Late-lumping backstepping control of partial differential equations

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Abstract

We consider in this paper three different partial differential equations (PDEs) that can be exponentially stabilized using backstepping controllers. For implementation, a finite-dimensional controller is generally needed. The backstepping controllers are approximated and it is proven that the finite-dimensional approximated controller stabilizes the original system if the order is high enough. This approach is known as late-lumping. The other approach to controller design for PDE's first approximates the PDE and then a controller is designed; this is known as early lumping. Simulation results comparing the performance of late-lumping and early-lumping controllers are provided.

Key words: partial differential equations; stabilization; backstepping, late-lumping.

1 Introduction

Controller design for partial differential equations (PDEs) typically needs to be done using a finite-dimensional, or lumped, approximation of the PDE. This approach is known as early lumping. It introduces questions of stability and performance of the designed system. However, for some PDEs, backstepping controllers can be directly designed using the PDE. Introduced in [49,50] for a general 1-D linear reaction-diffusion-advection PDE, it has been extended to a large number of boundary control problems: flow control [2,3,59], parabolic PDEs [54,55], or hyperbolic PDEs [5,17,24]. A complete history of the backstepping method and of its extensions has recently been given in [56]. The resulting controllers are explicit, in the sense that they are expressed as a linear functional of the distributed state at each instant. The (distributed) gains can be computed offline. Considering application of such controllers to industrial problems, in most cases, only an approximation of the state is available for controller design and the controller needs to be approximated.

This direct controller design approach is sometimes referred to as late lumping since the last step in the design is to approximate the controller by a finite-dimensional, or lumped parameter, system. The other approach is early-lumping where the controller design is based on a finite-dimensional approximation of the PDE. Numerous results ensuring the convergence of early-lumping controllers can be found in the literature; see for example[7,8,30,32–34,39,40] and the tutorial paper [42]. However, the question of the convergence of late-lumping backstepping controllers has not been well-investigated. In [58], a method for computing the bounded part of the control operator is proposed. It relies on a finite-dimensional approximation of the state and enables efficient computing of the feedback law. However, the unbounded part of the operator is not approximated and no guarantees of convergence are provided.

In this paper late lumping control is considered for three different systems that can be stabilized using backstepping control laws. The main contribution of this paper is to give sufficient conditions guaranteeing the convergence of backstepping-based late-lumping controllers for various examples: an unstable heat equation [50], a wave equation [51] and a general class of linear hyperbolic PDEs [17]. For each example, we consider an approximation of the state (that satisfies some specific assumptions) to design the control law. The resulting feedback system is mapped to a simpler target system us-

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ing backstepping-like transformations. An explicit Lyapunov function is used to prove exponential stability. The design is based on the boundary control formulation; the system is not converted to state space form. The performance of these late-lumping controllers are compared to early-lumping controllers in simulations using a high order approximation of the PDE as the system.

The paper is organized as follow. Section 2 provides the general framework and recalls existing results for early-lumping and late-lumping control. Some crucial assumptions concerning the state space and the approximating space are also given. We then prove for various examples (for which backstepping control laws have already been derived), that the approximated control laws still guarantee exponential stabilization. The heat equation is considered in Section 3, the wave equation in Section 4 and a general class of hyperbolic PDEs in Section 5. Some simulations results are given for each example: the late-lumping controller is compared in term of performance and control effort with a early-lumping controllers derived using a Galerkin approximation.

2 Presentation of the method

All the systems systems considered in this paper are boundary control systems [47]

$$\begin{aligned}&\frac{dz}{dt} = \mathfrak{A}z(t), \quad z(0) = z_0, \quad t \in [0, T]\\ &\mathfrak{B}z(t) = u(t), \end{aligned} \tag{1}$$

where $\mathfrak{A} : D(\mathfrak{A}) \subset \mathcal{Z} \mapsto \mathcal{Z}$ with $\mathcal{Z}$ a separable Hilbert space $\mathcal{Z}$ (the **state space**), $u(t) \in \mathcal{U}$, the Hilbert space $\mathcal{U}$ being the **input space**. The boundary control operator $\mathfrak{B} : D(\mathfrak{B}) \subset \mathcal{Z} \mapsto U$ is called the **control operator** of the system and satisfies $D(\mathfrak{A}) \subset D(\mathfrak{B})$. It is assumed that a unique solution to (1) with $u \equiv 0$ exists and is given by the semigroup $S(t)$. The initial condition z_0 is assumed to belong to $\mathcal{Z}$.

These systems can be rewritten in an abstract state space form, generally using unbounded control operators; that is, a control operator bounded to some Hilbert space larger than the state space and an observation operator bounded from a Hilbert space smaller than the state space [47]. There is an extensive literature dealing with systems having unbounded control operators; see for instance [18,19,21,23,31,45,47,57]).

It is not necessary though to convert to state space form [16]. The backstepping approach uses the boundary control formulation given by (1) and this formulation is used in approximation of the backstepping controller.

In this paper, the space $\mathcal{Z}$ has to satisfy the following additional assumption.

Assumption 1 *The domain of definition $D(\mathfrak{A})$ satisfies $D(\mathfrak{A}) \subset (\mathcal{H}^1([0, 1]))^p$ where p is a positive integer.*

The value of p depends on the particular PDE. Since the space $\mathcal{H}^1([0, 1])$ is embedded in the Holder space $\mathcal{C}^{0,\frac{1}{2}}([0, 1])$, using Morrey's inequality (see e.g [12, Theorem 9.12]), a direct consequence of Assumption 1 is the existence of an constant $\alpha > 0$ such that for all $z \in D(\mathfrak{A})$, for all $1 \leq i \leq p$,

$$\sup_{x\in[0,1]} |z_i(x)| \leq \alpha(||z_i||_{\mathcal{H}^1([0,1])})^p \tag{2}$$

Definition 1 *The system* (1) *is **exponentially stabilizable** if there exists $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ such that if $u(t) = Kz(t)$ the semigroup $\mathcal{S}$ associated to* (1) *is exponentially stable semigroup, i.e there exist $M \geq 1$ and $\omega > 0$ such that*

$$||\mathcal{S}(t)|| \leq Me^{-\omega t} \tag{3}$$

The early-lumping approach (also known as indirect controller design) consists in approximating the original PDE (1) using standard methods (such as finite elements for instance). This yields a system of ordinary differential equations. Controller design is based on this finite-dimensional approximation. Consider finite-dimensional subspace $\mathcal{Z}_n$ of the state-space $\mathcal{Z}$ and P_n the orthogonal projection $P_n : \mathcal{Z} \to \mathcal{Z}_n$ such that

$$\forall\ z\ \in \mathcal{Z},\ \lim_{n\to\infty} ||P_n z - z|| = 0. \tag{4}$$

The subspaces $\mathcal{Z}_n$ are equipped with the norm inherited from $\mathcal{Z}$. Considering this approximation scheme and defining the operator $A_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z})$ by some method method while $B_n = P_n\mathfrak{B}$, this leads to the following finite-dimensional approximation:

$$\begin{aligned}&\frac{d\tilde{z}}{dt} = A_n\tilde{z}(t), \quad \tilde{z}(0) = P_n z_0, \quad t \in [0, T].\\ &B_n z(t) = u(t)\end{aligned} \tag{5}$$

Denote $\mathcal{S}_n(t)$ the semigroups generated by A_n. We make the following classical assumption that ensure the uniform convergence on bounded intervals of the open-loop approximating state $\tilde{z}(t)$ to the exact state: for each $z \in \mathcal{Z}$, and all intervals of time $[t_1, t_2]$

$$\lim_{n\to\infty} \sup_{t\in[t_1,t_2]} ||\mathcal{S}_n(t)P_n z - \mathcal{S}(t)z|| = 0. \tag{6}$$

This assumption, which is often satisfied by ensuring that the conditions of the Trotter-Kato Theorem hold (see [25,43]), implies open loop convergence.

