Averaging on simple windows in deterministic optimal control

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Abstract

A windowed averaged scheme is defined for general control systems. The same method is used to average costs in optimal control problems (OCPs). A numerical parameter α can be computed, which expresses the distance between the original system and the averaged system in a weak sense.

Then, if we use the optimal control of the averaged OCP in the original OCP, the suboptimality of the control is bounded by an expression of the form $C\alpha^2$.

1 Introduction

Historically, the method of averaging was introduced to study the motion of celestial bodies by solving a simple two body equation which is perturbed by the influence of other bodies (see the section in [18] about the history of averaging). As developed in [17], the framework was that of the perturbation of an orbit by the small influence of a periodic input. Averaging was then generalized in a geometric framework, notably in [1, 2]. A comprehensive book on the subject is [18].

All of the previous references deal with systems of ordinary differential equations, without any notion of control. In this article we apply averaging on control systems, specifically in optimal control problems (OCPs). In this context, the question is: given a control that is optimal for an averaged OCP, will it be almost optimal for the non averaged OCP? In this case, it is not efficient to average a dynamical system and then add a control. Indeed, the "optimal" control obtained this way would vary slowly. A very simple counter example on a LQ problems shows in [8] that, in order to obtain a sub optimal control, the optimal control computed in the averaged problem must include fast varying components.

An early work [3] applies averaging to two point boundary value problems, but its application to optimal control is limited to what is essentially the LQ case. In [8], the method of averaging is applied to optimal control, both in open loop (in the periodic case) and in closed loop (study of the Hamilton-Jacobi equation under an ergodicity assumption). The study of the HJB equation is improved in [4]. Observe that, in these two references, the horizon is finite and the oscillatory input is fast. By contrast, the references [14, 13, 11, 12] consider an optimal control problem "in the long run" with averaging techniques. The convergence of the optimal cost is proved, but there is no study of the suboptimality of the optimal control of the averaged OCP when used in the original OCP. Optimal control of celestial objects (namely the optimal control of low thrust engines in space) have been studied from a geometric view point and in the periodic case in [6, 5, 7]. Averaging has also been used for similar problems [15, 10] in a spirit that is close to [8].

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To be complete, averaging has been used is stochastic optimal control, notably of Markov chains (see for instance [16, 22, 20]). Indeed, when there exists a cycle in a discrete state Markov chain that has high transition probabilities, then this cycle is gone through very fast and averaging can be applied.

A common feature of the previous literature is that

- it relies on a periodicity or ergodicity assumption
- it only provides *asymptotic* results.

By contrast, here

- the averaged cost and dynamics are obtained by numerical averaging on contiguous windows over the horizon $[0, T]$. These functions do not need to be periodic.
- we define a number α, which represents how close the original and averaged cost and dynamics are in a numerical "weak" sense, provided the functions are regular enough.The number α can by made small by using small windows. The number α only depends on the averaged problem solution and it can be defined for *any* smooth OCP.
- the number α^2, multipied by a number that depends essentially on the regularity and the convexity of the original functions, provides an error estimate between the optimal cost of the original problem, and the original cost obtained by using the optimal control of the averaged problem. This error estimate holds for *any* OCP provided that $\alpha \leq \frac{\beta}{2k_{J1}}$. The number β is part of the convexity assumption and k_{J1} depends on the regularity of the original functions.
- for controls that are better than the optimal control of the averaged problem, error estimates on the trajectories and controls are exhibited and are proportionnal to α.

The paper is organized as follows. Section 2 presents the original OCP. It then presents windowed averaging for functions, differential equations and control systems. It then presents the averaged OCP that is studied in this paper. This is where α is defined. Section 3 makes formal expansions in α of the state and of the costate of the nominal problem around the state and costate of the averaged problem. Auxiliary variables are introduced there. Section 4 introduces an auxiliary problem of optimization as well as new auxiliary variables. The main assumptions (bounded derivatives and convexity) are given before we state the auxiliary problem. The main result is exposed in section 5. It is a result on the control cost, trajectories and the optimal control. Section 6 is devoted to the proof of the main theorem. A conclusion is presented is section 7.

Note that, for a fluent reading of the paper, the detailed computations are in the appendices.

2 Problem statement

2.1 Nominal Problem

We wish to minimize the following Optimal Control Problem (OCP) :

$$\min_u \int_0^T L(x, u, t) dt \tag{1}$$

where x is is a finite dimensional state which satisfies the dynamics :

$$\frac{dx}{dt} = f(x, u, t) \ , \ x(0) = \chi_0 \tag{2}$$

where u is an integrable finite dimensional, unconstrained, control. The assumptions on f and L are given in section 4.1.

2.2 Averaged Problem

It is well known that, if the dependency of f or L with respect to time contains fast oscillations, the nominal problem may be difficult to solve. To avoid this, we define an averaged problem with the help of a very simple low pass filter. It is an averaging method for general functions, i.e. not necessarily periodic.

2.2.1 Filtering by windowed averaging

Definition 1. *Let N an integer, and g an integrable function on $[0, T]$. Let us subdivide $[0, T]$ into N intervals $[t_k, t_{k+1}]$, with:*

$$t_k = k\frac{T}{N} \ , \ k = 0..N \tag{3}$$

The averages of g on these intervals define the low pass filter LP on g:

$$LP[g](t) = \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} g(s)ds \ , \ t \in [t_k, t_{k+1}) \ , \ k = 0..N-1 \tag{4}$$

The difference $g - LP[g]$ defines the high pass filter HP on g:

$$HP = Id - LP \tag{5}$$

We then denote I[g] the function:

$$I[g](t) = \int_0^t HP[g](s)ds \ , \ t \in [0, T] \tag{6}$$

The upper bound of the function $I[g]$ is small when N is big and g is bounded.

Proposition 1. *The following bound holds for any bounded function g and any $t \in [0, T]$:*

$$|I[g](t)| \leq 2\|g\|_\infty \frac{T}{N} \tag{7}$$

Proof. Let us first prove that for any $k = 0..N$

$$\int_{t_k}^{t_{k+1}} HP[g](t)dt = 0$$

Indeed:

$$\int_{t_k}^{t_{k+1}} HP[g](t)dt = \int_{t_k}^{t_{k+1}} g(t)dt - \int_{t_k}^{t_{k+1}} \left[\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} g(s)ds\right] dt = 0$$

Hence, for $t \in [t_k t_{k+1}[$:

$$I[g](t) = \int_{t_k}^{t} HP[g](s)ds$$

But $|LP[g](s)| \leq \|g\|_\infty$, so that $|HP[g](s)| \leq 2\|g\|_\infty$. This gives finally:

$$|I[g](t)| \leq 2\|g\|_\infty (t - t_{n+1}) \leq 2\|g\|_\infty \frac{T}{N}$$

□

Example 1 Let us suppose that g is periodical with period $\frac{T}{N}$. Then $LP[g]$ is constant equal to the mean of g over a period. The function $HP[g]$ is a periodic signal with 0 average. The function $I[g]$ is the periodic antiderivative of $HP(g)$ with value 0 at 0. Its upper bound is of order $\frac{T}{N}$.

Example 2 Let us suppose that g is periodical with period ϵ a small divisor of T, and $N = 1$. Then $LP[g]$ is constant equal to the mean of g over the many periods in $[0, T]$. The function $HP[g]$ is a periodic signal of small period ϵ with 0 average. The function $I[g]$ is the periodic antiderivative of $HP(g)$ with value 0 at 0. Its upper bound is of order ϵ. This example shows that the upper bound of $I[g]$ can be small even with a small N.

2.2.2 Averaging errors for ordinary differential equation

Let us consider the ordinary differential equation:

$$\frac{dx}{dt} = f(x,t)\ ,\ x(0) = \chi_0 \tag{8}$$

with f Lipschitz with respect to x and integrable with respect to t.

The averaged ODE is defined as:

$$\frac{dx_0}{dt} = LP[f](x_0,t)\ ,\ x_0(0) = \chi_0 \tag{9}$$

with the low-pass filter on a function $g(x,t)$ defined as:

$$LP[g](\xi,t) = \frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}} g(\xi,s)ds\ ,\ t \in [t_k, t_{k+1})\ ,\ k = 0..N-1 \tag{10}$$

As a summary, $LP[g]$ averages g with respect to time on rectangular adjacent windows, leaving the state variable unchanged.

The state x_0 is well defined and bounded because f, and thus $LP[f]$, is Lipschitz.

Proposition 2. *Let $g(x,t)$ be bounded Lipschitz in x with a Lipschitz constant λ_g, and let's suppose that $f(x,t)$ is bounded. Let $I[g,x_0]$ the function of t defined by:*

$$I[g,x_0](t) = \int_0^t \left[g(x_0(s),s) - LP[g](x_0(s),s)\right] ds \tag{11}$$

where x_0 is the solution of the averaged ODE (9) Then the following bound holds:

$$|I[g](t)| \leq 2(\|g\|_\infty + \lambda_g T\|f\|_\infty)\frac{T}{N} \tag{12}$$

Proof. see appendix A. □

Proposition 3. *Let x defined by the ODE (8) and let x_0 defined by the averaged ODE (9). Let λ the Lipschitz constant of f in x and $\alpha = \sup_{t\in[0,T]}(|I[f](t)|)$. Then the following bound holds:*

$$\|x - x_0\|_\infty \leq \alpha\frac{e^{\lambda T}-1}{\lambda} \tag{13}$$

Proof. We have

$$\begin{aligned} |x(t) - x_0(t)| &= \left|\int_0^t f(x(s),s) - f(x_0(s),s) + f(x_0(s),s) - LP[f](x_0(s)\right| ds \\ &\leq \lambda\int_0^t |x(s)-x_0(s)|ds + \alpha \end{aligned}$$

which leads to the result thanks to the Gronwall lemma.

□

2.2.3 Low pass filtering of a controlled system

In the scope of this article, we define, for the integer $N > 0$, and any functions $u(t)$ and $g(x, u(t), t)$, an averaged function $LP[g, u]$

$$LP[g,u](x,t) = \frac{N}{T} \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} g(x,u(s),s)ds \; for \; t \in \left[\frac{kT}{N}, \frac{(k+1)T}{N}\right) \text{ and } k \in [0, N-1] \tag{14}$$

As a summary, $LP[g]$ averages g with respect to time (this includes the open loop control) on rectangular adjacent windows, leaving the state (or costate) unchanged.

2.2.4 Statement of the averaged problem

We define the averaged OCP which minimizes the cost

$$J_0(v) = \int_0^T LP[L,v](y(s),s)ds \tag{15}$$

where y is the state defined by

$$\frac{dy}{dt} = LP[f,v](y,t) \;,\; y(0) = \xi_0 \tag{16}$$

which is a well defined differential equation.

Assumption 1. *The averaged OCP admits an optimal control u_0 with a corresponding trajectory x_0.*

The trajectory x_0 is defined by the ODE:

$$\frac{dx_0}{dt} = LP[f,u_0](x_0,t) \;,\; x_0(0) = \xi_0 \tag{17}$$

2.2.5 Stationnarity condition for the averaged problem

Theorem 1. *Let H be the Hamiltonian of the nominal problem:*

$$H(x,u,p,t) = L(x,u,t) + p\, f(x,u,t) \tag{18}$$

Let p_0 be the costate of the averaged problem, defined along the optimal trajectory x_0 by the ODE with final condition:

$$\frac{dp_0}{dt} = -LP\left[\frac{\partial H}{\partial x}, u_0\right](x_0,p_0,t) \;,\; p_0(T) = 0 \tag{19}$$

Then the following stationarity condition holds:

$$\frac{\partial H}{\partial u}(x_0(t),u_0(t),p_0(t),t) = 0 \; a.e. \tag{20}$$

where a.e. stands for almost everywhere in $t \in [0,T]$.

Proof. see appendix B □

2.2.6 Introduction of a small α

For a function $g(x,u,t)$, the difference $g(x_0(t),u_0(t),t) - LP]g,u_0](x_0(t),t)$ is the result of high pass filtering $HP]g,u](x_0(t),t)$. Let's define the antiderivative:

$$I[g,x_0,u_0](t) = \int_0^t HP[g,u_0](x_0(s),s)ds \tag{21}$$

Then, as a consequence of Proposition 2, the following bound holds:

$$|I[g,x_0,u_0](t)| \leq (2\|g\|_\infty + \lambda_g T\|f\|_\infty)\frac{T}{N} \tag{22}$$

In other words, $\|I[g,x_0,u_0]\|_\infty$ can be made small if N is large, f is bounded and g is bounded and Lipschitz.

