

Let us see how to use this property to build an observer in the case where $Y = \mathbb{R}^P$ (which is predominant in applications). Consider the Luenberger-like observer:

$$L_n(z_n) = \exp\left(K_n(z_n - b)\right), \tag{28}$$

with $\exp() : \mathbb{R}^N \to G$ the exponential map (we recall that $\mathfrak{g}$ is identified to $\mathbb{R}^N$), $K_n \in \mathbb{R}^{N\times P}$ a gain matrix and b the target of the action, appearing in the observation (9). A consequence of Equation (27) is that ξ_n associated to errors (15), (16) has a very specific form:

Proposition 6. *Let e be the left-invariant error variable of a linear system on G observed via left actions. Consider a left linear pre-observer with $L_n()$ defined by* (28) *and let ξ denote the linearized errors* (26)*. Then we have:*

$$\begin{aligned} \xi_{n+1|n} &= F_n \xi_{n|n}, \\ \xi_{n+1|n+1} &= BCH\left[-K_n\left(\exp(\xi_{n+1|n}) \star b - b\right),\ \xi_{n+1|n}\right], \end{aligned} \tag{29}$$

where $BCH : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ denotes the Baker-Campbell-Hausdorff formula.

We see that only the update step is nonlinear once the error equation is mapped to the Lie algebra. This is extremely advantageous for stability analysis as in most cases, the propagation function is the one which changes over time due to inputs, not the observation function.

Being able to prove local convergence around any trajectory is already an achievement in the field of non-linear observers since local convergence is generally difficult to obtain, owing to the dependency of the linearized system on the estimated trajectory that is generally impossible to predict before having actually "seen" the observations y_n (and thus a gain tuning that works for any trajectory is hard to find). The following result, which is a (simplified) discrete time version for general groups of the main result of our previous work [2] devoted to continuous time and matrix groups, proves generalized linear observers on Lie groups achieve (at least) local asymptotic convergence.

Proposition 7. *Let $Y = \mathbb{R}^P$, H_n be defined through the first-order Taylor expansion* $\exp(\xi) \star b - b = H_n\xi + O(||\xi||^2)$*. Consider three invertible matrices $Q \in \mathbb{R}^{N\times N}$, $R \in \mathbb{R}^{P\times P}$, and $P_0 \in \mathbb{R}^{N\times N}$. Let K_n be defined through the recursion $P_{n+1|n} = F_n P_n F_n^T + Q$, $K_{n+1} = P_{n+1|n} H_n^T (H_n P_{n+1|n} H_n^T + R)^{-1}$, $P_{n+1} = P_{n+1|n} - K_{n+1} P_{n+1|n}$. Then, the linear observation observer defined through* (28) *converges locally around* any *trajectory if the pair (F_n, H_n) is uniformly observable, owing to standard convergence results on the Kalman filter [10]. Note that the latter tuning is inspired by the method of least squares (i.e., the Kalman filter) but other methods can be used.*

5 Characterization of general linear systems on groups

In Section 3.3, it was argued that systems of the form (8)-(9) generalize linear systems. Beyond the apparent transposition that consists of replacing linear maps with automorphisms and addition with group law, the resemblance to linear systems was confirmed by the results of Section 4.1 in which we proved those systems are analog to linear observers, in the sense that both propagation and update steps do not depend on the trajectory (i.e. autonomous estimation error evolution). In this section, we focus on the propagation step and we prove autonomy of the error is in fact equivalent to dynamics having the form (8). This is done through a theorem that provides seven different characterizations of systems with autonomous error propagation on groups.

To be more specific, in the linear case the key property that ensures autonomy of the error is the factorization property of linear maps, see (5). Consider now general dynamics on a group defined by:

$$x_{n+1} = \psi_n(x_n) \tag{30}$$

with $\psi_n : G \to G$. For the analog of (5) to hold, we need the propagated error to be a function of the error before propagation, that is, we need at each step n the existence of a function μ such that

$$\hat{x}_{n+1}^{-1} \cdot x_{n+1} = \psi_n(\hat{x}_{n+1})^{-1} \cdot \psi_n(x_n) = \mu(\hat{x}_n^{-1} \cdot x_n).$$

This is the autonomy property of the propagated error, and it corresponds to point (iv) in the list below. The following theorem proves this necessarily implies the general dynamics (30) be in fact of the form (8) indeed (point (i) below).