However (6) is not sufficient to guarantee that a control sequence u_n that stabilizes the approximations (5) wlll stabilize the original system and provide good performance (see [13,41,42]). For bounded control operators, a large number of tools and techniques are available for controller design using this approach; see for example[8,14,30,32,33,41,39] and the tutorial paper [42]). However, boundary control typically leads to an unbounded control operator when put in state space form and only a few results can be found in the literature[7,30]. We do not provide in this paper any general conditions guaranteeing the convergence of the early-lumping controller for unbounded control operator. However, to compare the results we obtain for late-lumping controller we derive for each example, without proving convergence or stabilization, two early-lumping controllers: a backstepping-like controller and a LQR controller.

Late-lumping control

For numerous systems, it is possible to directly derive from the PDE infinite-dimensional state feedback insuring stabilization, that is, to find an operator $K \in \mathcal{L}(\mathcal{Z},\mathcal{U})$ such that the semigroup associated to (1) along with the control law $u(t) = Kz(t)$ is stable. Examples include the backstepping controllers derived in [5,17,28,50], the flatness-based controllers derived in [38,46], the optimization controllers in [35], the controller in [36] based on a frequency-domain approach.

In design of a backstepping controller, an integral transformation is used to map the original system to a target system with desirable properties (in particular, this system is chosen stable). The control law ensuring the stabilization of the original system is then derived using this transformation. For application of these controllers to industrial problem for which sensors cannot be placed all along the system, it is necessary to derive an observer. In this paper, we only focus on the control aspects, neglecting the design of the observer. However, to reflect the fact that we do not have fully-distributed measurements, we assume that only an approximation of the state is available to synthesize the control law. More precisely, considering a stabilizing control law $u(t) = Kz$, the late-lumping assumption implies that the real control law that will be used is

$$u(t) = KP_n z = Kz^n, \tag{7}$$

denoting $z^n = P_n z$ where P_n is the orthogonal projection (4) onto some subspace. Our main contribution is to prove the uniform convergence of the late-lumping controller for different examples. Our proofs rely on the following assumption on the approximation sequence.

Assumption 2 *Let p be the integer in Assumption 1. There exists a sequence C_n such that $\forall z \in \mathcal{Z} \subset (\mathcal{H}^1([0,1]))^p$,*

(1) $\lim_{n\to\infty} C_n = 0$,

(2) $\forall n \in \mathbb{N},\ ||KP_n z - Kz|| \leq C_n ||z||_{(\mathcal{H}^1([0,1]))^p}$.

3 Unstable heat equation

We consider in this section the example of heat conduction in a rod of small cross-section. The rod is assumed thin enough so that the temperature can be assumed uniform across the section. We assume that the effects of heat loss and heat generation inside the rod are significant and have to be modeled (these terms can come from radiation, electrical resistivity). Moreover, we assume that the heat generation dominates the heat loss which makes the system unstable. The stabilization objective is achieved by applying a Neumann boundary control on one end and insulating the other. This yields (see [11,15,22]) the following parabolic PDE (unstable heat equation):

$$z_t(t,x) = z_{xx}(t,x) + \lambda z(t,x), \quad z(0,x) = z_0 \tag{8}$$

evolving in $\{(t,x)| \quad t > 0,\ x \in [0,1]\}$, with Neumann boundary conditions

$$z_x(t,0) = 0, \quad z_x(t,1) = u(t). \tag{9}$$

The parameter λ is assumed strictly positive so that the open-loop system (8)-(9) is unstable. The initial condition denoted z_0 is assumed to belong to $\mathcal{H}^1([0,1])$. For this system, various control laws ensuring exponential stabilization have already been designed (see [6,10,50]). In particular, in [50] a feedback control law is derived using the backstepping approach.

Late-lumping controller. We recall the main results of [50] in which is derived a control law that stabilizes the original infinite-dimensional system (8)-(9) using the backstepping method [29]. We assume then that only an approximation of the state is available for control design (late-lumping) and prove that the resulting control law stabilizes the original system. Let us consider the Volterra transformation

$$w(t,x) = z(t,x) - \int_0^x L(x,\xi)z(t,\xi)d\xi, \tag{10}$$

where the kernel $L(x,y)$ is defined on $\mathcal{T} = \{(x,y) \in [0,1]^2| \quad y \leq x\}$ by

$$L(x,y) = \begin{cases} -(\lambda+c)x\frac{I_1(\sqrt{(\lambda+c)(x^2-y^2)})}{\sqrt{(\lambda+c)(x^2-y^2)}}, & \text{if } x \neq y \\ -\frac{(\lambda+c)}{2}x \quad \text{if } x = y, \end{cases} \tag{11}$$

and where c is an arbitrary strictly positive constant. The function I_1 is the first modified Bessel function. The function L is two times differentiable on $\mathcal{T}$. In the following, we denote by R (bounded on $\mathcal{T}$) the derivative of L with respect to x, $R := L_x$.

Lemma 2 *[50, Theorems 5,8] There exist two constants C_1 and C_2 such that*

$$C_1||w||_{\mathcal{H}^1([0,1])} \leq ||z||_{\mathcal{H}^1([0,1])} \leq C_2||w||_{\mathcal{H}^1([0,1])} \quad (12)$$

Defining $K_{BS} \in \mathcal{L}(D(A), \mathbb{R})$ by

$$K_{BS}z = -\frac{(\lambda + c)}{2}z(1) + \int_0^1 R(1,\xi)z(\xi)d\xi, \quad (13)$$

we define the control law $u(t)$

$$u_{BS}(t) = K_{BS}z(t). \quad (14)$$

The transformation (10) *along with the control law* (14) *maps the original system* (8)-(9) *to the stable target system*

$$w_t(t,x) = w_{xx}(t,x) - cw(t,x), \quad (15)$$
$$w_x(t,0) = 0, \quad w_x(t,1) = 0. \quad (16)$$

Thus, for any initial condition $z_0 \in \mathcal{H}^1([0,1])$, the system (8)-(9) *with the control law* (14) *has a unique classical solution $z(t,x) \in C^{2,1}(([0,1]) \times (0,\infty))$ and is exponentially stable at the origin, $u(t,x) \equiv 0$ in the $\mathcal{L}^2([0,1])$ and $\mathcal{H}^1([0,1])$ norm. The control $u(t) = K_{BS}z(t)$ exponentially stabilizes the system* (8)-(9).

