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Variational Multiscale error estimator for anisotropic adaptive fluid mechanic simulations: application to Navier-Stokes problems

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Abstract

Keywords: CFD; VMS; Error Estimator; Mesh Adaptation; Navier-Stokes.

Contents

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1. Introduction

2. The incompressible Navier-Stokes equations in the Variational Multi-Scale framework

2.1. The incompressible Navier-Stokes equations

To fix the notations, let $\Omega \subset \mathbb{R}^d$ be the fluid domain, where d is the space dimension, and $\partial\Omega$ its boundary. The strong form of the incompressible Navier Stokes equations reads:

$$\begin{cases} \rho\,(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) - \nabla\cdot\boldsymbol{\sigma} = \mathbf{f} \\ \nabla\cdot\mathbf{v} = 0 \end{cases} \tag{1}$$

where $t \in [0, T]$ is the time, $\mathbf{v}(\mathbf{x}, t)$ the velocity, $p(\mathbf{x}, t)$ the pressure and ρ the density. The Cauchy stress tensor for a Newtonian fluid is given by:

$$\boldsymbol{\sigma} = 2\mu\,\boldsymbol{\varepsilon}(\mathbf{v}) - p\,\mathbf{I}_d, \tag{2}$$

with $\mathbf{I}_d$ the d-dimensional identity tensor and μ the dynamic viscosity. In order to close the problem, Eq. (1) are subject to the homogeneous Dirichlet boundary conditions.

The weak form of problem (1) combined with (2) is obtained by multiplication of a test function and integration by parts. Let $H^1(\Omega)$ be the Sobolev space of square integrable functions whose distributional derivatives are square integrable, and let $V \subset [H^1(\Omega)]^d$ be a functional space properly chosen according to the boundary conditions. Finally, let $Q =$

$\left\{q \in L^2(\Omega) : \int_\Omega q = 0\right\}$. By denoting $(\,\cdot\,,\,\cdot\,)$ the scalar product of the space $L^2(\Omega)$, the weak form of problem (1) on $\partial\Omega$ reads:

$$\left\{\begin{array}{l} \text{Find } (\mathbf{v}, p) \in V \times Q \text{ such that:} \\ \rho \; [(\partial_t \mathbf{v}, \mathbf{w}) + (\mathbf{v}\cdot\nabla\mathbf{v}, \mathbf{w})] + (2\mu\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w})) - (p, \nabla\cdot\mathbf{w}) = (\mathbf{f}, \mathbf{w}), \quad \forall \mathbf{w} \in V \\ (\nabla\cdot\mathbf{v}, q) = 0, \quad \forall q \in Q. \end{array}\right. \tag{3}$$

where ρ and μ are the density and the dynamic viscosity, respectively.

The standard Galerkin approximation consists in decomposing the domain Ω into N_{el} elements K such that they cover the domain. Therefore, the elements are either disjoint or share a complete edge (or face in 3D). Using a partition $\mathcal{T}_h$, the above-defined functional spaces V and Q are approached by finite dimensional spaces V_h and Q_h such that:

$$V_h = \{\mathbf{v}_h | \mathbf{v}_h \in C^0(\Omega)^n, \mathbf{v}_{h|K} \in P^1(K)^n,\ \forall K \in \mathcal{T}_h\} \tag{4}$$

$$Q_h = \{p_h | p_h \in C^0(\Omega)^n, p_{h|K} \in P^1(K)^n,\ \forall K \in \mathcal{T}_h\} \tag{5}$$

The Galerkin discrete problem consists therefore in solving the following mixed problem:

$$\left\{\begin{array}{l} \text{Find } (\mathbf{v}_h, p_h) \in V_h \times Q_h \text{ such that:} \\ \rho \; [(\partial_t \mathbf{v}_h, \mathbf{w}_h) + (\mathbf{v}_h\cdot\nabla\mathbf{v}_h, \mathbf{w}_h)] + (2\mu\boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_h)) - (p_h, \nabla\cdot\mathbf{w}_h) = (\mathbf{f}, \mathbf{w}_h), \quad \forall \mathbf{w}_h \in V_h \\ (\nabla\cdot\mathbf{v}_h, q_h) = 0, \quad \forall q_h \in Q_h. \end{array}\right. \tag{6}$$

It is well known that the stability of the semi-discrete formulation requires an appropriate choice of the finite element spaces V_h and Q_h that must fulfill a compatibility condition [1]. Accordingly, the standard Galerkin method using P1/P1 elements (i.e. the same piecewise linear space for V_h and Q_h) is not stable. Moreover, convection-dominant problems (i.e. problems where the convection term $\mathbf{v}\cdot\nabla\mathbf{v}$ is much larger than the diffusion term $\nabla\cdot\,(2\mu\;\boldsymbol{\varepsilon})$) also lead to a loss of coercivity in the formulation (3). This phenomenon manifests itself as oscillations that pollute the solution.