Theorem 2. *Let G be a group and $\psi : G \to G$ a function. Then the following seven properties are equivalent:*

(i) *There exists $\phi_l \in Aut(G)$ and $a \in G$ such that $\psi(g) = \phi_l(g) \cdot a$*

(ii) *There exists $a \in G$ and $\phi_r \in Aut(G)$ such that $\psi(g) = a \cdot \phi_r(g)$*

(iii) *ψ verifies $\forall g_1, g_2 \in G, \psi(g_1 \cdot g_2) = \psi(g_1) \cdot \psi(Id)^{-1} \cdot \psi(g_2)$*

(iv) *A function μ_l on G exists, such that $\forall g_1, g_2 \in G, \psi(g_1)^{-1} \cdot \psi(g_2) = \mu_l(g_1^{-1} \cdot g_2)$*

(v) *A function μ_r on G exists, such that $\forall g_1, g_2 \in G, \psi(g_1) \cdot \psi(g_2)^{-1} = \mu_r(g_1 \cdot g_2^{-1})$*

(vi) *There exists an automorphism $\mu_l \in Aut(G)$ such that $\forall g_1, g_2 \in G, \psi(g_1)^{-1} \cdot \psi(g_2) = \mu_l(g_1^{-1} \cdot g_2)$*

(vii) *There exists an automorphism $\mu_r \in Aut(G)$ such that $\forall g_1, g_2 \in G, \psi(g_1) \cdot \psi(g_2)^{-1} = \mu_r(g_1 \cdot g_2^{-1})$*

Note that, as a byproduct, we recover that linear systems are the only ones that ensure autonomous error propagation in the vector space case of Section 2.

The proof has been moved to the Appendix to improve readability. However, we reproduce it partially here, just to give an idea of why this theorem holds. Indeed, a particularly interesting implication is $(vi) \Rightarrow (i)$, that proves linear observation systems are the only ones that possess the autonomy related property (vi). This can be proved as follows. Applying (vi) to (Id, g) we obtain

$$\psi(Id)^{-1} \cdot \psi(g) = \mu_l(g)$$
$$\Rightarrow \psi(g) = \psi(Id) \cdot \mu_l(g) \cdot \psi(Id)^{-1} \cdot \psi(Id) = I_{\psi(Id)} \circ \mu_l(g) \cdot \psi(Id)$$

Setting $\phi = I_{\psi(Id)} \circ \mu_l$ (which is an automorphism as composition of automorphisms) and $a = \psi(Id)$ we obtain (i) indeed.

Remark 6. *A continuous time (and thus different) version of the equivalence between and (iii), (iv) and (v) already appears in our previous work [2], where the continuous analog to (iii) is referred to as the "group affine" property. So, the very intuitive definition of* (8) *turns out to be equivalent to the latter property, a fact never noticed before. Note also that, a continuous and local version of (iii) is introduced in [1] to define linear control systems on Lie groups, but this is done for control - not observation - purposes and no equivalence with error autonomy is shown nor exploited.*

6 Applications of the framework using the family of groups $SE_K(D)$

In this section, we would like to introduce a family of (Lie) groups that we denote $SE_K(D)$. It was introduced in [3] for arbitrary K and $D = 2$ or $D = 3$. Making use of it, many systems encountered in robotics and navigation are shown below to fit into the framework of linear observation systems.

6.1 The group family $SE_K(D)$

For $D \in \mathbb{N}$, let $SO(D)$ denotethe special orthogonal group of $\mathbb{R}^D$, that is, the set:

$$SO(D) = \{R \in \mathbb{R}^{D \times D},\ R^T R = Id,\ \det(R) = 1.\}$$

Definition 11 (The family $SE_K(D)$)**.** *For $K, D \in \mathbb{N}$, the group $SE_K(D)$ is defined as the set:*

$$SE_K(D) = \{(R, r_1, \cdots, r_K), R \in SO(D), r_1, r_2, \ldots, r_K \in \mathbb{R}^3\},$$

endowed with the following group law:

$$(R, r_1, \ldots r_K) \cdot (T, t_1, \ldots t_K) = (RT, Rt_1 + r_1, \cdots, Rt_K + r_K)\,.$$

In particular, the neutral element of $SE_K(D)$ is

$$Id = (I_3, 0_{3,1}, \ldots, 0_{3,1})\,,$$

and the inverse of an element $x = (R, r_1, \ldots, r_K)$ is

$$x^{-1} = \left(-R^T, -R^T r_1, \ldots, -R^T r_K\right).$$

Of course, we retrieve $SO(2)$ and $SO(3)$ for $K = 2,3$ and $D = 0$, $SE(2)$ and $SE(3)$ for $K = 2,3$, $D = 1$, the group $SE_2(3)$ introduced in [2] for inertial navigation, and the groups $SE_K(2)$ and $SE_K(3)$ shown in [3] to be the right framework for EKF-SLAM (note the first introduction of this group structure for nonlinear observer design for SLAM dates back to [8]).