Let us now consider an approximation scheme satisfying Assumption 2 and assume that only the $n \in \mathbb{N}^*$ first modes of the state are available to design the control. We denote P_n the projection on the approximating space. This means we consider the system (8)-(9) along with the control law

$$u^n_{BS}(t) = K_{BS}P_n z. \quad (17)$$

Theorem 3 *There exists $N \in \mathbb{N}$ such that for any $n \geq N$, for any initial condition $z_0 \in \mathcal{H}^1([0,1])$, the system* (8)-(9) *along with the approximated control law* (17) *is exponentially stable at the origin, $z(t,x) \equiv 0$ in the sense of the $\mathcal{L}^2([0,1])$-norm.*

PROOF. This theorem can be proved using [27, Theorem IX.2.4] since the semigroup is analytic perturbed by a small perturbation. However, this method cannot be extended for the other examples considered in this paper, contrary to the Lyapunov-based proof used here.

The main idea of the proof consists in mapping (8)-(9) along with the control law (17) to a simpler target system with a similar structure to (15)-(16) using the transformation (14). This target system is then proved to be exponentially stable for an order of approximation n large enough. This is done by the way of a Lyapunov function. Finally, due to inequality (12), this implies the exponential stability of the original system.

Let us consider (8)-(9) along with the control law (17). Similarly to [50], differentiating (10) with respect to space, we obtain

$$w_x(t,x) = z_x(t,x) - L(x,x)z(t,x) - \int_0^x R(x,\xi)z(t,\xi)d\xi.$$

and

$$\begin{aligned} w_{xx} &= z_{xx}(t,x) - L(x,x)z_x(t,x) - R(x,x)z(t,x) \\ &\frac{d}{dx}(L(x,x))z(t,x) - \int_0^x R_x(x,\xi)z(t,\xi)d\xi. \end{aligned}$$

Similarly, differentiating (10) with respect to time and using (8)

$$\begin{aligned} &w_t(t,x) = z_t(t,x) - \int_0^x L(x,\xi)z_t(t,\xi)d\xi \\ &= z_{xx}(t,x) + \lambda z(t,x) - L(x,x)z_x(t,x) + L_\xi(x,x)z(t,x) \\ &- \int_0^x L_{\xi\xi}(x,\xi)z(t,\xi) + \lambda L(x,\xi)z(t,\xi)d\xi. \end{aligned}$$

Thus, combining the two previous equations and using (11), we obtain

$$w_t(t,x) = w_{xx}(t,x) - cw(t,x). \quad (18)$$

Using (17) and (14), we obtain the following Neumann boundary conditions

$$w_x(t,0) = 0, \quad w_x(t,1) = u^n_{BS}(t) - u_{BS}(t). \quad (19)$$

Using Assumption 2 and inequality (12), we obtain

$$|K_{BS}P_n z - K_{BS}z| \leq C_n C_2||w||_{\mathcal{H}^1([0,1])}. \quad (20)$$

We now prove the stability of the system (18)-(19) with a Lyapunov analysis. Inspired by [50], let us consider the Lyapunov function candidate

$$V(t) = \int_0^1 w^2(t,x)dx. \quad (21)$$

Differentiating V with respect to time and integrating by part yields

$$\begin{aligned}\dot V(t) &= 2\int_0^1 w(t,x)(w_{xx}(t,x)-cw(t,x))dx\\ &= -2\int_0^1 w_x^2(t,x)dx - \int_0^1 2cw^2(t,x)dx\\ &+ 2w(t,1)(u_{BS}^n(t)-u_{BS}(t))\\ &\le -2\int_0^1 w_x^2(t,x)dx - \int_0^1 2cw^2(t,x)dx\\ &+ 2C_nC_2\alpha||w||^2_{\mathcal{H}^1([0,1])}, \end{aligned} \quad (22)$$

where we have used (1) and (20) to obtain the last inequality. Since C_n converges to zero, there exists $N\in\mathbb{N}$ such that for all $n\ge N$, $C_n \le \frac{\min(c,1)}{2C_2\alpha}$ This yields the existence of a constant δ such that

$$\dot V(t) \le -\delta V(t) \quad (23)$$

This implies the exponential stability of the system (18)-(19) in the sense of the $\mathcal{L}^2$-norm. Due to (12), the original state z has the same properties. This concludes the proof.

Early Lumping. We now give the abstract formulation of (8) in terms of operators. This abstract formulation, although it was not required for the design of the backstepping controller is useful while designing an early-lumping controller. Define $\mathcal{Z}=\mathcal{L}^2([0,1])$. We can rewrite the system in the abstract form as

$$\begin{aligned}&\dot z(t) = A_{\text{heat}}z(t), \quad z(0)=z_0.\\ &B_{\text{heat}}z(t)=u(t)\end{aligned} \quad (24)$$

The operator A_{heat} is defined by

$$\begin{aligned}A_{\text{heat}} : D(A_{\text{heat}}) \subset (\mathcal{H}^1([0,1])) \subset \mathcal{L}^2([0,1]) &\to \mathcal{L}^2([0,1])\\ z &\longmapsto z_{xx}+\lambda z,\end{aligned} \quad (25)$$

with $D(A_{\text{heat}}) = \{z \in \mathcal{H}^2([0,1]) |\ z_x(0)=0\}$, where $\mathcal{H}^2([0,1])$ indicates the Sobolev space of functions with weak second derivatives (see e.g [48]). Its domain of definition satisfies Assumption 1. We equip $D(A_{\text{heat}})$ with the scalar product associated with the graph norm $||z||_{D(A_{\text{heat}})} = ||z||_{\mathcal{L}^2[0,1]} + ||A_{\text{heat}}z||_{\mathcal{L}^2[0,1]}$, which is equivalent to the $\mathcal{H}^1([0,1])$-norm. The control operator $B_{heat}:\mathbb{R}\to[D(A)]'$ is $\delta(1)$ where $\delta(1)$ indicates evaluation at $x=1$.

The eigenfunctions ϕ_i $(i=0,...)$ of the operator A_{heat} form a Riesz basis for $\mathcal{L}^2(0,1)$. These eigenfunctions are (see [20]) by

$$\phi_k(x) = \begin{cases} 1 & \text{if } k=0\\ \sqrt{2}\cos(k\pi x) & \text{if } k\neq 0.\end{cases} \quad (26)$$

They form an orthogonal basis of $\mathcal{H}^1([0,1])$. Define $\chi_n = \text{span}_{k=0,\cdots,n}\{\phi_k\}$ and let P_n indicate the projection onto χ_n. Then define $z^n(t,x) = P_nz(t,x) = \sum_{k=0}^n z_k(t)\phi_k(x)$.
Define A_n by the Galerkin approximation

$$\langle A_n\phi_j,\phi_k\rangle = \langle A_{\text{heat}}\phi_j,\phi_k\rangle, \quad (j,k)\in[0,n]^2 \quad (27)$$

and $B_n = P_nB_{\text{heat}}$.In the following we denote $\mathbf{z}^n = \Big(z_0,\cdots z_n\Big)^T$, the concatenation of different projections of z on the space χ_n. Similarly, we denote $\mathbf{z}_0^n = \Big((P_nz_0)_0,\cdots(P_nz_0)_n\Big)^T$.