2.2. The Variational Multi-Scale formulation

In this work, we use a Variational MultiScale method[2] which circumvents both the previously stated problems through a Petrov-Galerkin approach. The basic idea is to consider that the unknowns can be split into two components, a coarse one and a fine one, corresponding to different scales or levels of resolution. First, we solve the fine scales in an approximate manner and then we replace their effect into the large-scale equation. We present here only an outline of the method, and the reader is referred to[3] for extensive details about the formulation.

2.2.1. Basic principles of the multiscale approach

Let us split the velocity and the pressure fields into resolvable coarse-scale and unresolved fine-scale components: $\mathbf{v} = \mathbf{v}_h + \mathbf{v}'$ and $p = p_h + p'$. The same decomposition can be applied to the weighting functions: $\mathbf{w} = \mathbf{w}_h + \mathbf{w}'$ and $q = q_h + q'$. Subscript h is used hereafter to denote the finite element (coarse) component, whereas the prime is used for the so called subgrid scale (fine) component of the unknowns. The enrichment of the functional spaces is performed as follows: $V = V_h \oplus V'$, $V_0 = V_{h,0} \oplus V_0'$ and $Q = Q_h \oplus Q'$. Thus, the finite element approximation for the time-dependent Navier-Stokes problem reads:

$$\left\{ \begin{array}{l} \text{Find}(\mathbf{v}, p) \in V \times Q \text{ such that:} \\ \rho \left(\partial_t(\mathbf{v}_h + \mathbf{v}'), (\mathbf{w}_h + \mathbf{w}') \right)_\Omega + \rho \left((\mathbf{v}_h + \mathbf{v}') \cdot \nabla (\mathbf{v}_h + \mathbf{v}'), (\mathbf{w}_h + \mathbf{w}') \right)_\Omega + (2\mu \boldsymbol{\varepsilon}(\mathbf{v}_h + \mathbf{v}') : \boldsymbol{\varepsilon}(\mathbf{w}_h + \mathbf{w} \\ \qquad\qquad - \left((p_h + p'), \nabla \cdot (\mathbf{w}_h + \mathbf{w}') \right)_\Omega = \left(\mathbf{f}, (\mathbf{w}_h + \mathbf{w}') \right)_\Omega , \quad \forall \mathbf{w} \in V_0 \\ \left(\nabla \cdot (\mathbf{v}_h + \mathbf{v}'), (q_h + q') \right)_\Omega = 0, \quad \forall q \in Q. \end{array} \right. \tag{7}$$

To derive the stabilized formulation, we split Eq. (7) into a large-scale and a fine-scale problem. Integrating by parts within each element, we obtain the so-called coarse-scale problem:

$$\left\{ \begin{array}{l} \rho \left(\partial_t(\mathbf{v}_h + \mathbf{v}'), \mathbf{w}_h \right)_\Omega + \rho \left((\mathbf{v}_h + \mathbf{v}') \cdot \nabla (\mathbf{v}_h + \mathbf{v}'), \mathbf{w}_h \right)_\Omega + \left(2\mu \boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_h) \right)_\Omega \\ \qquad\qquad - \left((p_h + p'), \nabla \cdot \mathbf{w}_h \right)_\Omega = (\mathbf{f}, \mathbf{w}_h)_\Omega , \quad \forall \mathbf{w}_h \in V_{h,0} \\ \left(\nabla \cdot (\mathbf{v}_h + \mathbf{v}'), q_h \right)_\Omega = 0, \quad \forall q_h \in Q_h. \end{array} \right. \tag{8}$$

and the so-called fine-scale problem:

$$\left\{ \begin{array}{l} \rho \left(\partial_t(\mathbf{v}_h + \mathbf{v}'), \mathbf{w}' \right)_K + \rho \left((\mathbf{v}_h + \mathbf{v}') \cdot \nabla (\mathbf{v}_h + \mathbf{v}'), \mathbf{w}' \right)_K + \left(2\mu \boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{w}') \right)_K \\ \qquad\qquad - \left((p_h + p'), \nabla \cdot \mathbf{w}' \right)_\Omega = (\mathbf{f}, \mathbf{w}')_\Omega , \quad \forall \mathbf{w}' \in V_0' \\ \left(\nabla \cdot (\mathbf{v}_h + \mathbf{v}'), q_h \right)_\Omega = 0, \quad \forall q' \in Q'. \end{array} \right. \tag{9}$$

where $(\cdot\,,\,\cdot)_\Omega$ represents the scalar product on the whole domain while $(\cdot\,,\,\cdot)_K$ is the scalar product on element K.