Each group $SE_K(D)$ comes with a family of natural group actions allowing to recover many of the observation functions of robotics and navigation:

Definition 12 (Actions of the group $SE_K(D)$)**.** *We call vector action of $SE_K(D)$ on $\mathbb{R}^D$ with parameters $(\gamma_1, \ldots, \gamma_K) \in \mathbb{R}^K$ the action of $x = (R, r_1, \ldots, r_K) \in SE_K(D)$ on $b \in \mathbb{R}^D$ defined by:*

$$x \star b = Rb + \sum_{i=1}^{K} \gamma_i r_i$$

It can be readily checked that this defines an action of $SE_K(D)$: let $x = (R, r_1, \ldots, r_K) \in SE_K(D)$, $x' = (T, t_1, \ldots, t_K) \in SE_K(D)$ and $b \in \mathbb{R}^D$. Then:

$$\begin{aligned} x \star (x' \star b) &= x \star \left(Tb + \sum_{i=1}^{K} \gamma_i t_i \right) \\ &= R \left(Tb + \sum_{i=1}^{K} \gamma_i t_i \right) + \sum_{i}^{K} \gamma_i r_i \\ &= RTb + \sum_{i=1}^{K} \gamma_i \left(Rt_i + r_i \right) \\ &= (x \cdot x') \star b \end{aligned}$$

where we used $x \cdot x' = (RT, Rt_1 + r_1, \ldots, Rt_K + r_K)$ from Definition 11.

This family of groups and actions will be used all over the present section, only changing K, D and the parameters $\gamma_1, \ldots, \gamma_K$ of the action. In applications, it might be the case that we need to extend the group action to account for multiple observations, through the following notion.

Proposition 8 (Direct product of group actions)**.** *We will call "product action" the action of a group G on cartesian products of sets on which the group acts separately. More precisely, letting $\star_1$ and $\star_2$ be two group actions of G on two sets Y_1 and Y_2 respectively, we can define an action $\star$ of G on $Y = Y_1 \times Y_2$ as follows:*

$$G \times Y = Y \tag{31}$$

$$x \star (b_1, b_2) = (x \star_1 b_1,\ x \star_2 b_2), \tag{32}$$

and the observation $y = (y_1, y_2) \in Y_1 \times Y_2$ fits into the generalized observation framework (9)*.*

Proof. We have to show that $\star$ satisfies (7). Let $x, x' \in G$, we have: $x\star(x' \star (b_1, b_2)) = x\star(x' \star_1 b_1, x' \star_2 b_2) = (x \star_1 (x' \star_1 b_1), x \star_2 (x' \star_2 b_2)) = ((x \cdot x') \star_1 b_1, (x \cdot x') \star_2 b_2) = (x \cdot x') \star (b_1, b_2)$. In term of observation functions, if we have two observations

$$y_n^1 = x_n \star_1 b_n^1 \quad \text{and} \quad y_n^2 = x_n \star_2 b_n^2$$

then we can stack them into a single observation $y_n = x_n \star b_n$ with $y_n = (y_n^1, y_n^2)$, $b_n = (b_n^1, b_n^2)$ and $\star$ defined as above. $\square$

6.2 Examples of linear dynamics on $SE_K(D)$

Let us see how common dynamics used in navigation and robotics can be cast into the framework of linear observation systems using the family of groups $SE_K(D)$.

6.2.1 2D non-honomic car with differential odometry

The vehicle is characterized by its orientation or heading $\theta_n \in \mathbb{R}$ and position $X_n \in \mathbb{R}^2$. Its speedometers measure velocities and heading change rate (through differential odometry), yielding the following dynamics:

$$\begin{aligned} \theta_{n+1} &= \theta_n + \omega_n \\ X_{n+1} &= X_n + R_{\theta_n} v_n, \end{aligned} \tag{33}$$

Representing θ_n through R_{θ_n}, the planar rotation of angle θ_n, we can embed this system into $SE_1(2) = SE(2)$:

$$x_n = (R_{\theta_n}, X_n) \in SE(2)$$

As well, the increments ω_n, v_n can be embedded into $SE(2)$ as $a_n = (R_{\omega_n}, v_n) \in SE(2)$, and the dynamics (33) becomes:

$$x_{n+1} = x_n \cdot a_n$$

6.2.2 3D Inertial navigation

The vehicle is characterized by its attitude $R_n \in SO(3)$ (the rotation matrix mapping the vehicle-fixed frame to a reference static frame), its velocity $V_n \in \mathbb{R}^3$ and position $X_n \in \mathbb{R}^3$. Its gyrometers measure angular rates $\Omega_t \in \mathbb{R}^3$ in continuous-time t, and its accelerometers the "specific force" $f_t \in \mathbb{R}^3$, i.e. vehicle acceleration minus gravity vector g. Both sensors are part of a measurement inertial unit (IMU) attached to the vehicle. In continuous time the dynamics of the system is:

$$\frac{d}{dt} R_t = \Omega_t, \quad \frac{d}{dt} V_t = R_t f_t + g, \quad \frac{d}{dt} X_t = V_t. \tag{34}$$

Unexpectedly enough (the result is non-trivial), the discretized dynamics of navigation above correspond to linear dynamics on the group.