The following open-loop convergence result is well-known.

Lemma 4 *[42, e.g.,Theorem 3.1] For each initial condition $z_0\in\mathcal{Z}$, the uncontrolled approximating state $z^n(t)$, converges uniformly on bounded intervals to the exact state $z(t)$.*

Using the Galerkin approximation (27) it becomes possible to derive early-lumping controllers that can be numerically compared with the late-lumping one. Inspired by the backstepping controller, a natural way to design an early-lumping controller is to approximate the (exponentially stable) target system (15)-(16), find the eigenvalues of the resulting ODE and place the eigenvalues of (27) on the same location. This sequence of control law will be denoted $u^n_{BS_{early}}$. Such a finite dimensional backstepping style method is proposed in [6] using a finite-difference discretization. The results obtained in [6] in simulation were quite unsatisfactory as the solver required a large number of modes to be efficient. The Galerkin approximation we propose leads to better results.

A second method to design an early lumping controller is linear quadratic control. Consider the quadratic functional

$$J(u^n,z_0) = \int_0^\infty \langle \mathbf{z}^n(t),\mathbf{z}^n(t)\rangle + \alpha((u^n)(t))^2dt, \quad (28)$$

where $\alpha>0$ is a tuning coefficient. Some convergence results can be found for parabolic equations with unbounded control operators [7,32]. The LQ controller associated with minimizing the cost (28) for the Galerkin approximation stabilizes the original PDE (8) if the number of modes n is large enough. Moreover it converges to the LQ-optimal controller for (8).

Simulation results. The following lemma is a direct consequence of Assumption 1 and of Jackson's inequality [26],[44, Exercise 1.5.14].

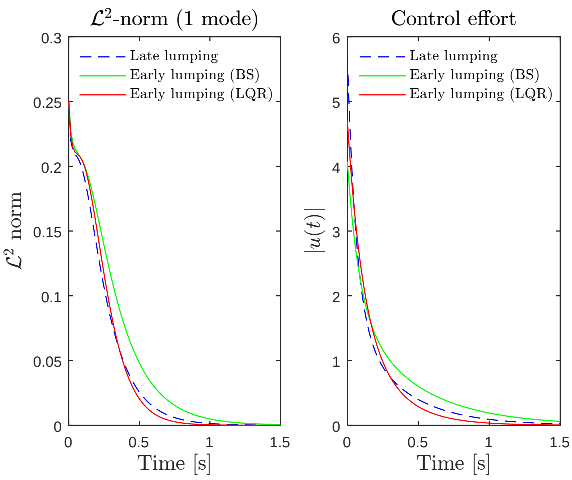

Fig. 1. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (heat equation, N=30, M=1)

Lemma 5 *The considered approximation scheme combined with the control law* (14) *satisfies Assumption 2.*

This implies (Theorem 3) the convergence of the late-lumping backstepping controller introduced in (14).

We now compare the controller given by (17) with the two early-lumping controller designed above. The real system is simulated using the same Galerkin approximation with the number of modes $N = 30$. The two control laws are designed using only $M < 30$ modes (different values of M will be used). We compare the time evolution of the $\mathcal{L}^2$ norm (performance) and the control effort for the three different controllers. The parameter λ is chosen to be equal to 3. The numerical parameters used for the design of the control laws are chosen as follow: $\alpha = 0.1$ and $c = 2$. The initial condition is defined by $z(0, x) = 0.25$. These simulations (see Figures 1-3) show better performance/control effort for the late-lumping backstepping controller compared to the early-lumping backstepping controller when only a few number of modes is used. For a large number of modes, the behaviors are similar. These simulation results also tend to show that the early-lumping LQR controller has a better performance/control effort trade-off compared to the two other controllers. Although this could be expected when using a large number of modes, this still holds even with a few number of modes.

4 Wave equation

A one-dimensional wave equation that is controlled from one end and contains instability at the other (free) end is considered in this section. This yields the following hyperbolic partial differential equation

$$z_{tt}(t, x) = z_{xx}(t, x), \tag{29}$$

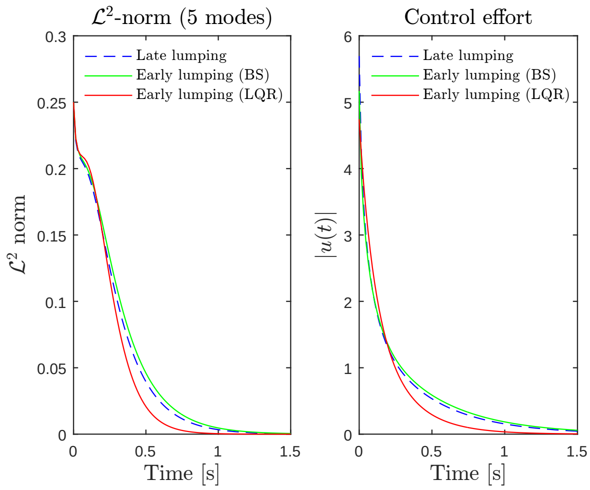

Fig. 2. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (heat equation, N=30, M=5)

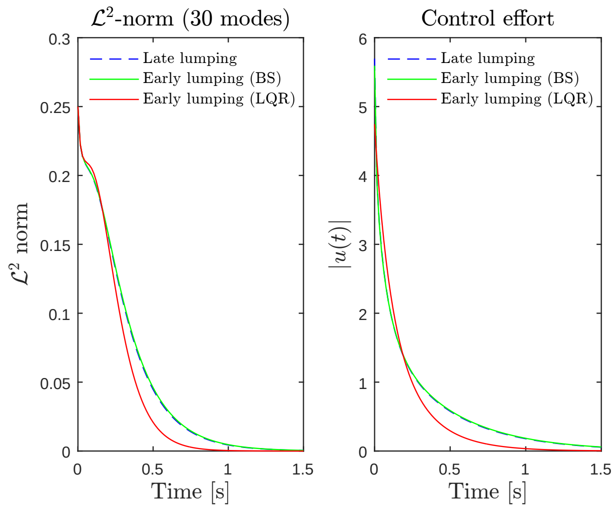

Fig. 3. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (heat equation, N=30, M=30)

evolving in $\{(t, x)| \quad t > 0, \ x \in [0, 1]\}$, with Neumann boundary conditions

$$z_x(t, 0) = -q z_t(t, 0), \quad z_x(t, 1) = u(t). \tag{30}$$

The parameter q is assumed different from -1 and strictly negative to avoid having an infinite number of eigenvalues in the right half plane (RHP). An infinite number of eigenvalues in the RHP would make impossible delay-robust stabilization (see [37]). The free end of the string is subject to a force proportional to the displacement, which physically may be the result of various phenomena. For instance, if the $x = 0$ end of the string is made of iron and is placed between two magnets of the same polarity, the string's end will be subject to a magnetic force which depends on its displacement. The initial condition denoted $(z^0, z_t^0) = (z(0,\cdot), z_t(0,\cdot))$ is assumed to belong to $\mathcal{H}^1([0,1]) \times \mathcal{H}^1([0,1])$. The system is stable but can still converge to a non-zero value $(z_1, 0)$. The objective of the control design is to ensure the stabilization to zero and also to increase the convergence rate.