Let's define similarly the backwards antiderivative

$$I^T[g,x_0,u_0](t) = \int_t^T HP[g,u_0](x_0(s),s)ds \tag{23}$$

Then, with a similar proof, the same inequality holds:

$$|I^T[g,x_0,u_0](t)| \leq (2\|g\|_\infty + \lambda_g T\|f\|_\infty)\frac{T}{N} \tag{24}$$

Consequently, $\|I^T[g,x_0,u_0]\|_\infty$ can also be made small if N is large, f is bounded and g is bounded and Lipschitz.

Definition 2. *For the rest of that document, we consider the small number α defined as:*

$$\alpha = \sup\left(\|I[f,x_0,u_0]\|_\infty\ ,\ \left\|I^T\left[\frac{\partial H}{\partial x},(x_0,p_0),u_0\right]\right\|_\infty\right) \tag{25}$$

This number is small because N is big and assumption 2 below holds (bounded functions and their derivatives).

3 A priori expansions and definitions of auxiliary variables

3.1 Notations

Definition 3. *We denote the following variables from the state x (x_0), the control u (u_0) and the costate p (p_0):*

$$\sigma = (x,u)\ ,\ \sigma_0 = (x_0,u_0)$$
$$w = (x,u,p)\ ,\ w_0 = (x_0,u_0,p_0)$$

Note: p is the costate of the nominal problem defined by the ODE with final condition:

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}(x,u,p,t)\ ,\ p(T) = 0 \tag{26}$$

Definition 4. *We denote the derivatives up to second order of functions with respect to the variables x, u or σ with indexes, on the model:*

$$f_x = \frac{\partial f}{\partial x}$$
$$H_{uu} = \frac{\partial^2 f}{\partial u^2}$$
$$H_{\sigma\sigma} = \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix}$$

3.2 Formal expansion in α

The variable α defined by the equation (25) is a small quantity. We thus develop the state x and the costate p at the first order in α.

3.3 Expansion in the state variable

The state variable x is the solution of the original dynamics equation (2). It is developed on the first order in α, as well as the original control u:

$$x = x_0 + \alpha x_1 \tag{27}$$

$$u = u_0 + \alpha u_1 \tag{28}$$

Note that the redundant definitions of x_0 and u_0 are consistent, as will be seen later.

We then distribute the coefficients in low (LP) and high (HP) frequencies signals:

$$x_0 = \bar{x_0} + \tilde{x_0}\ ,\ x_1 = \bar{x_1} + \tilde{x_1} \tag{29}$$

Because of the equations (27) and (29), and because the derivative of the high frequency signal $\tilde{x_1}$ is in $\frac{1}{\alpha}$ ([1]) , the derivative of x has the following expansion in α:

$$\frac{dx}{dt} = \left[\frac{d\bar{x_0}}{dt} + \frac{d\tilde{x_0}}{dt} + \alpha\frac{d\tilde{x_1}}{dt}\right] + \alpha\frac{d\bar{x_1}}{dt} \tag{30}$$

But, because x is the solution of the original dynamics equation (2), using the developments in α of x and u, and thanks to assumption 2, we have another development of $\frac{dx}{dt}$ at the first order in α:

$$\frac{dx}{dt} = f(x_0, u_0, t) + \alpha(f_u(x_0, u_0, t)u_1 + f_x(x_0, u_0, t)x_1) \tag{31}$$

Consequently, identifying the zero order terms in equations (30) and (31), we have:

$$\frac{d\bar{x_0}}{dt} + \frac{d\tilde{x_0}}{dt} + \alpha\frac{d\tilde{x_1}}{dt} = f(x_0, u_0, t)$$

But by definition of $\bar{x_0}$ as the low frequency part of x_0, we have:

$$\frac{d\bar{x_0}}{dt} = LP[f, u_0](x_0, t)$$

Consequently, by definition of HP, we have:

$$\frac{d\tilde{x_0}}{dt} + \alpha\frac{d\tilde{x_1}}{dt} = HP[f, u_0](x_0, t)$$

with initial value 0, that derives in:

$$\tilde{x_0} + \alpha\tilde{x_1} = I[f, x_0, u_0]$$

But $I[f, \bar{x_0}, u_0]$ is of order 1 in α as $\tilde{x_0}$ is of order 0. Thus $\tilde{x_0} = 0$, that gives $x_0 = \bar{x_0}$, and thus the definitions of x_0 and u_0 are consistent.

Moreover, we have a definition of $\tilde{x_1}$:

$$\alpha\tilde{x_1} = I[f, x_0, u_0] \tag{32}$$

so that the derivative of $\tilde{x_1}$ is $\frac{1}{\alpha}HP[f, u_0](x_0, t)$, that is in $\frac{1}{\alpha}$.

[1]The integral of HP is in α, so we consider that the derivative of HP is formally in $\frac{1}{\alpha}$

A consequence of the definition of $\alpha\tilde{x_1}$ in (32) and α in definition 2 is that:

$$\|\tilde{x_1}\|_\infty \leq 1 \tag{33}$$

3.4 Expansion in the costate variable

The costate variable p is the solution of the costate dynamics equation with ending condition (26). We develop it at the first oder in α:

$$p = p_0 + \alpha p_1 \tag{34}$$

We then distribute the coefficients in low (LP) and high (HP) frequencies signals:

$$p_0 = \bar{p_0} + \tilde{p_0}\ ,\ p_1 = \bar{p_1} + \tilde{p_1} \tag{35}$$

Because of the equations (34) and (35), and because the derivative of the high frequency signal $\tilde{p_1}$ is in $\frac{1}{\alpha}$, the derivative of p has the following expansion in α:

$$\frac{dp}{dt} = \left[\frac{d\bar{p_0}}{dt} + \frac{d\tilde{p_0}}{dt} + \alpha\frac{d\tilde{p_1}}{dt}\right] + \alpha\frac{d\bar{p_1}}{dt} \tag{36}$$

On the other hand, p is the solution of the costate dynamics equation with ending condition (26). Moreover, we have defined the developments in α of x, p and u.

Thus we have another development of $\frac{dp}{dt}$ at the first order in α:

$$\frac{dp}{dt} = -H_x(x_0,u_0,p_0,t) + \alpha(-H_{xu}(x_0,u_0,p_0,t)u_1 - H_{xx}(x_0,u_0,p_0,t)x_1 - f_x(x_0,u_0,t)p_1) \tag{37}$$

Consequently, identifying the zero order terms in equations (36) and (37), we have:

$$\frac{d\bar{p_0}}{dt} + \frac{d\tilde{p_0}}{dt} + \alpha\frac{d\tilde{p_1}}{dt} = -H_x(x_0,u_0,p_0,t)$$

But by definition of $\bar{p_0}$ as the low frequency part of p_0, we have:

$$\frac{d\bar{p_0}}{dt} = -LP[H_x,u_0](x_0,p_0,t)$$

Consequently, by definition of HP, we have:

$$\frac{d\tilde{p_0}}{dt} + \alpha\frac{d\tilde{p_1}}{dt} = -HP[H_x,u_0](x_0,p_0,t)$$

with final value 0, that derives in:

$$\tilde{p_0} + \alpha\tilde{p_1} = I^T[H_x,(x_0,p_0),u_0]$$

But $I[-H_x,x_0,p_0,u_0]$ is of order 1 in α as $\tilde{p_0}$ is of order 0. Thus $\tilde{p_0} = 0$, that gives $p_0 = \bar{p_0}$, and thus the definitions of p_0 are consistent.

Moreover, we have a definition of $\tilde{p_1}$:

$$\alpha\tilde{p_1} = I^T[H_x,(x_0,p_0),u_0] \tag{38}$$

so that the derivative of $\tilde{p_1}$ is $-\frac{1}{\alpha}HP[H_x,u_0](x_0,p_0,t)$, that is in $\frac{1}{\alpha}$.

A consequence of the definition of $\alpha\tilde{p_1}$ in (38) and α in definition 2 is that:

$$\|\tilde{p_1}\|_\infty \leq 1 \tag{39}$$

4 Auxiliary Problem

4.1 Assumptions

Assumption 2 (smoothness). *The derivatives, up to the third order, of f and L with respect to x and u are bounded by some $k > 0$.*

A consequence of that assumption is the the value α defined by equation (25) is well defined and small if N is sufficiently big (or the problem is periodic of small period).

Another consequence is that:

Proposition 4.

$$\left| f_{\sigma\sigma}(X,U,t) \left[\begin{array}{c} Y \\ V \end{array} \right]^2 \right| \leq k(|Y| + |V|)^2 \tag{40}$$

for any (X,U,Y,V).

Proof.

$$\begin{aligned} \left| f_{\sigma\sigma}(X,U,t) \left[\begin{array}{c} Y \\ V \end{array} \right]^2 \right| &= |f_{xx}(X,U,t)Y^2 + 2f_{xu}(X,U,t)YV + f_{uu}(X,U,t)V^2| \\ &\leq k(|Y| + |V|)^2 \end{aligned}$$

□

Moreover, as p_0 is the solution of the ODE with final condition (19), it is differentiable and thus continuous of the bounded interval $[0,T]$, so that it is bounded. So that another consequence of the assumption 2 is so:

Proposition 5. *The hamiltonian $H(x,u,p_0,t)$ and its derivatives up to the third order in u and x are bounded by a constant K.*

Proof. Take $K = (1 + \|p_0\|_\infty)k$. □

A consequence of that is:

Proposition 6.

$$\left| H_{\sigma\sigma}(X,U,p_0 t) \left[\begin{array}{c} Y \\ V \end{array} \right]^2 \right| \leq K(|Y| + |V|)^2 \tag{41}$$

for any (X,U,Y,V).

Proof. Similar proof as for $f_{\sigma\sigma}$. □

Assumption 3 (convexity). *There exists $\beta > 0$ so that for any (x,u), the following holds:*

$$H_{uu}(x,u,p_0,t) \geq \beta Id \tag{42}$$

and

$$\left[H_{xx} - H_{xu}(H_{uu})^{-1} H_{ux} \right](x,u,p_0,t) \geq 0 \tag{43}$$

The consequence of equation (42) is that H_{uu} is invertible and $\|H_{uu}^{-1}\|_\infty \leq \frac{1}{\beta}$ on any (x,u,p_0).

The consequence of equation (43) is :

Proposition 7.

$$H_{\sigma\sigma}(x,u,p_0,t) \geq 0 \tag{44}$$

Proof. We make a proof by contradiction.

Let's suppose that (44) is not true. Then there exists a negative eigenvalue $-\gamma$ of $H_{\sigma\sigma}(x,u,p_0,t)$, that is there exists an eigenvector $\begin{bmatrix} y \\ v \end{bmatrix}$ so that:

$$H_{\sigma\sigma}(x,u,p_0,t)\begin{bmatrix} y \\ v \end{bmatrix} = -\gamma \begin{bmatrix} y \\ v \end{bmatrix}$$

This implies that the two following equations hold:

$$\begin{aligned} H_{xx}(x,u,p_0,t)y + H_{xu}(x,u,p_0,t)v &= -\gamma y \qquad (45) \\ H_{ux}(x,u,p_0,t)y + H_{uu}(x,u,p_0,t)v &= -\gamma v \qquad (46) \end{aligned}$$

But $(\gamma Id + H_{uu)} \geq (\gamma+\beta)Id > 0$, so that it is invertible and the equation (46) can be solved in v, giving:

$$v = (\gamma Id + H_{uu}(x,u,p_0,t))^{-1} H_{ux} y \qquad (47)$$

Then, replacing v by its value in the equation (45), we have:

$$(H_{xx}(x,u,p_0,t) - H_{xu}(x,u,p_0,t)(\gamma Id + H_{uu}(x,u,p_0,t))^{-1} H_{ux}(x,u,p_0,t))y = -\gamma y \qquad (48)$$

But as $\gamma > 0$, we have the succession of inequalities:

$$(\gamma Id + H_{uu}(x,u,p_0,t)) \geq H_{uu}(x,u,p_0,t)$$

then

$$(\gamma Id + H_{uu}(x,u,p_0,t))^{-1} \leq H_{uu}^{-1}(x,u,p_0,t)$$

and then

$$\begin{aligned} &(H_{xx}(x,u,p_0,t) - H_{xu}(x,u,p_0,t)(\gamma Id + H_{uu}(x,u,p_0,t))^{-1} H_{0ux}(x,u,p_0,t)) \\ &\qquad \geq \left[(H_{xx} - H_{xu} H_{uu}^{-1} H_{ux}) \right](x,u,p_0,t) \geq 0 \end{aligned}$$

Thus the equation (48) can not hold, because $-\gamma < 0$ can not be an eigenvalue, and equation (44) is proved by contradiction. □