Proposition 9. *3D navigation equations* (34) *can be discretized to yield linear dynamics of the form* (8).

Proof. Between two time steps n and $n+1$ having time stamps t_n and $t_{n+1} = t_n + \tau_n$ (τ_n is the discretization step), we define integrated inputs as follows (notice the deliberate absence of g in the equation of $\bar{V}_t$):

$$\bar{R}_{t_n} = I_3, \bar{V}_{t_n} = 0, \bar{X}_{t_n} = 0,$$

$$\frac{d}{dt}\bar{R}_t = \Omega_t, \quad \frac{d}{dt}\bar{V}_t = \bar{R}_t f_t, \quad \frac{d}{dt}\bar{X}_t = \bar{V}_t$$

Then the corresponding discrete increments Ω_n, f_n, s_n are the solutions obtained at t_{n+1}: $\Omega_n = \bar{R}_{t_n}$, $f_n = \bar{V}_{t_n}$, $s_n = \bar{X}_{t_n}$. Now we use them to write the dynamics in discrete time:

$$\begin{aligned} R_{n+1} &= R_n \Omega_n \\ V_{n+1} &= V_n + R_n f_n + \tau_n g \\ X_{n+1} &= X_n + \tau_n V_n + R_n s_n \end{aligned} \tag{35}$$

We can embed this system into $SE_2(3)$, as well as the discrete increments, defining:

$$x_n = (R_n, V_n, X_n) \in SE_2(3) \tag{36}$$

$$u_n = (\Omega_n, f_n, s_n) \in SE_2(3) \tag{37}$$

$$g_n = (I_3, \tau_n g, \frac{1}{2}\tau_n^2 g) \in SE_2(3) \tag{38}$$

Then, we define the function $\phi'_n \in End\,(SE_2(3))$ as:

$$\begin{aligned} \phi'_n : SE_2(3) &\to SE_2(3) \\ (R, V, X) &\to (R, V, X + \tau_n V) \end{aligned} \tag{39}$$

which can be easily checked to be an automorphism of $SE_2(3)$. The dynamics (35) becomes:

$$x_{n+1} = g_n \cdot \phi'_n(x_n) \cdot u_n, \tag{40}$$

which can be re-written as:

$$x_{n+1} = g_n \cdot \phi'_n(x_n) \cdot g_n^{-1} \cdot g_n \cdot u_n = I_{g_n} \circ \phi'_n(x_n)(g_n \cdot u_n),$$

Finally, setting $\phi_n = I_{g_n} \circ \phi'_n$, i.e., $\phi_n(x_n)$ to be

$$\left(R_n, V_n + \tau_n(I_3 - R)g, X_n + \tau_n V_n + \frac{1}{2}\tau_n^2(I_3 - R)g\right) \tag{41}$$

(which is an automorphism as a composition of automorphisms) and

$$a_n = g_n \cdot u_n = \begin{pmatrix} \Omega_n, & \tau_n g + f_n, & \frac{1}{2}\tau_n^2 + s_n \end{pmatrix} \tag{42}$$

the dynamics (35) eventually writes:

$$x_{n+1} = \phi_n(x_n) \cdot a_n. \tag{43}$$

$\square$

6.2.3 Odometry based Simultaneous Localisation and Mapping (SLAM)

A static map of feature points can be included into the state without breaking the linear property of the dynamics. If we consider the 3D version of the wheeled robot model above, the vehicle is characterized by its attitude $R_n \in SO(3)$ (the rotation matrix mapping the vehicle-fixed frame to a reference static frame) and position $X_n \in \mathbb{R}^3$. Its odometers and/or gyrometers measure relative shifts $v_n \in \mathbb{R}^3$ and rotations $\Omega_n \in SO(3)$ of the vehicle, and a map of static feature points $(P_{k,n})_{1 \geq k \geq K}$ (k is the feature index while n is the time step), yielding the following dynamics:

$$\begin{aligned} R_{n+1} &= R_n \Omega_n \\ X_{n+1} &= X_n + R_n v_n \\ P_{1,n+1} &= P_{1,n} \\ &\vdots \\ P_{K,n+1} &= P_{K,n} \end{aligned} \tag{44}$$