Let us now give the state space formulation of (29).

$$\frac{d}{dt}\begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = A_{\text{wave}} \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix}, \quad B_{\text{wave}} z(t) = u(t), \quad (31)$$

$$\begin{pmatrix} z(0) & \dot{z}(0) \end{pmatrix} = \begin{pmatrix} z^0 & z_t^0 \end{pmatrix}. \quad (32)$$

The operator

$$A_{\text{wave}} : D(A_{\text{wave}}) \subset \mathcal{H}^1([0,1]) \times \mathcal{L}^2([0,1]) \to (\mathcal{L}^2([0,1]))^2$$
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & z_2 \\ \frac{d^2}{dx^2} z_1 & 0 \end{pmatrix}, \quad (33)$$

with

$$D(A_{\text{wave}}) = \{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathcal{H}^2(0,1) \times \mathcal{H}^1(0,1) | (z_1)_x(0) = - q z_2(0) \}$$

The operator A_{wave} is densely defined. We equip $D(A_{\text{wave}})$ with the scalar product associated with the norm $\mathcal{H}^1([0,1]) \times \mathcal{H}^1([0,1])$. The operator B_{wave} is defined on $[D(A_{wave})]\prime$ by $B_{wave} = [0,\ \delta(1)]^T$.

Late-lumping controller. A late-lumping backstepping controller will be used based on that described in [51]. Consider the Volterra transformation

$$w(t,x) = -\frac{1+qc}{q^2-1} z(t,x) + \frac{q(q+c)}{q^2-1} z(t,0) - \frac{q+c}{q^2-1} \int_0^x z_t(t,\xi) d\xi, \quad (34)$$

where the constant c is an arbitrary strictly positive constant such that $c \neq 1$ and $qc \neq -1$. We have the following lemma whose proof is straightforward.

Lemma 6 *There exist constants C_1 and C_2 such that*

$$C_1(||w||_{\mathcal{H}^1([0,1])} + ||w_t||_{\mathcal{H}^1([0,1])}) \leq (||z||_{\mathcal{H}^1([0,1])} + ||z_t||_{\mathcal{H}^1([0,1])}) \leq C_2(||w||_{\mathcal{H}^1} + ||w_t||_{\mathcal{H}^1([0,1])}). \quad (35)$$

Define $K_{BS} \in \mathcal{L}(D(A), \Re)$

$$K_{BS} z = \frac{c_0 q(q+c)}{1+qc} z(t,0) - c_0 z(t,1) - \frac{(q+c)}{1+qc} z_t(t,1) - \frac{c_0(q+c)}{1+qc} \int_0^1 z_t(t,\xi) d\xi, \quad (36)$$

$$u_{BS}(t) = K_{BS} z(t), \quad (37)$$

where c_0 is an arbitrary strictly positive coefficient (used to improve the convergence rate).

Lemma 7 *[51, Theorem 1] Transformation* (34) *along with the control law* (37) *maps the original system* (29)-(30) *to the following stable target system*

$$w_{tt}(t,x) = w_{xx}(t,x), \quad (38)$$

with Neumann boundary conditions

$$w_x(t,0) = c w_t(t,0), \quad w_x(t,1) = -c_0 w(t,1). \quad (39)$$

For any initial condition $(z(0,\cdot), z_t(0,\cdot)) \in \mathcal{H}^2(0,1) \times \mathcal{H}^1(0,1)$ compatible with the boundary conditions, the system (29)-(30) *along with the control law u_{BS} defined by* (37), *has a unique solution $(z, z_t) \in C([0,\infty), \mathcal{H}^1(0,1) \times \mathcal{L}^2(0,1))$ which is exponentially stable in the sense of the norm*

$$(\int_0^1 z_x(t,x)^2 dx + \int_0^1 z_t(t,x)^2 dx + z(t,1)^2)^2. \quad (40)$$

PROOF. System (38)-(39) can be obtained from (29)-(30) differentiating (34) with respect to space and time (see [51] for details). The rest of the proof is done through a Lyapunov analysis that can is detailed in [51].

Consider an approximation scheme satisfying Assumption 2 and assume that only the n first modes of the state are available to design the control (where $n \in \mathbb{N}$). We denote P_n the orthogonal projection on the approximating space. This means we consider the system (29)-(30) along with the following control law

$$u_{BS}^n(t) = K_{BS} P_n z. \quad (41)$$

We then have the following theorem.

Theorem 8 *There exists $N \in \mathbb{N}$ such that for any $n \geq N$, for any initial condition $(z(0,\cdot), z_t(0,\cdot)) \in \mathcal{H}^2(0,1) \times \mathcal{H}^1(0,1)$ compatible with the boundary conditions, the system* (29)-(30) *along with the approximated control law* (41) *is exponentially stable at the origin, $z(t,x) \equiv 0$ in the sense of the norm defined by* (40).

PROOF. This proof is similar to the one of Theorem 3. Let us consider (29)-(30) along with the control law (41). As in [51], differentiate twice (34) with respect to space to obtain

$$w_{xx}(t,x) = -\frac{1+qc}{q^2-1}z_{xx}(t,x) - \frac{q+c}{q^2-1}z_{tx}(t,x).$$

Similarly, differentiating twice (34) with respect to time and using (29), we obtain

$$w_t(t,x) = -\frac{1}{q^2-1}(-(1+qc)z_t(t,x) + q(q+c)z_t(t,0)$$
$$-(q+c)z_x(t,x) + (q+c)z_x(t,0))$$
$$w_{tt}(t,x) = -\frac{1}{q^2-1}(-(1+qc)z_{xx}(t,x) - (q+c)z_{tx}(t,0))$$

This yields the target system

$$w_{tt}(t,x) = w_{xx}(t,x), \tag{42}$$

with the following Neumann boundary conditions

$$w_x(t,0) = cw_t(t,0), \tag{43}$$
$$\begin{aligned} w_x(t,1) &= \frac{-1}{q^2-1}((1+qc)z_x(t,1) - (q+c)z_t(t,1)) \\ &= \frac{-1}{q^2-1}((1+qc)u(t) - (q+c)z_t(t,1)) \\ &= -c_0w(t,1) + \frac{1+qc}{q^2-1}(u_{BS}(t) - u^n_{BS}(t)). \end{aligned} \tag{44}$$

Using Assumption 2 and (35), we obtain

$$|K_{BS}P^nz - K_{BS}z| \leq C_nC_2(||(w,w_t)||_{\mathcal{H}^1([0,1])}). \tag{45}$$

We now prove the stability of the system (42)-(44) with a Lyapunov analysis. Inspired by [51], let us consider the Lyapunov function candidate

$$\begin{aligned} V(t) = \frac{1}{2}\int_0^1 w_x^2(t,x) + w_t^2(t,x)dx + \frac{c_0}{2}w(t,1)^2 \\ + \delta\int_0^1 (x-2)w_x(t,x)w_t(t,x)dx \end{aligned} \tag{46}$$