4.2 Auxiliary Problem Statement

Definition 5. *Let's define the following notations:*

$$H_{0\sigma\sigma} = H_{\sigma\sigma}(w_0,t) = \begin{bmatrix} H_{0xx} & H_{0xu} \\ H_{0ux} & H_{0uu} \end{bmatrix}$$

and

$$f_{0x} = f_x(\sigma_0,t) \ , \ f_{0u} = f_u(\sigma_0,t)$$

Then we define the auxiliary problem as the linear quadratic OCP with state y and control v:

$$\begin{aligned} \frac{dy}{dt} &= f_{0x}(y + \tilde{x_1}) + f_{0u} v \ , \ y(0) = 0 \qquad (49) \\ J_1(v) &= \int_0^T \left[\frac{1}{2} \begin{bmatrix} y & v \end{bmatrix} H_{0\sigma\sigma} \begin{bmatrix} y \\ v \end{bmatrix} + \tilde{p_1}\left(f_{0x} y + f_{0u} v\right) \right] dt \qquad (50) \end{aligned}$$

Proposition 8. *There exists an optimal cost v_1 for the auxiliary problem.*

Proof. It is a convex linear quadratic problem because $H_{0\sigma\sigma}$ is non-negative (equation (44)). □

Let's denote v_1 an optimal control, y_1 the trajectory corresponding to v_1 and q_1 the corresponding costate. Then y_1 follows the dynamics of the auxiliary problem:

$$\frac{dy_1}{dt} = f_{0x}(y_1 + \tilde{x_1}) + f_{0u}v_1 \ , \ y_1(0) = 0 \tag{51}$$

The hamiltonian of the auxiliary problem expands in:

$$H_1(y, v, q) = \frac{1}{2}\left(H_{0xx}y^2 + 2H_{0xu}yv + H_{0uu}v^2\right) + \tilde{p_1}\left(f_{0x}y + f_{0u}v\right) + q_1\left[f_{0x}(y + \tilde{x_1}) + f_{0u}v\right] \tag{52}$$

Thus the stationarity condition $\frac{\partial H_1}{\partial u}(y_1, v_1, p_1) = 0$ may be written the following way:

$$H_{0xu}y_1 + H_{0uu}v_1 + (\tilde{p_1} + q_1)f_{0u} = 0 \tag{53}$$

Moreover, the costate q_1 of the auxiliary problem follows the dynamics:

$$\frac{dq_1}{dt} = -H_{0xx}y_1 - H_{0xu}v_1 - (\tilde{p_1} + q_1)f_{0x} \ , \ q_1(T) = 0 \tag{54}$$

Moreover, we have:

Proposition 9. *y_1, v_1 and q_1 are bounded by a constant M.*

Proof. The auxiliary problem is smooth and convex. □

4.3 More auxiliary variables and their upper bounds

Definition 6. *For any $u \in \mathbf{L}^2_{[0,T]}$, x is the trajectory of the nominal dynamics (2).*
Let's then define the following notations:

$$\begin{aligned} \delta x &= x - x_0 & \tilde{x} &= \delta x - \alpha\tilde{x_1} \\ \delta u &= u - u_0 & \tilde{u} &= \delta u \\ \delta\sigma &= (\delta x, \delta u) & \rho(\lambda, \mu) &= \sigma_0 + \lambda\mu\delta\sigma \end{aligned}$$

Where (u_0, x_0) is a solution of the averaged problem and $\alpha\tilde{x_1}$ is $I[f, x_0, u_0]$ (equation (32)).

Upper bounds for r and v

Definition 7. *We define the following data:*

$$r = \tilde{x} - \alpha y_1 \ , \ v = \tilde{u} - \alpha v_1$$

$$Z[\lambda, \mu](t) = v + [H_{uu}^{-1}H_{ux}](\rho(\lambda, \mu), p_0, t)(r + \alpha\tilde{x_1})$$

$$z^2 = \int_0^1 \int_0^1 \lambda \|Z[\lambda, \mu]\|_2^2 d\lambda d\mu$$

Definition 8. *We define the following constants:*

$$k_{r1} = 4k^2 T e^{2k\left(1+\frac{K}{\beta}\right)} \quad , \quad k_{r2} = \frac{\left(\frac{K}{\beta} + \frac{1}{2}(1+2M)^2\alpha\right)^2}{\left(1+\frac{K}{\beta}\right)^2}\left(e^{k\left(1+\frac{K}{\beta}\right)} - 1\right)^2$$

$$k_{v1} = 6\left(2 + \frac{K^2 T k_{r1}}{\beta^2}\right) \quad , \quad k_{v2} = \frac{6K^2 T(k_{r2}+1)}{\beta^2}$$

with:

- k is introduced in assumption 2 abound the bounded derivatives of $f(x,u,t)$ and $L(x,u,t)$.

- $K = (1 + \|p_0\|_\infty)k$

- M is the upper bound of the optimal trajectory of the auxiliary problem introduced in section 4.2 (proposition 9).

- α is the small quantity defined in equation (25).

- β is the convexity constant of H_{uu} introduced in equation (42).

Proposition 10. *The following inequalities hold:*

$$\|r\|_\infty^2 \le k_{r1} z^2 + k_{r2}\alpha^2 \tag{55}$$

and:

$$\|v\|_2^2 \le k_{v1} z^2 + k_{v2}\alpha^2 \tag{56}$$

Proof. see appendix C.1 □

Upper bounds for $r - r_1$ and r_1

Definition 9. *r_1 is defined by the following dynamics:*

$$\frac{dr_1}{dt} = f(r_1 + x_0 + \alpha(\tilde{x_1} + y_1), v + u_0 + \alpha v_1, t) - f(x_0 + \alpha(\tilde{x_1} + y_1), u_0 + \alpha v_1, t)\ ,\ r_1(0) = 0 \tag{57}$$

Definition 10. *We define the following constants:*

$$k_{r5} = \frac{T}{2}(1+2M)^2(e^{kT} - 1)\ ,\ k_{r3} = 2k_{r1}\ ,\ k_{r4} = 2(k_{r2} + k_{r5}^2\alpha^2)$$

with k, K and M as in definition 8.

Proposition 11. *The following inequalities hold:*

$$\|r - r_1\|_\infty \le k_{r5}\alpha^2 \tag{58}$$

and

$$\|r_1\|_\infty^2 \le k_{r3} z^2 + k_{r4}\alpha^2 \tag{59}$$

Proof. see appendix C.2 □

5 Main theorem

Definition 11. *We define the following constants:*

$$\begin{aligned}
k_{J1} &= 4\sqrt{3}KMTk_{r1} + \left(\frac{3k}{2} + KM\right)Tk_{r3} + \left(\frac{3k}{2} + (2\sqrt{3}+1)KM\right)k_{v1} \\
k_{J2} &= 2\sqrt{3}KTk_{r5} + k(1+2M) + 2KTM(M+1) \\
&\quad + \frac{3k}{2}Tk_{r2} + \left(kM + 4\sqrt{3}KM\right)Tk_{r4} + \left(\frac{3k}{2} + (2\sqrt{3}+1)KM\right)k_{v2} \\
k_{J0} &= \left[e^{kT} - 1 + \frac{K}{2k^2T}\left(e^{kT} - 1\right)^2 + \frac{k}{2}\alpha\right] \\
k_J &= k_{J2} + k_{J0} \\
k_x &= \sqrt{2\left(\frac{2k_Jk_{r1}}{\beta} + k_{r2} + (M+1)^2\right)} \\
k_u &= \sqrt{2\left(\frac{2k_Jk_{v1}}{\beta} + k_{v2} + T^2M^2\right)}
\end{aligned}$$

where:

- k, K, M, α and β are as in definition 8.

- k_{r1}, k_{r2}, k_{v1} and k_{r2} are defined in definition 8

- k_{r3}, k_{r4} and k_{r5} are defined in definition 10.

Note that these constants depend only of k, K, M, α, β, and the horizon T.

Assumption 4.

$$\alpha \leq \frac{\beta}{2k_{J1}}$$

Theorem 2 (**Main Theorem**). *Considering the nominal problem in section 2.1, let $H(x, u, p, t)$ its Hamiltonian, and let $J^* = \inf_u J(u)$ be its infimum cost.*

Let u_0 a solution of the averaged problem described in section 2.2.4 with its trajectory x_0. Such a solution exists by assumption 1.

Let α be the small quantity defined in equation (25) and let β be the constant introduced in assumption 3.

Let the set of constants (k_{J1}, k_J, k_x, k_u) introduced in definition 11.

Then, under the set of assumptions listed in section 4.1 and the assumption 4, the following inequalities hold:

- the suboptimality of the real system commanded by u_0 is limited to:

$$J^* \leq J(u_0) \leq J^* + k_J \alpha^2 \tag{60}$$

- any trajectory x of the nominal problem for a u better than u_0 ($J(u) \leq J(u_0)$), is close to (x_0, u_0), with:

$$\|x - x_0\|_\infty \leq k_x \alpha \tag{61}$$

$$\|u - u_0\|_2 \leq k_u \alpha \tag{62}$$

6 Proof of the main result

6.1 Proof Process

To prove the main theorem, we proceed the following way.

The section 6.2 is devoted to the search of a lower bound of any real cost $J(u)$ of the nominal problem. That lower bound contains two integral terms that do not depend on u, a term in z^2 that is the only one depending on u, and a term in α^2, that dos not depend on u either.

The section 6.3 is devoted to the search of an upper bound of the real cost $J(u_0)$ of the nominal problem controlled by u_0. That upper bound contains the same two integral terms as in the lower bound of $J(u)$ and a term in α^2.

Then the section 6.4.1 uses the assumption $\alpha \leq \frac{\beta}{k_{J1}}$ to obtain a lower bound of $J(u)$ independent of u, so that it is also a lower bound for J^*. That lower bound is combined with the upper bound of $J(u_0)$ to prove the suboptimality in α^2 of $L(u_0)$ stated in the equation (60) of the first part of the main theorem 2.

Then the section 6.4.1 uses the stronger assumption $\alpha \leq \frac{\beta}{2k_{J1}}$ to obtain, for any u better than u_0, i.e. so that $J(u) \leq J(u_0)$, an upper bound of z^2. With that bound of z^2, upper bounds for $\|x - x_0\|_\infty$ and $\|u - u_0\|_2$ are found in the equations (61) and (62) of the second part of the main theorem, with the help of the definitions and bounds of r and v in section 4.3.

6.2 Lower bound on the real cost of the nominal problem

6.2.1 Expansion of the real cost $J(u)$

Proposition 12. *The cost $J(u) = \int_0^T L(x,u,t)dt$ for any command $u \in \mathbf{L}^2_{[0,T]}$ expands the following way:*

$$J(u) = \int_0^T L(x_0,u_0,t)dt + \alpha \int_0^T H_{0x}\tilde{x_1}dt + \int_0^T HP[H_x,u_0](x_0,p_0,t)\tilde{x}dt \\ + \int_0^T \int_0^1 \int_0^1 \lambda H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)(\delta\sigma)^2 d\lambda d\mu dt \tag{63}$$

Proof. see appendix D. □

6.2.2 Lower Bound of the third term of the expansion of $J(u)$ in equation (63)

Proposition 13. *The following inequality holds:*

$$\begin{aligned}\int_0^T HP[H_x,u_0](x_0,p_0,t)\tilde{x}dt \quad &\geq \quad \alpha \int_0^T \tilde{p_1}[f_{0x}r_1 + f_{0u}v]dt - \frac{3k}{2}(Tk_{r3}+k_{v1})\alpha z^2 \\ &- \quad \left[2KTk_{r5} + k(1+2M) + \frac{k}{2}(3(Tk_{r4}+k_{v2}) + 4T(1+2M)^2)\alpha\right]\alpha^2\end{aligned} \tag{64}$$

Proof. See appendix E. □

6.2.3 Lower Bound of the fourth term of the expansion of $J(u)$ in equation (63)

Proposition 14. *The following inequality holds:*

$$\begin{aligned}\int_0^T \int_0^1 \int_0^1 \lambda H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)(\delta\sigma)^2 d\lambda d\mu dt \quad &\geq \quad -2KM\left[TM + \sqrt{3}(2Tk_{r2} + k_{v2} + T(M^2+2))\alpha\right]\alpha^2 \\ &+ \left[\beta - 2\sqrt{3}KM(2Tk_{r1}+k_{v1})\alpha\right]z^2 \\ &+ \alpha \int_0^T \tilde{p_1}\left[y_1,v_1\right]H_{0\sigma\sigma}\begin{bmatrix} r+\alpha\tilde{x_1} \\ v \end{bmatrix} dt\end{aligned} \tag{65}$$

Proof. see appendix F. □

6.2.4 Bound in absolute value for the sum of the integral terms of the right hand sides of equations (64) and (65)

Proposition 15. *Let's define R as the sum of the integral terms of the right hand sides of equations (64) and (65):*

$$R = \alpha \int_0^T \tilde{p_1}\left[f_{0x}r_1 + f_{0u}v + \left[y_1,v_1\right]H_{0\sigma\sigma}\begin{bmatrix} r+\alpha\tilde{x_1} \\ v \end{bmatrix}\right] dt$$

Then the following inequality holds:

$$|R| \leq KM[Tk_{r3}+k_{v1}] + [2KTM + (2KTMk_{r5} + kTMk_{r4} + kMk_{v2})\alpha]\alpha^2 \tag{66}$$

Proof. see appendix G. □

6.2.5 Lower Bound of the real cost $J(u)$

Lemma 1. *A lower bound of the cost $J(u)$ of the nominal system for any control u is given by:*

$$J(u) \geq \int_0^T L(x_0, u_0, t)dt + \alpha \int_0^T H_{0x}\tilde{x_1}dt + (\beta - k_{J1}\alpha)z^2 - k_{J2}\alpha^2 \tag{67}$$

where k_{J1} and k_{J2} are defined in definition 11.