We can embed this system into $SE_{1+K}(3)$:

$$x_n = (R_n, X_n, P_{1,n}, \dots, P_{K,n}) \in SE_{1+K}(3) \tag{45}$$

As well, the increments Ω_n, v_n can be embedded into $SE_{1+K}(3)$ as

$$a_n = (\Omega_n, v_n, 0_{2,1}, \dots, 0_{2,1}) \in SE_{1+K}(3) \tag{46}$$

and the dynamics (33) writes :

$$x_{n+1} = x_n \cdot a_n$$

6.2.4 3D IMU-aided SLAM

As well, adding a static map to the inertial navigation equations (36),(41),(42) is straightforward and yield linear dynamics, and this has never been noticed before. In this case, the state becomes:

$$x_n = (R_n, V_n, X_n, P_{1,n}, \dots, P_{K,n}) \in SE_{2+K}(3) \tag{47}$$

The increment becomes:

$$a_n = (\Omega_n, V_n, X_n, 0_{2,1}, \dots, 0_{2,1}) \in SE_{2+K}(3) \tag{48}$$

The automorphism Φ_n becomes:

$$\begin{aligned} \phi_n(x_n) = &(R_n, V_n + (R_n - I_3)\tau_n g, \\ &X_n + \tau_n V_n + \frac{1}{2}(R - I_3)\tau_n^2 g, P_{1,n}, \dots, P_{K,n}) \end{aligned} \tag{49}$$

and the 3D inertial navigation with simultaneous map estimation writes:

$$x_{n+1} = \phi_n(x_n) \cdot a_n \tag{50}$$

6.2.5 SLAM with moving features

For any previously considered dynamics on $SE_K(D)$, a map containing moving points having constant (but unknown) velocity, can be added to the state without breaking the linear property of the dynamics. Assume navigating using the 3D inertial navigation model (46), (48), (49) while tracking points $P_1, \dots, P_k$ having constant velocity vectors $Q_1, \dots, Q_k$ (note that tracking both static and moving points would not be a problem). The state becomes an element of $SE_{2+K}(3)$:

$$x_n = (R_n, V_n, X_n, P_{1,n}, \dots, P_{K,n}, Q_{1,n}, \dots, Q_{K,n},) \tag{51}$$

The increment becomes:

$$a_n = (\Omega_n, V_n, X_n, 0_{2,1}, \dots, 0_{2,1}) \in SE_{2+2K}(3) \tag{52}$$

The automorphism Φ_n becomes $\phi_n(x_n) = (R_n, V_n + (R - I_3)\tau_n g, X_n + \tau_n V_n + \frac{1}{2}(R_n - I_3)\tau_n^2 g, P_{1,n} + \tau_n Q_{1,n}, \dots, P_{K,n} + \tau_n Q_{K,n}, Q_{1,n}, \dots, Q_{K,n})$, and the 3D inertial navigation in a map with possibly moving features again writes:

$$x_{n+1} = \phi_n(x_n) \cdot a_n \tag{53}$$

6.3 Examples of linear observation observations as group actions of $SE_K(D)$

Here, we assume the state is embedded into $SE_K(D)$. Using vector actions of this family of groups using Definition 12 with suitable parameters $(\gamma_i)_{1 \geq i \geq K}$, we can easily express many standard observation functions of robotics, navigation, and localization.

6.3.1 Measurements through left group actions

Considering $SE_K(D)$ as we use it, i.e., R generally represents the rotation mapping the body frame to the global frame, and the remaining vectors generally denote quantities of the global frame. The first type of measurements is associated to absolute measurements. Assume indeed (for example) that we work on 3D inertial navigation with state space $x_n = (R_n, V_n, X_n) \in SE_2(3)$. Let us see what kind of observations we obtain using Definition 12. To this end, consider an observation y_n defined as:

$$y_n = x_n \star b$$

If we choose the parameters $\gamma_1 = 1, \gamma_2 = 0$ and the target $b = 0_{3,1}$ we obtain a velocity measurement:

$$y_n = V_n.$$

If we choose the parameters $\gamma_1 = 0, \gamma_2 = 1$ and the target $b = 0_{3,1}$ we obtain a position measurement:

$$y_n = X_n.$$

If we choose the parameters $\gamma_1 = 0, \gamma_2 = 1$ and a nonzero target $b = b_0 \in \mathbb{R}^3$ we obtain a position measurement of a point with known lever arm b_0 with respect to the vehicle frame origin, for example a GNSS antenna having a lever arm with respect to the IMU:

$$y_n = X_n + R_n b_0.$$

6.3.2 Measurements through right group actions

They are measurements being relative to the body. Indeed assume (for example) that we work on 3D inertial-aided (or IMU-aided) SLAM, with state space $x_n = (R_n, V_n, X_n, P_{1,n}, \cdots, P_{K,n}) \in SE_{2+K}(3)$ as in (47). Let us see what observation functions can be modeled as vector group actions of $SE_2(3)$ on $\mathbb{R}^3$ with suitable parameters $(\gamma_i)_{1 \geq i \geq K}$. To this end, consider an observation y_n defined as:

$$y_n = x_n^{-1} \star b,$$

where the star $\star$ a vector action of $SE_{2+K}(3)$ to be defined through its parameters $(\gamma_i)_{1 \geq i \geq K}$. Note that, it is a right action (using the inverse, see Section 4.3).