Using the Cauchy Schwartz and Young's inequalities, one can show that for sufficiently small δ, there exist $m_1 > 0$ and $m_2 > 0$ such that

$$m_1U \leq V \leq m_2U, \tag{47}$$

where $U = ||w_x||^2 + ||w_t||^2 + w^2(1)$. In the following, we will assume that δ is small enough so that (47) is satisfied. In particular, we assume that $\delta \leq \frac{c}{1+c^2}$. For such a δ, V is positive definite. Differentiating V with respect to time and integrating by part yields

$$\begin{aligned} \dot{V}(t) &= \int_0^1 w_x(t,x)w_{tx}(t,x) + w_t(t,x)w_{xx}(t,x)dx \\ &+ \delta\int_0^1 (x-2)w_{xt}w_t + (x-2)w_xw_{xx}dx + c_0w_t(t,1)w(t,1) \\ &= -w_t(t,1)w_x(t,1) + w_t(t,1)w_x(t,1) - w_t(t,0)w_x(t,0) \\ &+ \frac{1+qc}{q^2-1}(K_{BS}z - K_{BS}P^nz)w_t(t,1) \\ &+ \frac{\delta}{2}(-w_x^2(t,1) + 2w_x^2(t,0) - w_t^2(t,1) + 2w_t^2(t,0)) \\ &- \frac{\delta}{2}(\int_0^1 w_x^2(t,x) + w_t^2(t,x)dx) \end{aligned}$$

Thus,

$$\begin{aligned} \dot{V} &\leq -\frac{\delta}{2}(\int_0^1 w_x^2(t,x) + w_t^2(t,x)dx) - (c - \delta(1+c^2))w_t^2(t,0) \\ &- c_0^2\frac{\delta}{2}w^2(t,1) - \frac{\delta}{2}(\frac{1+qc}{q^2-1})^2(K_{BS}z - K_{BS}P^nz)^2 + (K_{BS}z- \\ &K_{BS}P^nz)\frac{1+qc}{q^2-1}w_t(t,1) + c_0\delta\frac{1+qc}{q^2-1}(K_{BS}z - K_{BS}P^nz)w(t,1) \\ &\leq -\frac{\delta}{2}(\int_0^1 w_x^2(t,x) + w_t^2(t,x)dx) + c_0^2\frac{\delta}{4}w^2(t,1) \\ &+ \frac{\delta}{4}||w_t||^2_{\mathcal{H}^1([0,1])} + (\frac{1+qc}{q^2-1})^2(\frac{\alpha^2}{\delta} + \frac{\delta}{2})(K_{BS}z - K_{BS}P^nz)^2, \end{aligned}$$

where we have used (2) and Young's inequality in the last line. Using (45) leads to

$$\begin{aligned} \dot{V} &\leq -\frac{\delta}{4}(\int_0^1 w_x^2(t,x) + w_t^2(t,x)dx) - c_0^2\frac{\delta}{4}w^2(t,1) \\ &C_n^2C_2^2(\frac{1+qc}{q^2-1})^2(\frac{\alpha^2}{\delta} + \frac{\delta}{2})(||w||_{\mathcal{H}^1([0,1])} + ||w_t||_{\mathcal{H}^1([0,1])})^2. \end{aligned}$$

Since C_n converges to zero, using Young's and Poincarre's inequality, there exists $M > 0$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\dot{V}(t) \leq -MV(t) \tag{48}$$

This implies the exponential stability of the system (18)-(19) in the sense of the norm defined in (40). Due to (35), the original state z has the same properties. This concludes the proof.

Simulations To implement the system in simulation, and to design early lumping controllers, a Galerkin approximation based on eigenfunctions is again used. The approximation scheme is based on a Riesz basis. Consider the family ϕ_k defined for all $k \in \mathbb{N}^*$ by

$$\phi_k(x) = \begin{pmatrix} \phi_k^1(x) \\ \phi_k^2(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{k\pi}\cos(k\pi x) \\ \cos(k\pi x) \end{pmatrix} \quad (49)$$

Define ϕ_0 and $\phi_{0,1}$ as

$$\phi_0(x) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T, \quad \phi_{0,1}(x) = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \quad (50)$$

The family $\{\phi_{0,1}, \phi_k,\ k \in \mathbb{N}\}$ forms a Riesz basis on $D(A_{wave})$ [20]. Let us consider $n \in \mathbb{N}$, we define $\chi_n = \text{span}\{\text{span}_{i=-n,\cdots,n}\{\phi_i\}, \phi_{0,1}\}$ and denote P_n, the orthogonal projection onto χ_n. The space χ_n is equipped with the $\mathcal{H}^1$-norm.

Due to Jackson's theorem, this approximation scheme satisfies Assumption 2. This implies convergence of the late-lumping backstepping controller.

Using this approximation scheme, it is straightforward to design early-lumping backstepping controller and early-lumping LQR controllers, following a procedure identical to that described for the heat equation. However, since the underlying semigroup is not analytic, the convergence or performance of the controllers on the PDE is not guaranteed.

The late lumping backstepping controller (41) was compared with the two early-lumping controllers. The real system is simulated using approximation with a number of modes $N = 40$. The control laws are designed using $M < 40$ modes. We compare the time evolution of the $\mathcal{L}^2$ norm (performance) and the control effort for the three controllers. The parameters are chosen as follow: $q = -\frac{1}{2}, \alpha = 0.5$, $c = 0.8$ and $c_0 = 1.05$. The choice of these parameters is motivated by an effort to have similar performance in terms of the $\mathcal{L}^2$ norm of the late-lumping backstepping controller and the early-lumping LQR controller when only one mode is used. The initial conditions are defined by $z^0(x) = 1 + \frac{1}{N\pi}\cos(N\pi x)$ and $z_t^0(x) = 1 + \cos(N\pi x)$. Comparing Figures 4-6, it is apparent that the early-lumping backstepping controller and the early-lumping LQR controller have similar behavior in this respect. However, when few modes are used, the late-lumping backstepping controller achieves similar performance with less control effort in later times.

5 Two linear coupled hyperbolic PDEs

We consider in this section two linear first-order hyperbolic PDEs which appear for instance in Saint-Venant equations, heat exchangers equations and other linear

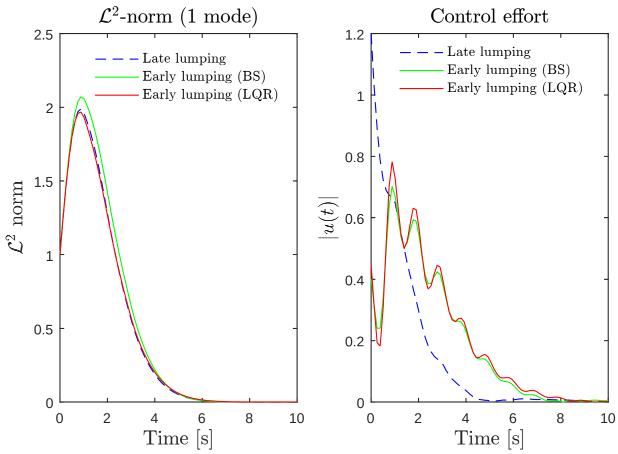

Fig. 4. Time evolution of the $\mathcal{L}^2$-norm of the state w and of the control efforts for different controllers (wave equation, N=40, M=1)