Proof. This is a consequence of equations (64), (65) and (66) $\square$

6.3 Upper bound on the cost of the nominal system controlled by u_0

6.3.1 Expansion of the cost $J(u_0)$

Definition 12. *Let x^0 be the trajectory of the nominal problem controlled by u_0. It is defined by the dynamics:*

$$\frac{dx^0}{dt} = f(x^0, u_0, t)\ ,\ x^0(0) = \chi_0 \tag{68}$$

For that trajectory, we set the notations:

$$\delta x^0 = x^0 - x_0 \qquad\qquad \tilde{x}^0 = \delta x^0 - \alpha\tilde{x_1}$$
$$\rho^0(\lambda, \mu) = x_0 + \lambda\mu\ \delta x^0$$

Where (u_0, x_0) is the solution of the averaged problem and $\alpha\tilde{x_1}$ is $I[f, x_0, u_0]$ (equation (32)).

Proposition 16. *The cost $J(u_0) = \int_0^T L(x^0, u_0, t)dt$ for the optimal command u_0 of the averaged problem expands the following way:*

$$\begin{aligned} J(u_0) = & \int_0^T L(x_0, u_0, t)dt + \alpha \int_0^T H_{0x}\tilde{x_1}dt + \int_0^T HP[H_x, u_0](x_0, p_0, t)\tilde{x}^0 dt \\ & + \int_0^T \int_0^1 \int_0^1 \lambda H_{xx}(\rho^0(\lambda, \mu), u_0, p_0, t)(\delta x^0)^2 d\lambda d\mu dt \end{aligned} \tag{69}$$

Proof. It is a consequence of proposition 12 with $u = u_0$, so that $\delta u = 0$. $\square$

6.3.2 Comparison of x^0 and x_0

Proposition 17. *The following inequality hold for $\delta x^0 = x^0 - x_0$:*

$$|\delta x^0| \leq \frac{\alpha}{kT}\left(e^{kT} - 1\right) \tag{70}$$

Proof. As $\delta x^0 = x^0 - x_0$, x^0 follows the dynamics (68) and x_0 follows the averaged dynamics (17), we have the following integral equation:

$$\begin{aligned} \delta x^0 &= \int_0^t [f(x^0, u_0, t) - LP[f, u_0](x_0, t)]dt \\ &= \int_0^t [f(x^0, u_0, t) - f(x_0, u_0, t)]dt + \int_0^t [f(x_0, u_0, t) - LP[f, u_0](x_0, t)]dt \end{aligned}$$

Thus the following inequality holds:

$$|\delta x^0| \leq \|f_x\|_\infty \left(\int_0^t |\delta x^0| ds\right) + |I[f, x_0, u_0](t)| \leq kT \left(\int_0^t |\delta x^0| ds\right) + \alpha$$

Equation (70) follows from Gronwall lemma. $\square$

6.3.3 Upper Bound of the third term of the development of $J(u_0)$ (69)

Proposition 18. *The following inequality holds:*

$$\left|\int_0^T HP[H_x, u_0](x_0, p_0, t)\tilde{x}^0 dt\right| \leq \left(e^{kT} - 1 + \frac{k}{2}\alpha\right)\alpha^2 \tag{71}$$

Proof. By definition of $\tilde{x}^0$, we have

$$\tilde{x}^0 = x^0 - x_0 - \alpha\tilde{x_1},$$

so that:

$$\begin{aligned}\frac{d\tilde{x}^0}{dt} &= f(x_0 + \delta x^0, u_0, t) - LP[f, u_0](x_0, t) + HP[f, u_0](x_0, t)\\ &= f(x_0 + \delta x^0, u_0, t) - f(x_0, u_0, t)\end{aligned}$$

Thus, by Taylor expansion of $f(x^0, u_0, t)$ with integral remainder, we have:

$$\frac{d\tilde{x}^0}{dt} = f_x(x_0, u_0, t)\delta x^0 + \int_0^1\int_0^1 \lambda f_{xx}((x_0 + \lambda\mu(\delta x^0), u_0, t)(\delta x^0)^2 d\lambda d\mu$$

Thus, thanks to proposition 17, we have:

$$\left|\frac{d\tilde{x}^0}{dt}\right| \leq (e^{kT} - 1)\alpha + \frac{k}{2}\alpha^2 \tag{72}$$

But an integration by part, together with the fact that $HP[H_x, u_0](x_0, p_0, t) = -\alpha\frac{d\tilde{p_1}}{dt}$ and that $\tilde{x}^0(0) = \tilde{p_1}(T) = 0$ leads to:

$$\int_0^T HP[H_x, u_0](x_0, p_0, t)\tilde{x}^0 dt = \alpha\int_0^T \tilde{p_1}\frac{d\tilde{x}^0}{dt}dt \tag{73}$$

Including equation (72) and the fact that $\|\tilde{p_1}\|_\infty \leq 1$ into equation (72) proves equation (71) □

6.3.4 Upper Bound of the fourth term of the development of $J(u_0)$ (69)

Proposition 19. *The following inequality holds:*

$$\left|\int_0^T\int_0^1\int_0^1 \lambda H_{xx}(\rho^0(\lambda,\mu), u_0, p_0, t)(\delta x^0)^2 d\lambda d\mu dt\right| \leq \frac{K}{2k^2T}\left(e^{kT} - 1\right)^2\alpha^2 \tag{74}$$

Proof. This is a consequence of the proposition 17. □

6.3.5 Upper Bound of $J(u_0)$

Inserting equations (71) and (74) into equation (69 leads to:

Lemma 2. *An upper bound of the cost $J(u_0)$ of the nominal system controlled by u_0 is given by:*

$$J(u_0) \leq \int_0^T L(x_0, u_0, t)dt + \alpha\int_0^T H_{0x}\tilde{x_1}dt + k_{J0}\alpha^2 \tag{75}$$

where k_{J0} is defined in definition 11.

6.4 Proof of the main theorem

The main theorem is proved in two steps. The first step compares the costs to estimate the suboptimality of the real system cotrolled by u_0. The second step compares the trajectories and controls better than u_0 to estimate how close they are from the trajectory and control dealed by u_0.

6.4.1 Comparison of the real cost controlled by u_0 and the infimum cost of the real system

Let's now use the assumption $\alpha \leq \frac{\beta}{2k_{J1}}$ of the first part of the Main Theorem 2 into the equation (67) of Lemma 1. The term in z^2 is then non negative, and we get the lower bound independent of u:

$$J(u) \geq \int_0^T L(x_0, u_0, t)dt + \alpha \int_0^T H_{0x}\tilde{x_1}dt - k_{J2}\alpha^2$$

As that lower bound holds for any u, it is also a lower bound for the infimum cost $J^* = \inf_u J(u)$:

$$J^* \geq \int_0^T L(x_0, u_0, t)dt + \alpha \int_0^T H_{0x}\tilde{x_1}dt - k_{J2}\alpha^2$$

so that:

$$\int_0^T L(x_0, u_0, t)dt + \alpha \int_0^T H_{0x}\tilde{x_1}dt \leq J^* + k_{J2}\alpha^2$$

Inserting this equation into the equation (75) of Lemma 2, together with the fact that $J(u_0) \geq J^*$, by definition of J^*, proves the suboptimality equation (60) in he Main Theorem 2, since $k_J = K_{J0} + k_{J2}$.

6.4.2 Comparison of the controls and trajectories with and without $u = u_0$

Let's consider a control u better than u_0, i.e. such that $J(u) \leq J(u_0)$.

Let's now use in a stronger manner the assumption $\alpha \leq \frac{\beta}{2k_{J1}}$ of the second part of the Main Theorem 2 into the equation (67) of Lemma 1. The coefficient z^2 is then lower than $\frac{\beta}{2}$, and we get the lower bound dependent of z^2, that depends on u:

$$J(u) \geq \int_0^T L(x_0, u_0, t)dt + \alpha \int_0^T H_{0x}\tilde{x_1}dt + \frac{\beta}{2}z^2 - k_{J2}\alpha^2$$

Thus, together with the equation (75) of Lemma 2, we have the list of inequalities:

$$\begin{aligned} \int_0^T L(x_0, u_0, t)dt + \alpha \int_0^T H_{0x}\tilde{x_1}dt + \frac{\beta}{2}z^2 - k_{J2}\alpha^2 \quad &\leq \quad J(u) \leq J(u_0 \\ &\leq \quad \int_0^T L(x_0, u_0, t)dt + \alpha \int_0^T H_{0x}\tilde{x_1}dt + k_{J0}\alpha^2 \end{aligned}$$

So that, with the definition of $k_J = K_{J0} + k_{J2}$, we have

$$z^2 \leq \frac{2k_J}{\beta}\alpha^2 \tag{76}$$

On the other hand, by definition of r, we have:

$$x - x_0 = r + \alpha(\tilde{x_1} + y_1)$$

so that, together with the equation (55):

$$\|x - x_0\|_\infty^2 \leq 2\left(\|r\|_\infty^2 + \alpha^2(1+M)^2\right) \leq 2\left(k_{r1}z^2 + (k_{r2}(1+M)^2)\alpha^2\right)$$

Introducing equation (76) into that equation leads to equation (61) of the second part of the Main Theorem 2.

Now let's consider the fact that, by definition of v:

$$u - u_0 = v + \alpha v_1$$

so that:

$$\|u - u_0\|_2^2 \leq 2\left(\|v\|_2^2 + \alpha^2 T^2 M^2\right) \leq 2\left(k_{v1}z^2 + (k_{v2}T^2 M)^2\alpha^2\right)$$

Introducing equation (76) into that equation leads to equation (62) of the second part of the Main Theorem 2.

7 Conclusion

We have shown that the method of averaging can be used very simply by performing averages of the dynamics on adjacent intervals. Its efficiency, notably in optimal control, is measured by α and by the convexity β of the cost function. Using α as a measure of the efficiency of averaging amounts to saying that, from this point of view, the simple integrator acts as reference for all state space models. An important point is that, provided that $\alpha \leq \frac{\beta}{2k_{J1}}$, the estimates (60,61,62) in the main theorem 2 hold *for any system.*

This method of averaging has been applied to the guidance of a low thrust satellite in the non keplerian case [23]. In this case, we cannot use periodic averaging on orbits because of the influence of the sun and of the moon, which have different periods.

A Proof of Proposition 2 about the upper bound of $I[g, x_0](t)$

$I[g, x_0](t)$ is defined by equation (11) and x_0 is the solution of the averaged ODE (9).

Let's fix t and let K be so that $t \in [t_K, t_{K+1})$. Then we have:

$$I[g, x_0](t) = \sum_{k+0}^{K-1} I_k[g, x_0](t_{k+1}) + I_K[g, x_0](t) \tag{77}$$

where $I_j[g, x_0](\tau)$ is defined for $\tau \in [t_j, t_{i+1})$ as:

$$I_j[g, x_0](\tau) = \int_{t_j}^{\tau} \left[g(x_0(s), s) - \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} g(x_0(s), \sigma) d\sigma \right] ds$$

We can develop the terms $I_k[g, x_0](t_{k+1})$ and $I_K g, x_0(t)$ in subtracting and adding the values at t_k.

$$\begin{aligned} I_k[g, x_0](t_{k+1}) \quad &= \quad \int_{t_k}^{t_{k+1}} \left[g(x_0(t_k, s) + \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} g(x_0(t_k), \sigma) d\sigma \right] ds \\ &+ \int_{t_k}^{t_{k+1}} [g(x_0(s), s) - g(x_0(t_k), s)] ds \\ &+ \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} [g(x_0(s), \sigma) - g(x_0(t_k), \sigma)] d\sigma ds \end{aligned}$$

The first term is equal to 0.

The terms integrated once and twice in the second and third terms are both lower or equal to $\lambda_g \|x_0(s) - x_0(t_k\|$ because g is Lipschitz in x, with Lipschitz constant λ_g.