If we choose all parameters γ_i equal to zero and a non-zero target $b = -b_0 \in \mathbb{R}^3$ (representing for instance the magnetic field) we obtain a partial attitude mesurement:

$$y_n = R_n^T b_0$$

If we choose the $\gamma_1 = -1$ and all other γ_i equal to zero, and also take a zero target $b = 0_{3,1}$ we obtain a velocity measurement in the vehicle frame:

$$y_n = R_n^T V_n$$

If we choose the $\gamma_2 = 1$, all other γ_i equal to zero, and a (possibly) nonzero target $b = -\tilde{P} \in \mathbb{R}^3$ we obtain a relative measurement of a landmark with *known* absolute position $\tilde{P}$:

$$y_n = R_n^T(\tilde{P} - x_n)$$

If we choose the all parameters γ_i equal to zero except γ_2 and γ_{2+k} for one index $k \leq K$, and set $\gamma_2 = 1$, $\gamma_{2+k} = -1$ and $b = 0_{3,1}$ we obtain a relative position measurement of unknown feature k which is part of the state:

$$y_n = R_n^T(P_{k,n} - X_n)$$

Note that this extends to the observation of an arbitrary number of observations using Proposition 8. Finally, if we track moving features and the state has K additional velocity states as above, choosing all parameters γ_i to be equal to zero except γ_1 and γ_{2+K+k} for one index $k \leq K$, and setting $\gamma_1 = 1$, $\gamma_{2+K+k} = -1$ and $b = 0_{3,1}$ we obtain a relative velocity measurement of feature k:

$$y_n = R_n^T(Q_{k,n} - V_n)$$

We have thus proved the following result.

Proposition 10. *Combining the dynamics of Section 6.2 and observations* either *of Section 6.3.1* or *Section 6.3.2, yields linear systems on groups, and the corresponding generalized linear observers readily have the autonomous error equations property. Furthermore, under uniform observability conditions, there exists a class of gains L_n ensuring local asymptotical convergence around* any *trajectory.*

The latter part is a mere consequence of the fact that $SE_K(D)$ is a Lie group, and of Proposition 7.

6.4 Linear observation formulation of some classical problems

Combining the dynamics and observations of Sections 6.2 and 6.3 we see that a wide range of systems fit into our framework. For the article to be self-contained and useful to practitioners, we enumerate quite a few of them.

6.4.1 GPS-aided inertial navigation

The state space of equations(35) modeling attitude, position and velocity can be embedded into $SE_2(3)$ as (36). Then, with ϕ_n and a_n defined as (41) and (42) and defining $\star$ as the vector action (see Def. 12) of $SE_2(3)$ with parameters $\gamma_1 = 0, \gamma_2 = 1$ and target $b = 0_{3,1}$, the dynamics (35) with position observations becomes:

$$x_{n+1} = \Phi_n(x_n) \cdot a_n, \quad y_n = x_n \star b_n$$

6.4.2 3D SLAM with a wheeled robot

The state space of equations (44) attitude and position can be embedded into $SE_{1+K}(3)$ as (45). Then, whith a_n defined as (46) and defining $\star$ as the vector action (see Def. 12) of $SE_{1+K}(3)$ with parameters $\gamma_1 = -1, \gamma_{1+k} = 1$ for a given feature index k, $\gamma_i = 0$ for other indices i and target $b_n = 0_{3,1}$, the dynamics (44) with relative observations of landmark k becomes:

$$x_{n+1} = x_n \cdot a_n, \quad y_n = x_n^{-1} \star b_n$$

Any number of features can be observed at each time step, as stacking group actions still gives a group action (see Proposition 8).

6.4.3 IMU-aided 3D SLAM

The state space of equations(35) modeling attitude, position and velocity, when adding K static landmarks $(P_{k,n})_{1 \leq k \leq K}$, can be embedded into $SE_{2+K}(3)$ as (47). Then, whith ϕ_n and a_n defined as (49) and (48) and defining $\star$ as the vector action (see Def. 12) of $SE_{2+K}(3)$ with all parameters γ_i equal to zero except $\gamma_2 = -1, \gamma_{2+k} = 1$, and target $b = 0_{3,1}$, the dynamics (35) with K static landmarks $(P_{k,n})_{1\leq k\leq K}$ and relative feature position measurement of landmark k becomes:

$$x_{n+1} = \Phi_n(x_n) \cdot a_n, \quad y_n = x_n^{-1} \star b_n$$

Any number of features can be observed at each time step, as stacking group actions still gives a group action (see Proposition 8).

7 Conclusion

In this paper, we have provided a unifying and versatile framework for the definition of linear systems on Lie groups that encompass various prior works, but focusing on the discrete time case. The approach is supported by sound mathematical properties and comes with complete characterizations. Its versatility is shown through applications to a wide variety of problem from navigation and robotics using a single family of groups $SE_K(D)$.