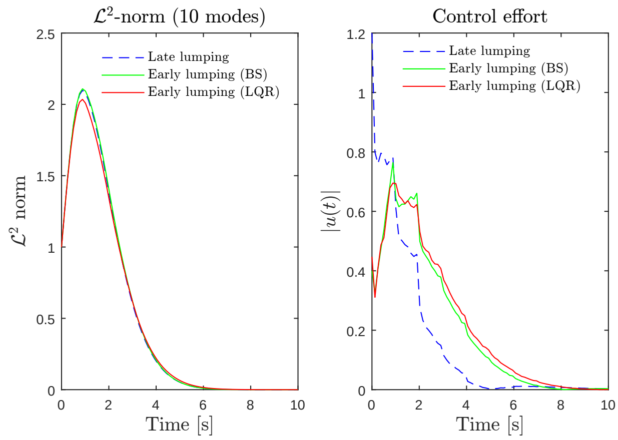

Fig. 5. Time evolution of the $\mathcal{L}^2$-norm of the state w and of the control efforts for different controllers (wave equation, N=40, M=10)

hyperbolic balance laws (see [9]):

$$w_t(t,x) + \lambda w_x(t,x) = \sigma^{+-} z(t,x) \quad (51)$$
$$z_t(t,x) - \mu z_x(t,x) = \sigma^{-+} w(t,x), \quad (52)$$

evolving in $\{(t,x)| \quad t > 0,\ x \in [0,1]\}$, with the following linear boundary conditions

$$w(t,0) = qz(t,0), \quad z(t,1) = u(t), \quad (53)$$

with constant coupling terms σ^{-+} and σ^{+-} and constant velocities λ and μ. The boundary coupling term q is assumed non null. Depending on the value of σ^{+-}, σ^{-+} and q, the system may be unstable [9] (the eigenvalues can curve over). The initial conditions denoted w_0 and z_0 are assumed to belong to $\mathcal{H}^1([0,1])$ and satisfy

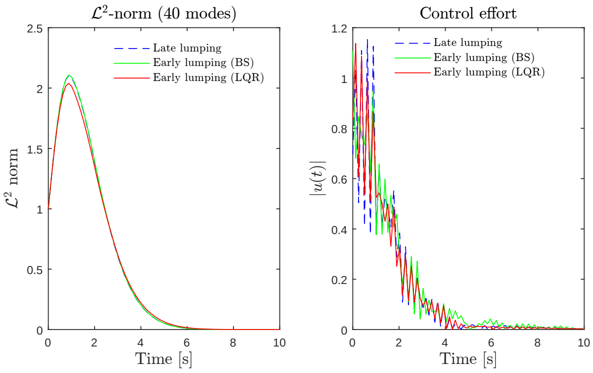

Fig. 6. Time evolution of the $\mathcal{L}^2$-norm of the state w and of the control efforts for different controllers (wave equation, N=40, M=40)

the compatibility conditions. As proved in [4], the system (51)-(53) is delay-robustly stabilizable and has a finite number of poles in the right half-plane.

Late-lumping controller. In [17] a control law that stabilizes the original infinite-dimensional system (51)-(53) using the backstepping method [29] is derived. Consider the Volterra transformation

$$\gamma(t,x) = w(t,x) - \int_0^x (K^{uu}(x,\xi)w(\xi) + K^{uv}(x,\xi)z(\xi))d\xi, \quad (54)$$
$$\beta(t,x) = z(t,x) - \int_0^x (K^{vu}(x,\xi)w(\xi) + K^{vv}(x,\xi)z(\xi))d\xi, \quad (55)$$

where the kernels $K^{uu}, K^{uv}, K^{vu}, K^{vv}$ are defined on $\mathcal{T} = \{(x,\xi) \in [0,1]^2 | \quad \xi \leq x\}$ by a set of hyperbolic PDEs (see [17]). We have the following lemma, whose proof is straightforward.

Lemma 9 *There exist constants C_1 and C_2 such that*

$$C_1(||\gamma||_{\mathcal{H}^1([0,1])} + ||\beta||_{\mathcal{H}^1([0,1])}) \leq ||z||_{\mathcal{H}^1([0,1])} + ||w||_{\mathcal{H}^1([0,1])}, \quad (56)$$

$$(||z||_{\mathcal{H}^1([0,1])} + ||w||_{\mathcal{H}^1([0,1])}) \leq C_2(||\gamma||_{\mathcal{H}^1([0,1])} + ||\beta||_{\mathcal{H}^1([0,1])}). \quad (57)$$

Define the control law

$$u_{BS}(t) = K_{BS}\begin{pmatrix} w & z \end{pmatrix}^T, \quad (58)$$

$$K_{BS}\begin{pmatrix} w \\ z \end{pmatrix} = \int_0^1 K^{vu}(1,\xi)w(\xi) + K^{vv}(1,\xi)z(\xi)d\xi. \quad (59)$$

Lemma 10 *[17, Theorems 3.2] Transformation (54)-(55) along with the control law (58) maps the original system (51)-(53) to the following stable target system*

$$\gamma_t(t,x) = -\lambda\gamma_x(t,x) \quad (60)$$
$$\beta_t(t,x) = \mu\beta_x(t,x) \quad (61)$$

with the following boundary conditions

$$\gamma(t,0) = q\beta(t,0), \quad \beta(t,1) = 0. \quad (62)$$

For any initial condition $(w(0,\cdot), z(0,\cdot)) \in \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1)$ that satisfies the compatibility conditions, the system (51)-(53) along with the control law u_{BS} defined by (58), has a unique solution $(w,z) \in C([0,\infty), \mathcal{H}^1(0,1) \times \mathcal{H}^1(0,1))$ which is exponentially stable in the sense of the $\mathcal{L}^2$-norm. As proved in [17], using the control law (58), the system actually reaches its zero equilibrium in finite time $t_f = \frac{1}{\lambda} + \frac{1}{\mu}$.

PROOF. System (60)-(62) can be obtained from (51)-(53), differentiating the invertible Volterra transformation (54)-(55) with respect to space and time and using integration by parts (see [17] for details). Since the origin of the (γ,β) system is L^2-exponentially stable with an arbitrary large exponential decay rate (see [17]), we conclude, using the fact that the Volterra transformation is invertible can the origin of the (w,z)-system is also L^2 exponentially stable with an arbitrary large exponential decay rate.

Let us consider an approximation scheme satisfying Assumption 2. Denoting by P_n the projection on the approximating space, consider the system (51)-(53) along with the control law

$$u^n_{BS}(t) = K_{BS}P_n\begin{pmatrix} w & z \end{pmatrix}^T = K_{BS}P^n\begin{pmatrix} w & z \end{pmatrix}^T. \quad (63)$$

We then have the following theorem.

Theorem 11 *There exists $N \in \mathbb{N}$ such that for any $n \geq N$, for any initial condition $z_0 \in \mathcal{H}^1([0,1])$, the system (51)-(53) along with the approximated control law (63) is exponentially stable at the origin.*

PROOF. This proof is similar to that of Theorem 8. Let us consider (51)-(53) along with the control law (63).