Moreover, because x_0 is the solution of the averaged ODE (9), we have:

$$\|x_0(s) - x_0(t_k\| \leq \|f\|_\infty (s - t_k)$$

Thus, we have:

$$|I_k[g, x_0](t_{k+1})| \leq 2\lambda_g \|f\|_\infty \int_{t_k}^{t_{k+1}} (s - t_k) ds$$

That is:

$$|I_k[g, x_0](t_{k+1})| \leq 2\lambda_g \|f\|_\infty \left(\frac{T}{N} \right)^2 \tag{78}$$

Let's now develop $I_K[g,x_0](t)$ in an analog way starting from t_K:

$$\begin{aligned} I_K[g,x_0](t) &= \int_{t_K}^{t}\left[g(x_0(t_K,s)+\frac{1}{t_{K+1}-t_K}\int_{t_K}^{t_{K+1}} g(x_0(t_K),\sigma)d\sigma\right]ds \\ &+\int_{t_K}^{t}[g(x_0(s),s)-g(x_0(t_K),s)]ds \\ &+\frac{1}{t_{K+1}-t_K}\int_{t_K}^{t}\int_{t_K}^{t_{K+1}}[g(x_0(s),\sigma)-g(x_0(t_K),\sigma)]d\sigma ds \end{aligned}$$

The first term is not 0 here, but it is lower or equal to $2\|g\|_\infty\frac{T}{N}$, and the other terms are bounded as the ones of $I_k[g,x_0](t_{k+1})$.

Thus we have:

$$|I_K[g,x_0](t)|\leq \|g\|_\infty\frac{T}{N}+2\lambda_g\|f\|_\infty\left(\frac{T}{N}\right)^2 \tag{79}$$

Finally, because $K+1\leq N$, the inequations (78) and (79) yield to the proposition 2.

B Proof of the stationarity condition of the averaged problem

B.1 Averaged two boundaries problem

Let u_0 be the optimal control of the averaged problem (16) and let x_0 be the corresponding trajectory. x_0 is defined by the ODE with initial condition (17). Let p_0 be the costate of the optimal trajectory, defined by the ODE with final condition (19), with H the hamiltonian (18).

The system constituted of the of equations (17) and (19) is a two boundaries problem. It is defined as the two boundaries problem corresponding to the averaged optimal control problem.

B.2 First variation in the direction of δu

Let $\delta u\in\mathbf{L}^2_{[0,T]}$ a scalar square integrable function on $[0,T]$.

Let $\epsilon>0$ and let $u_\epsilon=u_0+\epsilon\delta u$ the variation of u_0 in the direction of δu.

As $J_0(u_0)=\min_u(J_0(u)$, the following stationarity condition holds:

$$\forall\ \delta u\in\mathbf{L}^2_{[0,T]},\left(\frac{dJ_0(u_\epsilon)}{d\epsilon}\right)_{\epsilon=0}=0 \tag{80}$$

Let x_ϵ be the trajectory corresponding to u_ϵ, defined by the dynamics equation 16 with $v=u_\epsilon$:

$$\frac{dx_\epsilon}{dt}=LP[f,u_\epsilon](x_\epsilon,t)\ ,\ x_\epsilon(0)=\xi_0 \tag{81}$$

Let δx be the variation trajectory corresponding to the direction δu given by $\delta x=\left(\frac{dx_\epsilon}{d\epsilon}\right)_{\epsilon=0}$.

Lemma 3. *δx respects the following dynamics function:*

$$\frac{d(\delta x)}{dt}=LP\left[\frac{\partial f}{\partial x},u_0\right](x_0,t)\delta x+LP\left[\frac{\partial f}{\partial u}\delta u,u_0\right](x_0,t) \tag{82}$$

Proof.

$$\frac{d(\delta x)}{dt}=\frac{d}{dt}\left(\frac{dx_\epsilon}{d\epsilon}\right)_{\epsilon=0}=\left(\frac{d}{d\epsilon}\frac{dx_\epsilon}{dt}\right)_{\epsilon=0}=\left(\frac{d}{d\epsilon}LP[f,u_0+\epsilon\delta u](x_\epsilon,t)\right)_{\epsilon=0}$$

Let k be so that $[t_k, t_{k+1})$). Then:

$$\begin{aligned}
\frac{d(\delta x)}{dt} &= \left(\frac{d}{d\epsilon}\left(\frac{1}{t_{k+1}-t_k}+\int_{t_k}^{t_{k+1}} f(x_\epsilon(t), u_0(s)+\epsilon\delta u(s), s)ds\right)\right)_{\epsilon=0} \\
&= \frac{1}{t_{k+1}-t_k}+\int_{t_k}^{t_{k+1}} \frac{d}{d\epsilon}\left(f(x_\epsilon(t), u_0(s)+\epsilon\delta u(s), s)\right)_{\epsilon=0} ds \\
&= \frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}} \left(\frac{\partial f}{\partial x}(x_{0(t),u_0(s),s}\delta x(t)+\frac{\partial f}{\partial u}(x_{0(t),u_0(s),s}\delta u(s)\right) ds \\
&= LP\left[\frac{\partial f}{\partial x}, u_0\right](x_0,t)\delta x + LP\left[\frac{\partial f}{\partial u}\delta u, u_0\right](x_0,t)
\end{aligned}$$

□

B.3 Proof of the stationarity result

Let's make use of Equation (80) in developing $\left(\frac{dJ_0(u_\epsilon)}{d\epsilon}\right)_{\epsilon=0}$ for a given δu.

Lemma 4. *The derivative at* 0 *of* $J_0(u_\epsilon)$ *in* ϵ *is related to the hamiltonian by the following equation:*

$$\left(\frac{dJ_0(u_\epsilon)}{d\epsilon}\right)_{\epsilon=0} = \int_0^T LP\left[\frac{\partial H}{\partial u}\delta u, u_0\right](x_0, p_0, t)dt \tag{83}$$

Proof. Let's use the equation (15) and then commute the differentiation and integration:

$$\left(\frac{dJ_0(u_\epsilon)}{d\epsilon}\right)_{\epsilon=0} = \left(\frac{d}{d\epsilon}\left[\int_0^T LP[L,u_\epsilon](x_\epsilon,t)dt\right]\right)_{\epsilon=0} = \int_0^T\left(\frac{d}{d\epsilon}\left[LP[L,u_\epsilon](x_\epsilon,t)\right]\right)_{\epsilon=0} dt$$

For any $k \in [0, N-1]$ and for any $t \in [t_k, t_{k+1})$, we have the definition (14) of LP:

$$LP[L,u_\epsilon](x_\epsilon,t) = \frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}} L(x_\epsilon(t), u_\epsilon(s), s)ds$$

Thus if we commute again the integration and the differentiation:

$$\left(\frac{d}{d\epsilon}\left[LP[L,u_\epsilon](x_\epsilon,t)\right]\right)_{\epsilon=0} = \frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}}\left(\frac{dL(x_\epsilon(t),u_\epsilon(s),s)}{d\epsilon}\right)_{\epsilon=0} ds$$

But by definition of x_ϵ, u_ϵ, δx and δu, we have:

$$\left(\frac{dL(x_\epsilon(t),u_\epsilon(s),s)}{d\epsilon}\right)_{\epsilon=0} = \frac{\partial L}{\partial x}(x_0(t),u_0(s),s)\delta x(t)+\frac{\partial L}{\partial u}(x_0(t),u_0(s),s)\delta u(s)$$

Thus by averaging on $[t_k, t_{k+1}]$, the result is (LP is linear):

$$\left(\frac{d}{d\epsilon}\left[LP[L,u_0](x_\epsilon,t)\right]\right)_{\epsilon=0} = LP\left[\frac{\partial L}{\partial x},u_0\right](x_0(t),t)\delta x(t) + LP\left[\frac{\partial L}{\partial u}\delta u, u_0\right](x_0(t),t)$$

But because of the dynamics (19) of the averaged costate p_0, we have, with the definition (18) of the Hamiltonian:

$$LP\left[\frac{\partial L}{\partial x},u_0\right](x_0(t),t) = -\frac{dp_0}{dt} - p_0 LP\left[\frac{\partial f}{\partial x},u_0\right](x_0(t),t)$$

Thus integrating $\left(\frac{d}{d\epsilon}\left[LP[L,u_0](x_\epsilon,t)\right]\right)_{\epsilon=0}$ between 0 and T, we obtain:

$$\left(\frac{dJ_0(u_\epsilon)}{d\epsilon}\right)_{\epsilon=0} = \int_0^T -\frac{dp_0}{dt}\delta x - \int_0^T p_0 LP\left[\frac{\partial f}{\partial x},u_0\right](x_0(t),t)\delta x(t)dt + \int_0^T LP\left[\frac{\partial L}{\partial u}\delta u, u_0\right](x_0(t),t)dt \tag{84}$$

Let's make an integration by part for the first term of that equation:

$$\int_0^T -\frac{dp_0}{dt}\delta x = -\left[p_0\delta x\right]_0^T + \int_0^T \frac{d(\delta x)}{dt}p_0$$

The variation of $p_0\delta x$ between 0 and T is null because $\delta x(0) = 0$ and $p_0(T) = 0$. Thus, with the dynamics of δx given by the Lemma 3, the following holds:

$$\int_0^T -\frac{dp_0}{dt}\delta x = \int_0^T p_0 LP\left[\frac{\partial f}{\partial x}, u_0\right](x_0,t)\delta x(t)dt + \int_0^T p_0 LP\left[\frac{\partial f}{\partial u}\delta u, u_0\right](x_0,t)dt$$

Let's insert this equation in the first term of equation (84).It results in:

$$\left(\frac{dJ_0(u_\epsilon)}{d\epsilon}\right)_{\epsilon=0} = \int_0^T \left[LP\left[\frac{\partial L}{\partial u}\delta u, u_0\right](x_0,t) + p_0 LP\left[\frac{\partial f}{\partial u}\delta u, u_0\right](x_0,t)\right]dt$$

This proves the equation (83) by definition of the Hamiltonian. □

Let's now make use of equation (83). Let's first fix t and let k be so that $t \in [T_k, t_{k+1})$. Then we have:

$$LP\left[\frac{\partial H}{\partial u}\delta u, u_0\right](x_0,p_0,t) = \frac{1}{t_{k+1}-t_k}\int_{t_k}^{t_{k+1}} \frac{\partial H}{\partial u}(x_0(t),u_0(s),p_0(t),t)\delta u(s)ds$$

Let's specialize δu as a "needle variation":

$$\delta u = \frac{t_{k+1}-t_k}{\eta}\mathbf{1}_{[t,t+\eta]}\delta v$$

with $\eta > 0$ so that $t + \eta < t_{k+1}$ and $\delta v \in \mathbf{L}^2_{[0,T]}$.

Then we have:

$$LP\left[\frac{\partial H}{\partial u}\delta u, u_0\right](x_0,p_0,t) = \frac{1}{\eta}\int_t^{t+\eta} \frac{\partial H}{\partial u}(x_0(t),u_0(s),p_0(t),t)\delta v(s)ds \underset{\eta\to 0}{\overset{a.e.}{\longrightarrow}} \frac{\partial H}{\partial u}(x_0(t),u_0(t),p_0(t),t)\delta v(t)$$

More precisely, the limit is the value of the function at t everywhere the function is continue, that is for any t possibly except for a countable number of "jumps". As any countable set is negligible, the limit holds almost everywhere.