A Proof of Theorem 2

Proof. We first show $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (vi) \Rightarrow (i)$:

1. $(i) \Rightarrow (ii)$ If (i) is verified we have $\psi(g) = \phi_l(g) \cdot a = a \cdot a^{-1} \cdot \phi_l(g) \cdot a = a \cdot I_a \circ \phi_l(g)$. Setting $\phi_r = I_a \circ \phi_l$ we obtain (ii), $I_a \circ \phi_l$ being a automorphism as the composition of two automorphisms.

2. $(ii) \Rightarrow (iii)$ We just have to assume (ii) and check (iii): $\phi(g_1 \cdot g_2) = a \cdot \phi_r(g_1 \cdot g_2) = a \cdot \phi_r(g_1) \cdot \phi_r(g_2) = a \cdot \phi_r(g_1) \cdot \phi_r(Id)^{-1} \cdot \phi_r(g_2)$, where we have used the automorphisms properties $\phi_r(g_1 \cdot g_2) = \phi_r(g_1) \cdot \phi_r(g_2)$ and $\phi_r(Id) = Id$. Now we insert the element $a^{-1} \cdot a = Id$ then add brackets to regroup terms: $\phi(g_1 \cdot g_2) = a \cdot \phi_r(g_1) \cdot \phi_r(Id)^{-1} \cdot a^{-1} \cdot a \cdot \phi_r(g_2) = [a \cdot \phi_r(g_1)] \cdot [a \cdot \phi_r(Id)]^{-1} \cdot [a \cdot \phi_r(g_2)] = \psi(g_1) \cdot \psi(Id)^{-1} \cdot \psi(g_2)$.

3. $(iii) \Rightarrow (iv)$ For all g, applying (iii) to (g, g^{-1}) as (g_1, g_2) we obtain $\psi(Id) = \psi(g) \cdot \psi(Id)^{-1} \cdot \psi(g^{-1})$, thus $\psi(g)^{-1} = \psi(Id)^{-1} \cdot \psi(g^{-1}) \cdot \psi(Id)^{-1}$. Now, for all g_1, g_2 we have $\psi(g_1)^{-1} \cdot \psi(g_2) = \psi(Id)^{-1} \cdot \psi(g_1^{-1})\psi(Id)^{-1} \cdot \psi(g_2) = \psi(Id)^{-1} \cdot \psi(g_1^{-1} \cdot g_2)$ applying (iii) again. Setting $\mu_l(g) = \psi(Id)^{-1} \cdot \psi(g)$ we obtain (iv)

4. $(iv) \Rightarrow (vi)$ Applying (iv) to (Id, g) we obtain $\psi(Id)^{-1}\psi(g) = \mu_l(g)$, applying to (g^{-1}, Id) we obtain $\psi(g^{-1})^{-1} \cdot \psi(Id) = \mu_l(g)$. To show μ_l is a automorphism we compute $\mu_l(g_1 \cdot g_2) = \mu_l((g_1^{-1})^{-1} \cdot g_2) = \psi(g_1^{-1})^{-1} \cdot \psi(g_2)$ (using (iv)) which is, using the two relations we just derived: $\left[\mu_l(g_1) \cdot \psi(Id)^{-1}\right] \cdot [\psi(Id) \cdot \mu_l(g_2)]$, i.e., $\mu_l(g_1) \cdot \mu_l(g_2)$.

5. $(vi) \Rightarrow (i)$ Applying (vi) to (Id, g) we obtain $\psi(Id)^{-1} \cdot \psi(g) = \mu_l(g)$, i.e. $\psi(g) = \psi(Id) \cdot \mu_l(g) \cdot \psi(Id)^{-1} \cdot \psi(Id) = I_{\psi(Id)} \circ \mu_l(g) \cdot \psi(Id)$. Setting

$\phi = I_{\psi(Id)} \circ \mu_l$ (which is an automorphism as composition of automorphisms) and $a = \psi(Id)$ we obtain (i).

The proofs for $(iii) \Rightarrow (v) \Rightarrow (vii) \Rightarrow (iii)$ are similar. □

B Further links and differences with the theory of symmetry preserving observers

As said before, [7] (and to some extent [9] as well in a partial way) considers a different class of systems, called invariant systems, where the group is allowed to act on the inputs of the system, that is, (using a discrete time analog) systems of the form

$$x_{n+1} = f(x_n, u_n)$$

with x_n in G, having the property (called invariance of the system) that

$$g \cdot f(x_n, u_n) = f(g \cdot x_n, g \star u_n) \tag{54}$$

with $\star$ a group action. Using this formalism, the navigation example of [9] where one seeks to estimate the velocity and orientation of a body in space using various sensors, was proved to lead to an autonomous error, due to the following reason. The discrete time counterpart of this example system is of the form

$$x_{n+1} = b_n \cdot x_n \cdot v_n \tag{55}$$

which can be re-written $x_{n+1} = f(x_n, u_n)$ with $u_n = (v_n, b_n)$. By defining the following group action on the inputs $g \star (v_n, b_n) = g \cdot b_n \cdot g^{-1}$ we recover indeed that the dynamics fulfills the definition of invariance (54) since $g \cdot f(x_n, u_n) = g \cdot b_n \cdot x_n \cdot v_n = g \cdot b_n \cdot g^{-1} \cdot g \cdot x_n \cdot v_n = f(g \cdot x_n, g \star u_n)$, owing to associativity.