Using the results from [4], this system can be mapped to

$$\gamma_t(t,x) = -\lambda\gamma_x(t,x) \quad \beta_t(t,x) = \mu\beta_x(t,x) \tag{64}$$

$$\gamma(t,0) = q\beta(t,0), \quad \beta(t,1) = (K_{BS}P^n - K_{BS})\begin{pmatrix} w & z \end{pmatrix}^T. \tag{65}$$

Since the approximation scheme satisfies Assumption 2, we obtain

$$|(K_{BS}P^n - K_{BS})\begin{pmatrix} w \\ z \end{pmatrix}| \leq C_n C_2(||(\gamma,\beta)||_{\mathcal{H}^1}). \tag{66}$$

We now prove the stability of the system (64)-(65) with a Lyapunov analysis. Let us consider the Lyapunov function candidate

$$V(t) = \int_0^1 \frac{1}{\lambda}e^{-\nu x}\gamma^2(t,x) + \frac{q^2}{\mu}e^{\nu x}\beta(t,x)dx \tag{67}$$

where ν is a strictly positive parameters. Using the Cauchy Schwartz and Young's inequalities, one can show that there exist $m_1 > 0$ and $m_2 > 0$ such that

$$m_1(||\gamma||^2 + ||\beta||^2) \leq V \leq m_2(||\gamma||^2 + ||\beta||^2). \tag{68}$$

Differentiating V with respect to time and integrating by part yields

$$\begin{aligned}\dot{V}(t) &= -\int_0^1 \nu e^{-\nu x}\gamma^2(t,x) + \nu q^2 e^{\nu x}\beta^2(t,x)dx \\ &+ [-e^{-\nu x}\gamma^2(t,x) + q^2e^{\nu x}\beta^2(t,x)]_0^1 \\ &\leq -\int_0^1 \nu e^{-\nu x}\gamma^2(t,x) + \nu q^2 e^{\nu x}\beta^2(t,x)dx \\ &+ q^2e^{\nu}((K_{BS}P^n - K_{BS})\begin{pmatrix} w \\ z \end{pmatrix})^2 \\ &\leq -\int_0^1 \nu e^{-\nu x}\gamma^2(t,x) + \nu q^2 e^{\nu x}\beta^2(t,x)dx \\ &+ C_n^2 C_2^2 q^2 e^{\nu}(||\gamma||^2 + ||\beta||^2).\end{aligned} \tag{69}$$

Since C_n converges to zero, we easily obtain using (66) that there exists $M > 0$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\dot{V}(t) \leq -MV(t) \tag{70}$$

This implies the exponential stability of the system (64)-(65). Due to (57), the original state (z, w) has the same properties. This concludes the proof.

Simulations. The real system is simulated using the Galerkin's approximation with a number of modes $N =$ 40. The basis we use for the approximating spaces is the same as the one introduced in the previous section (i.e the family ϕ_k defined in equation (49)-(50)).

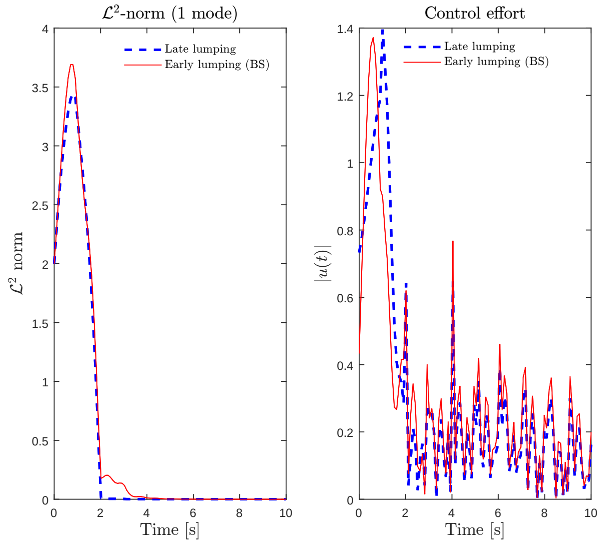

Fig. 7. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (system (51)-(53), N=40, M=1)

We finally compare the controller (63) with two early-lumping controllers, designed similarly to those in the previous sections. The control laws are designed using only $M < 40$ modes. The system parameters are chosen as follow: $\sigma^{+-} = 0$, $\sigma^{-+} = 1$, $q = 1$. The initial conditions are defined by $w_0(x) = z_0(x) = 1$. The LQR early-lumping controller did not stabilize the system when using more than 10 modes. Therefore, in Figure 7-8, we compare the time evolution of the $\mathcal{L}^2$ norm (performance) and the control effort for only the early-lumping backstepping controller and the late-lumping backstepping controller. The late-lumping backstepping controller still stabilizes the system in **finite-time** even with a few number of modes. The early-lumping backstepping controller also stabilizes the system (even with one mode) but the performance are not as good. However, when the number of modes increases, we obtain similar results in term of performance and control efforts for the two controllers.

6 Concluding remarks

In this paper we have considered different systems that can be stabilized by a backstepping control law. We have proved that under some assumptions, these controllers still ensure exponential stabilization when an approximation of the state is used. This has been done through a Lyapunov analysis, using the backstepping method as an analysis tool.

The late lumping backstepping controllers were com-

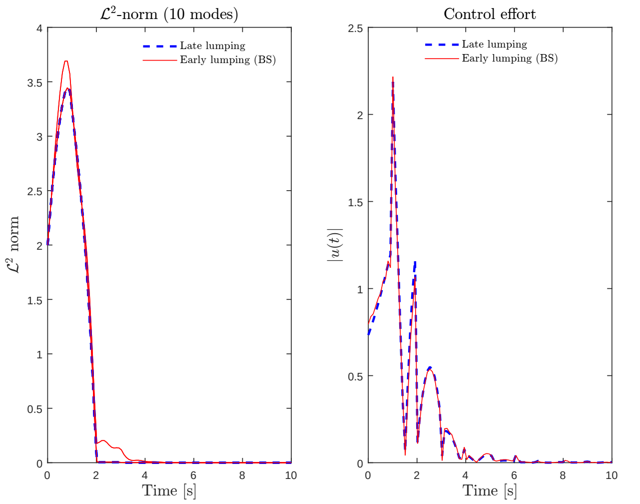

Fig. 8. Time evolution of the $\mathcal{L}^2$-norm and of the control efforts for different controllers (system (51)-(53), N=40, M=10)

pared in simulations with early-lumping controllers. Note that stability of the closed loop systems with early lumping controllers has not been established for the two wave equation examples. All controllers performed well for the heat equation. But for the wave equation (section 4) the late lumping controller was able to stabilize the system with a smaller number of modes than the early lumping controllers. For the hyperbolic system considered in section 5, performance was tuned to be similar but the late lumping controller required less control effort when a small number of modes was used.

The presented results raise important questions about the comparison between late-lumping and early-lumping controllers. In particular, robustness properties or computational efforts are not considered here. A current limitation for a deeper analysis is the lack of results to analyze the stability properties of early-lumping controller for unbounded control operators. This work is a first step towards practical applications of backstepping controllers. The question of the late-lumping backstepping controller-observer or the extension to systems of larger dimensions (using the results of [1,52,53]) has not been considered in this paper and will be the focus of future work.

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