Thus, because of the equations (80) and (83), we have:

$$\int_0^T \frac{\partial H}{\partial u}(x_0(t),u_0(t),p_0(t),t)\delta v(t) = 0$$

and this is true for any $\delta v \in \mathbf{L}^2_{[0,T]}$. This proves the stationnarity result:

$$\frac{\partial H}{\partial u}(x_0(t),u_0(t),p_0(t),t) = 0 \; a.e.$$

C Proof of the inequalities on r, v, $r - r_1$ and r_1

C.1 Proof of proposition 10

C.1.1 Upper bound for r

Proposition 20. *The dynamics of r is the following:*

$$\begin{aligned}\frac{dr}{dt} &= f(r + x_0 + \alpha(\tilde{x_1} + y_1), v + u_0 + \alpha v_1, t) - f(x_0 + \alpha(\tilde{x_1} + y_1), u_0 + \alpha v_1, t) \\ &\quad + \alpha^2 \int_0^1\int_0^1 \lambda f_{\sigma\sigma}(x_0 + \lambda\mu\alpha(x_1 \tilde{+} y_1, v + u_0 + \alpha v_1, t)\begin{bmatrix}\tilde{x_1} + y_1 \\ v_1\end{bmatrix}^2 d\lambda d\mu\end{aligned} \tag{85}$$

Proof. By definition of r and v, we have:

$$\begin{aligned} r = x - x_0 - \alpha(\tilde{x_1} + y_1), &\quad \textit{so that}: \quad x = r + x_0 + \alpha(\tilde{x_1} + y_1) \\ v = u - u_0 - v_1, &\quad \textit{so that}: \quad u = v + u_0 + v_1 \end{aligned}$$

Moreover:

$$\begin{aligned} \frac{dr}{dt} &= \frac{dx}{dt} - \frac{dx_0}{dt} - \alpha\frac{d\tilde{x_1}}{dt} - \alpha\frac{dy_1}{dt} \\ &= f(r + x_0 + \alpha(\tilde{x_1} + y_1), v + u_0 + v_1, t) - LP[f, u_0](x_0, t) \\ &\quad -HP[f, u_0](x_0, t) - \alpha[f_{0x}(y1 + \tilde{x_1}) + f_{0u}v_1] \\ &= f(r + x_0 + \alpha(\tilde{x_1} + y_1), v + u_0 + v_1, t) - f(x_0, u_0, t) \\ &\quad -\alpha[f_{0x}(y1 + \tilde{x_1}) + f_{0u}v_1] \end{aligned} \tag{86}$$

But a Taylor expansion of $f(x_0 + \alpha(\tilde{x_1} + y_1), u_0 + \alpha v_1, t)$ is so:

$$f(x_0(\tilde{x_1} + y_1), u_0 + \alpha v_1, t) = f(x_0, u_0, t) + \alpha[f_{0x}(y1 + \tilde{x_1}) + f_{0u}v_1] \tag{87}$$

$$+\alpha^2 \int_0^1 \int_0^1 \lambda f_{\sigma\sigma}(x_0 + \lambda\mu\alpha(x_1 \tilde{+} y_1, v + u_0 + \alpha v_1, t) \begin{bmatrix} \tilde{x_1} + y_1 \\ v_1 \end{bmatrix}^2 d\lambda d\mu$$

Introducing equation (87) in equation (86) proves equation (85). □

Proposition 21. *The following inequality holds:*

$$\|r\|_\infty \leq ke^{k\left(1+\frac{K}{\beta}\right)} \left[\sqrt{T}\|Z(\lambda, \mu)\|_2 + T\left(\frac{K}{\beta} + \frac{1}{2}((1 + 2M)^2\alpha\right)\alpha\right] \tag{88}$$

Proof. Equation (40) about the upper bound of $f_{\sigma\sigma}$ leads to:

$$\begin{aligned} \left| f_{\sigma\sigma}(x_0 + \lambda\mu\alpha(x_1 \tilde{+} y_1, v + u_0 + \alpha v_1, t) \begin{bmatrix} \tilde{x_1} + y_1 \\ v_1 \end{bmatrix}^2 \right| &\leq k(|\tilde{x_1} + y_1| + |v_1|)^2 \\ &\leq k(|\tilde{x_1}| + |y_1| + |v_1|)^2 \end{aligned}$$

Hence:

$$\left| f_{\sigma\sigma}(x_0 + \lambda\mu\alpha(x_1 \tilde{+} y_1, v + u_0 + \alpha v_1, t) \begin{bmatrix} \tilde{x_1} + y_1 \\ v_1 \end{bmatrix}^2 \right| \leq k(1 + 2M)^2 \tag{89}$$

Moreover, the function f is Lipschitz in x and u with Lipschitz constant the bound of the derivatives k, so that:

$$|f(r + x_0 + \alpha(\tilde{x_1} + y_1), v + u_0 + \alpha v_1, t) - f(x_0 + \alpha(\tilde{x_1} + y_1), u_0 + \alpha v_1, t)| \leq k(|r| + |v|) \tag{90}$$

The equations (89) and (90) in the equation (85) give, together with the fact that $r(0) = 0$ the following inequality:

$$|r(t)| \leq \int_0^t (|r(s)| + |v(s)|)ds + \frac{kT}{2}(2M + 1)^2\alpha^2 \tag{91}$$

But by definition of $Z[\lambda, \mu]$, we have:

$$v = Z[\lambda, \mu] - [H_{uu}^{-1}H_{ux}](\rho(\lambda, \mu), p_0, t)(r + \alpha\tilde{x_1}) \tag{92}$$

so that:

$$|v(s)| \leq |Z[\lambda, \mu](s)| + \|H_{uu}^{-1}\|_\infty \|H_{ux}\|_\infty (|r(s)| + \alpha|\tilde{x_1}(s)|)$$

But the assumption 3 proves that H_{uu} is invertible and that H_{uu}^{-1} is bounded by $\frac{1}{\beta}$, so that:

$$|v(s)| \leq |Z[\lambda,\mu](s)| + \frac{K}{\beta}(|r(s)| + \alpha) \tag{93}$$

Including equation(93) in equation (91) leads to:

$$|r(t)| \leq k\left(1+\frac{K}{\beta}\right)\int_0^t (|r(s)|)ds + \frac{kKT}{\beta}\alpha + \frac{kT}{2}(2M+1)^2\alpha^2 + k\int_0^T |Z[\lambda,\mu](t)||dt \tag{94}$$

But Cauchy property leads to:

$$\int_0^T |Z[\lambda,\mu](t)||dt \leq \sqrt{T}\|Z(\lambda,\mu)]\|_2 \tag{95}$$

Including equation (95) in equation (94) lead to:

$$|r(t)| \leq k\left(1+\frac{K}{\beta}\right)\int_0^t (|r(s)|)ds + k\sqrt{T}\|Z(\lambda,\mu)\|_2 + kT\left[\frac{K}{\beta} + \frac{1}{2}(2M+1)^2\alpha\right]\alpha \tag{96}$$

Equation (96), together with Gronwall lemma, proves equation (88). □

Proof of Equation (55) Let's take the square of equation (88):

$$\|r\|_\infty^2 \leq 2k^2 e^{2k\left(1+\frac{K}{\beta}\right)}\left[T\|Z(\lambda,\mu)\|_2^2 + T^2\left(\frac{K}{\beta} + \frac{1}{2}((1+2M)^2\alpha\right)^2\alpha^2\right]$$

Let's now multiply by λ and integrate relatively to λ and μ between 0 and 1:

$$\frac{1}{2}\|r\|_\infty^2 \leq 2k^2 e^{2k\left(1+\frac{K}{\beta}\right)}\left[Tz^2 + \frac{1}{2}T^2\left(\frac{K}{\beta} + \frac{1}{2}((1+2M)^2\alpha\right)^2\alpha^2\right]$$

Multiplying by 2 that equation proves Equation (55).

C.1.2 Upper bound for v

Let's apply the triangular inequality for the $\mathbf{L}_2$ norm to the expression of v (92):

$$\|v\|_2 \leq \|Z[\lambda,\mu]\|_2 + \|[H_{uu}^{-1}H_{ux}](\rho(\lambda,\mu),p_0,t)(r+\alpha\tilde{x_1})\|_2$$

But:

$$\|[H_{uu}^{-1}H_{ux}](\rho(\lambda,\mu),p_0,t)(r+\alpha\tilde{x_1}\|_2 \leq \sqrt{T}\|[H_{uu}^{-1}H_{ux}](\rho(\lambda,\mu),p_0,t)(r+\alpha\tilde{x_1})\|_\infty \leq \frac{K\sqrt{T}}{\beta}(\|r\|_\infty + \alpha\|\tilde{x_1}\|_\infty)$$

Thus, taking the squares:

$$\|v\|_2^2 \leq 3\left[\|Z[\lambda,\mu]\|_2 + \frac{K^2T}{\beta^2}(\|r\|_\infty^2 + \alpha^2\|\tilde{x_1}\|_\infty^2)\right]$$

Let's multiply by λ and integrate relatively to λ and μ between 0 and 1:

$$\frac{1}{2}\|v\|_2^2 \leq 3\left[z^2 + \frac{K^2T}{2\beta^2}(\|r\|_\infty^2 + \alpha^2\right] \leq 3\left[\left(1+\frac{K^2Tk_{r1}}{2\beta^2}\right)z^2 + \frac{K^2T}{2\beta^2}(1+k_{r2})\alpha^2\right]$$

Multiplying by 2 that equation proves Equation (56).

C.2 Proof of proposition 11

r follows the dynamic (85) with $r(0) = 0$ and r_1 follows the dynamic (57).

Thus $r - r_1$ follows the dynamics:

$$\begin{aligned}\frac{d(r-r_1)}{dt} &= f(r + x_0 + \alpha(\tilde{x_1} + y_1), v + u_0 + \alpha v_1, t) - f(r_1 + x_0 + \alpha(\tilde{x_1} + y_1), v + u_0 + \alpha v_1, t) \\ &\quad +\alpha^2 \int_0^1 \int_0^1 \lambda f_{\sigma\sigma}(x_0 + \lambda\mu\alpha(x_1 \tilde{+} y_1, v + u_0 + \alpha v_1, t) \begin{bmatrix} \tilde{x_1} + y_1 \\ v_1 \end{bmatrix}^2 d\lambda d\mu\end{aligned}$$

Thus, in a similar way than for the upper bound of r, the following inequality holds:

$$|r(t) - r_1(t)| \leq k \int_0^t |r(s) - r_1(s)| ds + \frac{kT}{2}(2M+1)^2\alpha^2$$

The equation (58) follows from Gronwall lemma.

The equation (59) is then the consequence of equations (55) and (58), together with:

$$\|r_1\|_\infty^2 \leq 2(\|r\|_\infty^2 + \|r - r_1\|_\infty^2)$$

D Proof of the expansion of the real cost (proposition 12)

Proposition 22. *$L(x,u,t)$ expands the following way:*

$$L(x,u,t) = L(x_0,u_0,t) + L_x(x_0,u_0,t)\delta x + L_u(x_0,u_0,t)\delta u + \int_0^1 \int_0^1 \lambda L_{\sigma\sigma}(\rho(\lambda,\mu),t)(\delta\sigma)^2 d\lambda d\mu \tag{97}$$

Proof. It is a Taylor expansion of $L(x,u,t)$ with integral remainder. □

Proposition 23. *The dynamics $\frac{d\tilde{x}}{dt}$ of $\tilde{x}$ expands the following way:*

$$\frac{d\tilde{x}}{dt} = f_x(x_0,u_0,t)\delta x + f_u(x_0,u_0,t)\delta u + \int_0^1 \int_0^1 \lambda f_{\sigma\sigma}(\rho(\lambda,\mu),t)(\delta\sigma)^2 d\lambda d\mu \tag{98}$$

Proof. By definition of $\tilde{x}$, we have

$$\tilde{x} = x - x_0 - \alpha\tilde{x_1},$$

so that:

$$\begin{aligned}\frac{d\tilde{x}}{dt} &= f(x_0 + \delta x, u_0 + \delta u, t) - LP[f,u_0](x_0,t) + HP[f,u_0](x_0,t) \\ &= f(x_0 + \delta x, u_0 + \delta u, t) - f(x_0,u_0,t)\end{aligned}$$

The equation (98) follows by Taylor expansion of $f(x,u,t)$ with integral remainder. □

Proposition 24. *$L(x,u,t))$ rewrites the following way:*

$$L(x,u,t) = L(x_0,u_0,t) + H_{0x}(\tilde{x} + \alpha\tilde{x_1}) - p_0\frac{d\tilde{x}}{dt} + \int_0^1 \int_0^1 \lambda H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)(\delta\sigma)^2 d\lambda d\mu \tag{99}$$

Proof. We have the following identities:

$$\delta x = \tilde{x} + \alpha\tilde{x_1}$$

and:

$$L_x(x_0,u_0,t) = H_{0x} - p_0 f_x(x_0,u_0,t)$$

Moreover, because of the stationary condition of the averaged problem, we have:

$$L_u(x_0, u_0, t) = -p_0 f_u(x_0, u_0, t)$$

Finally we change the integral remainder of the expansion of $L(x, u, t)$ in equation (97) with:

$$L_{\sigma\sigma}(\rho(\lambda, \mu), t) = H_{\sigma\sigma}(\rho(\lambda, \mu), p_0, t) - p_0 f_{\sigma\sigma}(\rho(\lambda, \mu), t)$$

Thus equation (97) leads to:

$$\begin{aligned} L(x, u, t) = L(x_0, u_0, t + H_{0x}(\tilde{x} + \alpha\tilde{x_1}) - p_0 \left[f_x(x_0, u_0, t) + f_u(x_0, u_0, t) \right. \\ \left. + \int_0^1 \int_0^1 \lambda f_{\sigma\sigma}(\rho(\lambda, \mu), t)](\delta\sigma)^2 d\lambda d\mu \right] + \int_0^1 \int_0^1 \lambda H_{\sigma\sigma}(\rho(\lambda, \mu), p_0, t)(\delta\sigma)^2 d\lambda d\mu \end{aligned}$$

Inserting equation (98) in this equation proves proposition 24. □

Proposition 25. *The following equality holds:*

$$\int_0^T -p_0 \frac{d\tilde{x}}{dt} dt = -\int_0^T LP[H_x, u_0](x_0, t)\tilde{x} dt \tag{100}$$

Proof. Let's make an integration by part:

$$\int_0^T -p_0 \frac{d\tilde{x}}{dt} dt = -[p_0 \tilde{x}]_0^T + \int_0^T \frac{dp_0}{dt} \tilde{x} dt$$

This leads to the equation (100) because $\tilde{x}(0) = 0$, $p_0(T) = 0$ and:

$$\frac{dp_0}{dt} = -LP[H_x, u_0](x_0, t)$$

□

Now inserting the equation (100) in the equation (99) integrated between 0 and T leads to equation (63), which ends the proof of proposition 12.