But, (55) also defines linear dynamics on the group in the sense (8) of the present article, by letting $\phi_n(x) = b_n \cdot x_n \cdot b_n^{-1}$, which is an endormorphism, and $a_n = b_n \cdot v_n$. So it is logical that the equation error be autonomous indeed. Note that, however, the definition (54) of an invariant system of [9, 7] is broader than ours (8), since the group can act in any desired way on the inputs u_n, that can correspond to control inputs or more generally to any parameter of the system. But then, autonomous error equations are then not guaranteed, unless f has a specific form, namely $f(x_n, u_n) = x_n \cdot u_n$ in equation (12) of [7].

A last remark regarding the links to [7] is that in the latter paper the notion of "permanent trajectory" is introduced. Around such trajectories, the error system becomes time-invariant, and so will the linearized error system in the case where G is a Lie group. By tuning the gains using Proposition 7, the gains will then converge to fixed values, around permanent trajectories. In the present framework, permanent trajectories are trajectories of the dynamical system (8) such that ϕ_n, a_n are held constant over time.

C Continuous time case and exact discretization

In [2], the continuous analog to (iii) is referred to as the "group affine" property. For a continuous-time system on a Lie group G defined by

$$\frac{d}{dt}x_t = f_t(x_t) \tag{56}$$

where $f(x_t)$ is now a tangent vector to G at x_t, the affine property requires that for all $a, b \in G$ and all $t \in \mathbb{R}$ we have

$$f_t(a \cdot b) = a \cdot f(b) + f(a) \cdot b - a \cdot f(Id) \cdot b \tag{57}$$

where in the right hand side we also use "$\cdot$" to define the tangent maps induced by group multiplications. Note also that, a local version of (57) is introduced in [1] to define linear control systems on Lie groups. We have the following result.

Proposition 11. *Consider the continuous time dynamics* (56) *on a Lie group G, and assume it satisfies the group affine condition* (57)*. Then, given discrete time instants $t_1 < t_2 < t_3 \cdots$, there exists a sequence of automorphisms ϕ_n and group elements $a_n \in G$ such that the system admits an* exact *discretization in the form of* (8)*, i.e.,*

$$x_{t_{n+1}} = \phi_n(x_{t_n}) \cdot a_n$$

Proof. According to equation (60) of [2], combined with Lemma 2 of [2], the error between two arbitrary trajectories satisfies $e_{t_{n+1}} = \phi_n(e_{t_n})$ with ϕ_n an automorphism, and thus property (vi) of Theorem 2 is satisfied. And by Theorem 2 it is equivalent to dynamics of form (8). $\square$

D Controlling linear observation systems on groups

Consider a linear observation system (8)-(9) on G, and assume it may be controlled through a control input u_n as follows

$$\begin{aligned} x_{n+1} &= \phi_n(x_n) \cdot a_n \cdot u_{n+1} \\ y_n &= x_n \star b_n \end{aligned}$$

Consider a reference trajectory for the system that one may want to track of the form

$$x^r_{n+1} = \phi_n(x^r_n) \cdot a_n \cdot u^r_{n+1}$$

Let $e_n = (x^r_n)^{-1} \cdot x_n$ be the tracking error. Define the controller as

$$u_{n+1} = g(\rho_{n+1}, (x^r_n)^{-1} \star y_n), \quad \rho_{n+1} = g(\rho_n, (x^r_n)^{-1} \star y_n)$$

with g, f freely chosen by the user. We have the following result

Proposition 12. *The tracking error system is* independent *of the reference trajectory x_n^r, in the sense that we have*

$$e_{n+1} = (u_{n+1}^r)^{-1} \cdot a_n^{-1} \cdot \phi_n(e_n) \cdot a_n \cdot g(\rho_{n+1}, e_n \star b) \tag{58}$$
$$\rho_{n+1} = f(\rho_n, e_n \star b) \tag{59}$$

Proof. We have indeed using the automorphism property of ϕ_n and the group action property of $\star$

$$e_{n+1} = (x_{n+1}^r)^{-1} \cdot x_{n+1} = (u_{n+1}^r)^{-1} \cdot a_n^{-1} \cdot \phi_n((x_n^r)^{-1} \cdot x_n) \cdot a_n \cdot g(\rho_{n+1}, ((x_n^r)^{-1} \cdot x_n) \star b).$$

□

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