E Proof of the lower bound of the third term (Proposition 13)

Proposition 26. *The following inequality holds:*

$$\int_0^T HP[H_x, u_0](x_0, p_0, t)\tilde{x} dt \geq -(2KTk_{r5} + k(1 + 2M))\alpha^2 + \alpha \int_0^T \tilde{p_1} \frac{dr_1}{dt} dt \tag{101}$$

Proof. By definition of r, we have:

$$\tilde{x} = r + \alpha y_1 = (r - r_1) + (r_1 + \alpha y_1)$$

So that

$$\int_0^T HP[H_x, u_0](x_0, p_0, t)\tilde{x} dt = \int_0^T HP[H_x, u_0](x_0, p_0, t)(r - r_1) dt + \int_0^T HP[H_x, u_0](x_0, p_0, t)(r_1 + \alpha y_1) dt$$

But because of equation (58), we have

$$\left| \int_0^T HP[H_x, u_0](x_0, p_0, t)(r - r_1) dt \right| \leq T \| HP[H_x, u_0] \|_\infty k_{r5} \alpha^2 \leq 2KTk_{r5}\alpha^2$$

Moreover, by definition of $\alpha\tilde{p_1}$, we have:

$$HP[H_x,u_0](x_0,p_0,t) = -\alpha\frac{d\tilde{p_1}}{dt}(t)$$

So that the following inequality holds:

$$\int_0^T HP[H_x,u_0](x_0,p_0,t)\tilde{x}dt \geq -2KTk_{r5}\alpha^2 - \alpha\int_0^T \frac{d\tilde{p_1}}{dt}(\tilde{x_1}+\alpha y_1)dt$$

If we make an integration by part and use the fact that $\tilde{p_1}(T) = \tilde{x_1}(0) = y_1(0) = 0$, we get:

$$\int_0^T HP[H_x,u_0](x_0,p_0,t)\tilde{x}dt \geq -2KTk_{r5}\alpha^2 + \alpha\int_0^T \tilde{p_1}\left(\frac{d\tilde{x_1}}{dt}+\alpha\frac{dy_1}{dt}\right)dt$$

y_1 follows the dynamics (51) and $\|\tilde{p_1}\|_\infty \leq 1$, so that:

$$\left|\alpha^2\int_0^T \tilde{p_1}\frac{dy_1}{dt}dt\right| \leq k\alpha^2(\|y_1+\tilde{x_1}\|_\infty + \|v_1\|_\infty \leq k(2M+1)\alpha^2$$

So that the inequality (101) is proved. $\square$

Proposition 27. *The dynamics of r_1 expands the following way:*

$$\begin{aligned}
\frac{dr_1}{dt} &= [f_{0x}r_1 + f_{0u}v] \\
&+ \int_0^1\int_0^1 \lambda f_{\sigma\sigma}(x_0+\lambda\mu(r_1+\alpha(\tilde{x_1}+y_1)),u_0+\lambda\mu(v+\alpha v_1),t)\begin{bmatrix} r_1+\alpha(\tilde{x_1}+y_1) \\ v+\alpha v_1\end{bmatrix}^2 d\lambda d\mu \\
&-\alpha^2\int_0^1\int_0^1 \lambda f_{\sigma\sigma}(x_0+\lambda\mu\alpha(\tilde{x_1}+y_1),u_0+\lambda\mu\alpha v_1,t)\begin{bmatrix}\tilde{x_1}+y_1 \\ \alpha v_1\end{bmatrix}^2 d\lambda d\mu
\end{aligned} \tag{102}$$

Proof. The dynamics of r r_1 (equation (57)) is the difference between the quantities $f(r_1+x_0+\alpha(x_1+y_1),v+u_0+\alpha v_1,t)$ and $f(x_0+\alpha(x_1+y_1,u_0+\alpha v_1,t)$.

Let's make the Taylor expansions of these quantities at (x_0,u_0):

$$\begin{aligned}
f(r_1+x_0 &+ \alpha(x_1+y_1),v+u_0+\alpha v_1,t) \\
&= f(x_0,u_0,t) \\
&\quad +f_{0x}(r_1+\alpha(x_1+y_1)) + f_{0u}(v+\alpha v_1) \\
&\quad + \int_0^1\int_0^1 \lambda f_{\sigma\sigma}(x_0+\lambda\mu(r_1+\alpha(\tilde{x_1}+y_1)),u_0+\lambda\mu(v+\alpha v_1),t)\begin{bmatrix} r_1+\alpha(\tilde{x_1}+y_1) \\ v+\alpha v_1\end{bmatrix}^2 d\lambda d\mu
\end{aligned}$$

and

$$\begin{aligned}
f(x_0 &+ \alpha(x_1+y_1),u_0+\alpha v_1,t) \\
&= f(x_0,u_0,t) \\
&\quad +\alpha[f_{0x}(x_1+y_1)) + f_{0u}v_1] \\
&\quad +\alpha^2\int_0^1\int_0^1 \lambda f_{\sigma\sigma}(x_0+\lambda\mu\alpha(\tilde{x_1}+y_1),u_0+\lambda\mu\alpha v_1,t)\begin{bmatrix}\tilde{x_1}+y_1 \\ v_1\end{bmatrix}^2 d\lambda d\mu
\end{aligned}$$

The simplifications between the two expansions while we take their differences gives the dynamics of r_1 (102). $\square$

The consequence of that dynamics expansion is that:

$$\alpha\int_0^T \tilde{p_1}\frac{dr_1}{dt}dt = \alpha\int_0^T \tilde{p_1}[f_{0x}r_1 + f_{0u}v]dt$$
$$+\alpha\int_0^T \tilde{p_1}\left[\int_0^1\int_0^1 \lambda f_{\sigma\sigma}(x_0+\lambda\mu(r_1+\alpha(\tilde{x_1}+y_1)), u_0+\lambda\mu(v+\alpha v_1), t)\begin{bmatrix} r_1+\alpha(\tilde{x_1}+y_1) \\ v+\alpha v_1\end{bmatrix}^2 d\lambda d\mu\right]dt$$
$$-\alpha^3\int_0^T \tilde{p_1}\left[\int_0^1\int_0^1 \lambda f_{\sigma\sigma}(x_0+\lambda\mu\alpha(\tilde{x_1}+y_1), u_0+\lambda\mu\alpha v_1, t)\begin{bmatrix} \tilde{x_1}+y_1 \\ v_1\end{bmatrix}^2 d\lambda d\mu\right]dt \tag{103}$$

Proposition 28. *Let's define the terms:*

$$R_1 = \alpha\int_0^T \tilde{p_1}\left[\int_0^1\int_0^1 \lambda f_{\sigma\sigma}(x_0+\lambda\mu(r_1+\alpha(\tilde{x_1}+y_1)), u_0+\lambda\mu(v+\alpha v_1), t)\begin{bmatrix} r_1+\alpha(\tilde{x_1}+y_1) \\ v+\alpha v_1\end{bmatrix}^2 d\lambda d\mu\right]dt$$

and

$$R_2 = \alpha^3\int_0^T \tilde{p_1}\left[\int_0^1\int_0^1 \lambda f_{\sigma\sigma}(x_0+\lambda\mu\alpha(\tilde{x_1}+y_1), u_0+\lambda\mu\alpha v_1, t)\begin{bmatrix} \tilde{x_1}+y_1 \\ v_1\end{bmatrix}^2 d\lambda d\mu\right]dt$$

Then the following inequalities hold:

$$|R_1| \leq \frac{3k}{2}[(k_{r3}T + k_{v1}]\alpha z^2 + [(k_{r4}T + k_{v2} + T(1+2M)^2]\alpha^3 \tag{104}$$

and

$$|R_2| \leq \frac{kT}{2}(1+2M)^2\alpha^3 \tag{105}$$

Proof. Equation (40) about the upper bound of $f_{\sigma\sigma}$ leads to:

$$\begin{aligned}\left|f_{\sigma\sigma}(x_0+\lambda\mu\alpha(\tilde{x_1}+y_1), u_0+\lambda\mu\alpha v_1, t)\begin{bmatrix} r_1+\alpha(\tilde{x_1}+y_1 \\ v+\alpha v_1\end{bmatrix}^2\right| &\leq 3k[|r_1+\alpha(\tilde{x_1}+y_1)|+|v+\alpha v_1|]^2 \\ &\leq 3k[|r_1|+|v|+\alpha(|\tilde{x_1}+y_1|+|v_1|)]^2 \\ &\leq 3k[r_1^2+v^2+\alpha(|\tilde{x_1}+y_1|+|v_1|)^2]\end{aligned}$$

Thus, together with the fact that $\tilde{p_1}\leq 1$, we have:

$$\begin{aligned}|R_1| &\leq \frac{3k\alpha}{2}[\|r_1\|_2^2+\|v\|_2^2+\alpha^2T(2M+1)^2] \\ |R_1| &\leq \frac{3k\alpha}{2}[\|r_1\|_\infty^2+\|v\|_2^2+\alpha^2T(2M+1)^2]\end{aligned}$$

Equations (59) and (56) then lead to equation (104).

Equation (40) about the upper bound of $f_{\sigma\sigma}$ leads also to:

$$\left|f_{\sigma\sigma}(x_0+\lambda\mu\alpha(\tilde{x_1}+y_1), u_0+\lambda\mu\alpha v_1, t)\begin{bmatrix} r_1+\alpha(\tilde{x_1}+y_1 \\ v+\alpha v_1\end{bmatrix}^2\right| \leq k(|\tilde{x_1}+y_1|+|v_1|)^2 \leq k(1+2M)^2$$

That proves equation (105) by triple integration and multiplication by α^3. □

The equations (104) and (105) included in equation (105) (103) lead to:

$$\begin{aligned}\alpha\int_0^T \tilde{p_1}\frac{dr_1}{dt}dt &= \alpha\int_0^T \tilde{p_1}[f_{0x}r_1+f_{0u}v]dt \\ &\quad -\frac{3k}{2}(k_{r2}T+k_{v1})Maz^2 - \frac{k}{2}[3(k_{r4}T+k_{v2})+4T(1+2M)^2]\alpha^3\end{aligned}$$

Introducing that equation into equation (101) leads to equation (64) and thus the proposition 13.

F Proof of the lower bound of the fourth term (Proposition 14)

Proposition 29. *The following inequality holds:*

$$\begin{aligned}\int_0^T\int_0^1\int_0^1 \lambda H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)(\delta\sigma)^2 d\lambda d\mu dt &\geq -2KTM^2\alpha^2+\beta z^2 \\ &+2\alpha\int_0^T\int_0^1\int_0^1 \lambda\,[y_1,v_1]\,H_{\sigma\sigma}(\rho(\lambda,\mu),p_0t)\begin{bmatrix} r+\alpha\tilde{x_1} \\ v\end{bmatrix} d\lambda d\mu dt\end{aligned} \tag{106}$$

Proof. Let's use the fact that:

$$\delta\sigma=\begin{bmatrix} r+\alpha\tilde{x_1} \\ v\end{bmatrix}+\alpha\begin{bmatrix} y_1 \\ v_1\end{bmatrix}$$

to expand $H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)(\delta\sigma)^2$:

$$\begin{aligned}H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)(\delta\sigma)^2 &= H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)\begin{bmatrix} r+\alpha\tilde{x_1} \\ v\end{bmatrix}^2+\alpha^2 H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)\begin{bmatrix} y_1 \\ v_1\end{bmatrix}^2 \\ &+2\alpha[y_1,v_1]H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)\begin{bmatrix} r+\alpha\tilde{x_1} \\ v\end{bmatrix}\end{aligned} \tag{107}$$

The second term is easily upper bounded in absolute value (equation (41) about the upper bound of $H_{\sigma\sigma}$):

$$\left|H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)\begin{bmatrix} y_1 \\ v_1\end{bmatrix}^2\right|\leq K(|y_1|+|v_1|)^2\leq 4KM^2$$

so that:

$$\left|\int_0^T\int_0^1\int_0^1 H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)\begin{bmatrix} y_1 \\ v_1\end{bmatrix}^2\right| d\lambda d\mu dt\leq 2KTM^2 \tag{108}$$

To upper bound the first term, let's expand it in its components:

$$\begin{aligned}H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)\begin{bmatrix} r+\alpha\tilde{x_1} \\ v\end{bmatrix}^2 &= H_{xx}(\rho(\lambda,\mu),p_0,t)(r+\alpha\tilde{x_1})^2+2(r+\alpha\tilde{x_1})H_{xu}(\rho(\lambda,\mu),p_0,t)v \\ &+H_{uu}(\rho(\lambda,\mu),p_0,t)v^2\end{aligned} \tag{109}$$

Now let's use the definition of $Z[\lambda,\mu](t)=v+[H_{uu}^{-1}H_{ux}](\rho(\lambda,\mu),p_0,t)(r+\alpha\tilde{x_1}$ to expand the terms in v and v^2:

$$\begin{aligned}2(r+\alpha\tilde{x_1})H_{xu}(\rho(\lambda,\mu),p_0,t)v &= 2(r+\alpha\tilde{x_1})H_{xu}(\rho(\lambda,\mu),p_0,t)Z[\lambda,\mu] \\ &-2[H_{ux}H_{uu}^{-1}H_{xu}](\rho(\lambda,\mu),p_0,t)(r+\alpha\tilde{x_1})^2\end{aligned} \tag{110}$$

and

$$\begin{aligned}H_{uu}(\rho(\lambda,\mu),p_0,t)v^2 &= H_{uu}(\rho(\lambda,\mu),p_0,t)Z[\lambda,\mu]^2 \\ &+[H_{ux}H_{uu}^{-1}H_{xu}](\rho(\lambda,\mu),p_0,t)(r+\alpha\tilde{x_1})^2 \\ &-2(r+\alpha\tilde{x_1})H_{xu}(\rho(\lambda,\mu),p_0,t)Z[\lambda,\mu]\end{aligned} \tag{111}$$

Introducing equations (110) and (111) in equation (109) leads to:

$$\begin{aligned}H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)\begin{bmatrix} r+\alpha\tilde{x_1} \\ v\end{bmatrix}^2 &= [H_{xx}-H_{ux}H_{uu}^{-1}H_{xu}](\rho(\lambda,\mu),p_0,t)(r+\alpha\tilde{x_1})^2 \\ &+H_{uu}Z[\lambda,\mu]^2\end{aligned}$$

Because of assumption 3 on the convexity, this proves that:

$$H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)\left[\begin{array}{c} r+\alpha\tilde{x_1} \\ v \end{array}\right]^2 \geq \beta Z[\lambda,\mu]^2$$

so that:

$$\int_0^T\int_0^1\int_0^1 H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)\left[\begin{array}{c} r+\alpha\tilde{x_1} \\ v \end{array}\right]^2 d\lambda d\mu dt \geq \beta z^2 \tag{112}$$

Including equations (113) and (108) into equation (107) leads to equation (106). ☐

Proposition 30. *Let's denote:*

$$J_{41} = 2\alpha\int_0^T\int_0^1\int_0^1 \lambda\,[y_1,v_1]\,[H_{\sigma\sigma}(\rho(\lambda,\mu),p_0 t) - H_{\sigma\sigma}(w_0,t)]\left[\begin{array}{c} r+\alpha\tilde{x_1} \\ v \end{array}\right] d\lambda d\mu dt$$

Then the following bound holds:

$$|J_{41}| \leq 4\sqrt{3}KM\alpha\left[\left((Tk_{r1}+\frac{1}{2}k_{v1}\right)z^2 + \left(Tk_{r2}+\frac{1}{2}k_{v2}+\frac{T}{2}(M^2+2)\right)\alpha^2\right] \tag{113}$$

Proof. Let's expand $[y_1,v_1]\,[H_{\sigma\sigma}(\rho(\lambda,\mu),p_0 t)H_{\sigma\sigma}(w_0,t)]\left[\begin{array}{c} r+\alpha\tilde{x_1} \\ v \end{array}\right]$ into its coordinates:

$$\begin{aligned}
[y_1,v_1]\,[H_{\sigma\sigma}(\rho(\lambda,\mu),p_0 t)H_{\sigma\sigma}(w_0,t)]\left[\begin{array}{c} r+\alpha\tilde{x_1} \\ v \end{array}\right] \quad = \quad & y_1[H_{xx}(\rho(\lambda,\mu),p_0 t) - H_{xx}(w_0,t)](r+\alpha\tilde{x_1}) \\
& +y_1[H_{xu}(\rho(\lambda,\mu),p_0 t) - H_{xu}(w_0,t)]v \qquad (114) \\
& +v_1[H_{ux}(\rho(\lambda,\mu),p_0 t) - H_{ux}(w_0,t)](r+\alpha\tilde{x_1}) \\
& +v_1[H_{uu}(\rho(\lambda,\mu),p_0 t) - H_{uu}(w_0,t)]v
\end{aligned}$$

But the second derivatives of $H(x,u,p_0,t)$ are Lipschitz in (x,u) of Lipschitz constant K because the third derivatives of $H(x,u,p_0,t)$ are bounded by K. Thus with the definition of $\rho(\lambda,\mu)$ and w_0, equation (114) leads to:

$$\begin{aligned}
& \left|[y_1,v_1]\ [H_{\sigma\sigma}(\rho(\lambda,\mu),p_0 t)H_{\sigma\sigma}(w_0,t)]\left[\begin{array}{c} r+\alpha\tilde{x_1} \\ v \end{array}\right]\right| \\
& \quad \leq \lambda\mu K\sqrt{(r+\alpha\tilde{x_1})^2+\alpha^2 v_1^2}(|y_1||r+\alpha\tilde{x_1}| + |y_1||v| + |v||r+\alpha\tilde{x_1}| + |v_1||v| \\
& \quad \leq K\sqrt{(r+\alpha)^2+\alpha^2M^2}(2M(|r|+\alpha)+2M|v|) \\
& \quad \leq 2MK\sqrt{2r^2+2\alpha^2+M^2\alpha^2}\sqrt{3(r^2+v^2+\alpha^2)} \\
& \quad \leq 2MK\sqrt{3}\ (2r^2+v^2+(M^2+2)\alpha^2)
\end{aligned}$$

Thus, with a triple integration after multiplication by λ, we get:

$$|J_{41}| \leq 4\sqrt{3}KM\alpha\left[T\|r\|_\infty^2 + \frac{T}{2}\|v\|_2^2 + ((M^2+2)\alpha^2\right]$$

Then, using equations (55) and (56) lead to equation (113). ☐

Applying propositions 29 and 30 lead to:

$$\begin{aligned}
\int_0^T\int_0^1\int_0^1 \lambda H_{\sigma\sigma}(\rho(\lambda,\mu),p_0,t)(\delta\sigma)^2 d\lambda d\mu dt \quad \geq \quad & -2KM\left[TM+\sqrt{3}(2Tk_{r2}+k_{v2}+T(M^2+2))\alpha\right]\alpha^2 \\
& +\left[\beta - 2\sqrt{3}KM(2Tk_{r1}+k_{v1})\alpha\right]z^2 \\
& +2\alpha\int_0^T\int_0^1\int_0^1 \lambda\,[y_1,v_1]\,H_{\sigma\sigma}(w_0,t)\left[\begin{array}{c} r+\alpha\tilde{x_1} \\ v \end{array}\right] d\lambda d\mu dt
\end{aligned}$$

That proves proposition 14 because $H_{0\sigma\sigma} = H_{\sigma\sigma}(w_0,t)$ does not depend on λ and μ.

G Proof of Bound in absolute value for the sum of the integral terms of the right hand sides of equations (64) and (65)

Thanks to the fact that $r + \alpha\tilde{x_1} = r_1 + ((r - r_1) + \alpha\tilde{x_1})$, we have $R = R_3 + R_4$, where R_3 and R_4 are defined as:

$$R_3 = \alpha \int_0^T \tilde{p_1} \left[f_{0x} r_1 + f_{0u} v + [y_1, v_1]\, H_{0\sigma\sigma} \left[\begin{array}{c} r_1 \\ v \end{array} \right] \right] dt$$

and

$$R_4 = \alpha \int_0^T [y_1, v_1]\, H_{0\sigma\sigma} \left[\begin{array}{c} r_1 + \alpha\tilde{x_1} \\ 0 \end{array} \right] dt$$

Proposition 31. *The following inequality holds:*

$$|R_3| \leq kM \left((Tk_{r3} + k_{v1})\alpha z^2 + (Tk_{r4} + k_{v2})\alpha^3 \right) \tag{115}$$

Proof. Let's develop $[y_1, v_1]\, H_{0\sigma\sigma} \left[\begin{array}{c} r_1 \\ v \end{array} \right]$ into coordinates:

$$[y_1, v_1]\, H_{0\sigma\sigma} \left[\begin{array}{c} r_1 \\ v \end{array} \right] = H_{0xx} y_1 r_1 + H_{0ux} y_1 v + H_{0xu} v_1 r_1 + H_{0uu} v_1 v$$

so that:

$$\begin{aligned} &\tilde{p_1} \left[f_{0x} r_1 + f_{0u} v + [y_1, v_1]\, H_{0\sigma\sigma} \left[\begin{array}{c} r_1 \\ v \end{array} \right] \right] \\ &\quad = (f_{0x}\tilde{p_1} + H_{0xx} y_1 + H_{0xu} v_1) r_1 + (f_{0u}\tilde{p_1} + H_{0ux} y_1 + H_{0uu} v_1) v \end{aligned}$$

Now let's use the costate dynamics (54) of the auxiliary problem, together with its stationarity condition (53). Then we get:

$$\tilde{p_1} \left[f_{0x} r_1 + f_{0u} v + [y_1, v_1]\, H_{0\sigma\sigma} \left[\begin{array}{c} r_1 \\ v \end{array} \right] \right] = -\frac{dq_1}{dt} r_1 - q_1 \left[f_{0x} r_1 + f_{0u} v \right]$$

An integration by parts, together with the fact that $r_1(0) = q_1(T) = 0$ leads to:

$$R_3 = \alpha \int_0^T q_1 \left(\frac{dr_1}{dt} - [f_{0x} r_1 + f_{0u} v] \right)$$

But r_1 follows the dynamics (57), so that a Taylor expansion of $f(r_1 + x_0 + \alpha(\tilde{x_1} + y_1), v + u_0 + \alpha v_1, t)$ at $(x_0 + \alpha(\tilde{x_1} + y_1), u_0 + \alpha v_1)$ leads to:

$$\left(\frac{dr_1}{dt} - [f_{0x} r_1 + f_{0u} v] \right) = \int_0^1 \int_0^1 \lambda f_{\sigma\sigma}(x_0 + \alpha(\tilde{x_1} + y_1) + \lambda\mu r_1, u_0 + \alpha v_1 + \lambda\mu v, t) \left[\begin{array}{c} r_1 \\ v \end{array} \right]^2 d\lambda d\mu$$

so that, thanks to equation (40) about the bound of $f_{\sigma\sigma}$:

$$\left| q_1 \left(\frac{dr_1}{dt} - [f_{0x} r_1 + f_{0u} v] \right) \right| \leq \frac{1}{2} kM (|r_1| + |v|)^2 \leq kM (r_1^2 + v^2)$$

An integration on $[0, T]$ and a multiplication by α leads to

$$|R_3| \leq kM (T\|r_1\|_\infty^2 + \|v\|_2^2)$$

That equation, together with the bounds on r_1 (59) and v (56) lead to equation (115). □

Proposition 32. *The following inequality holds:*

$$|R_4| \leq 2KTM(1 + k_{r5}\alpha)\alpha^2 \tag{116}$$

Proof. Let's develop $[y_1, v_1]\, H_{0\sigma\sigma} \begin{bmatrix} r_1 + \alpha\tilde{x_1} \\ 0 \end{bmatrix}$ into coordinates:

$$[y_1, v_1]\, H_{0\sigma\sigma} \begin{bmatrix} r_1 + \alpha\tilde{x_1} \\ 0 \end{bmatrix} = (H_{0xx}y_1 + H_{0uu}v_1)\,((r - r_1) + \alpha\tilde{x_1})$$

Thus, thanks to the bound on $r - r_1$ (58), its absolute value can be bounded:

$$\left|[y_1, v_1]\, H_{0\sigma\sigma} \begin{bmatrix} r_1 + \alpha\tilde{x_1} \\ 0 \end{bmatrix}\right| \leq 2KM(\|r - r_1\|_\infty + \alpha) \leq 2KM(k_{r5}\alpha^2 + \alpha) \leq 2KM(1 + k_{r5}\alpha)\alpha$$

An integration between 0 and T and a multiplication by α lead to equation (116). $\square$

Proof of equation (66) This is a consequence of equations (115) and (116), together with the fact that $R = R_3 + R_4$.

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