To derive our stabilized formulation, we first solve the fine scale problem (9), defined on the sum of element interiors and written in terms of the time-dependent large scale variables. Then we substitute the fine scale solution back into the coarse problem (8), thereby *eliminating appearance of the fine-scale while still modelling their effects.* As in [3]; we recall here 3 important remarks/assumptions that have to be made:

- by considering the small scale velocity as bubble functions vanishing on the boundaries of the element, terms involving integrals over the element interior boundaries will be neglected,
- we neglect the second derivatives of the weighting function in the momentum residuals of (9),
- as the fine-scale space is assumed to be H^1-orthogonal to the finite element space, crossed viscous terms vanish in (8) and (9).

2.2.2. The fine scale sub-problem

Under several assumptions about the time-dependency and the non-linearity of the momentum equation of the sub-scale system detailed in [3], the fine-scale solutions $\mathbf{v}'$ and p' can be written in terms of the time-dependent large-scale variables using residual-based terms that are derived consistently. For all $K \in \mathcal{T}_h$, we have:

$$\begin{aligned} &\mathbf{v}'|_K = \tau_K \boldsymbol{\mathcal{R}}_{\mathrm{M}} \\ &p' \approx \tau_{\mathrm{C}} \boldsymbol{\mathcal{R}}_C \end{aligned} \tag{10}$$

where the momentum residual $\boldsymbol{\mathcal{R}}_{\mathrm{M}}$ and the continuity residual $\mathcal{R}_{\mathrm{C}}$ are expressed as:

$$\begin{aligned} &\boldsymbol{\mathcal{R}}_{\mathrm{M}} = \mathbf{f} - \rho\partial_t \mathbf{v}_h - \rho \mathbf{v}_h \cdot \nabla \mathbf{v}_h - \nabla p_h \\ &\mathcal{R}_{\mathrm{C}} = -\nabla \cdot \mathbf{v}_h \end{aligned} \tag{11}$$

In this work, we adopt the definition proposed by Codina in [4] for the stabilizing coefficient:

$$\tau_K = \left[\left(\frac{2\rho\|\mathbf{v}_h\|_K}{h_K} \right)^2 + \left(\frac{4\mu}{h_K^2} \right)^2 \right]^{-\frac{1}{2}}, \tag{12}$$

$$\tau_{\mathrm{C}} = \left[\left(\frac{\mu}{\rho} \right)^2 + \left(\frac{c_2\|\mathbf{v}_h\|_K}{c_1 h_K} \right)^2 \right]^{\frac{1}{2}} \tag{13}$$

where h_K is the characteristic length of the element and c_1 and c_2 are algorithmic constants. We take them as $c_1 = 4$ and $c_2 = 2$ for linear elements. $\|\mathbf{v}_h\|_K$ is the coarse scale velocity norm on the element, defined by:

$$\|\mathbf{v}_h\|_K = \sqrt{\mathbf{v}_{x,h}^2 + \mathbf{v}_{y,h}^2} \tag{14}$$

2.2.3. The coarse scale sub-problem

Let us consider the coarse scale problem (8). Taking into account the assumptions prescribed in [3] and recalled in Section 2.2.1 for the fine scale fields, the large-scale system becomes:

$$\left\{ \begin{array}{l} \rho\left(\partial_t \mathbf{v}_h, \mathbf{w}_h\right)_\Omega + \left(\rho\mathbf{v}_h \cdot \nabla\mathbf{v}_h, \mathbf{w}_h\right)_\Omega + \left(\rho\mathbf{v}_h \cdot \nabla\mathbf{v}', \mathbf{w}_h\right)_\Omega + \left(2\mu\boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_h)\right)_\Omega \\ -\left(p_h, \nabla\cdot\mathbf{w}_h\right)_\Omega - \left(p', \nabla\cdot\mathbf{w}_h\right)_\Omega = \left(\mathbf{f}, \mathbf{w}_h\right)_\Omega, \quad \forall \mathbf{w}_h \in V_{h,0} \\ \left(\nabla\cdot\mathbf{v}_h, q_h\right)_\Omega + \left(\nabla\cdot\mathbf{v}', q_h\right)_\Omega = 0, \quad \forall q_h \in Q_h. \end{array} \right. \tag{15}$$

Then, integrating by parts the third term in the first equation and the second term in the second equation of (15) and substituting the expressions of both the fine-scale pressure and the fine-scale velocity of (10), the large-scale system reads:

$$\left\{ \begin{array}{l} \rho\left(\partial_t \mathbf{v}_h, \mathbf{w}_h\right)_\Omega + \left(\rho\mathbf{v}_h \cdot \nabla\mathbf{v}_h, \mathbf{w}_h\right)_\Omega - \sum_{K\in\mathcal{T}_h} \left(\tau_K \boldsymbol{\mathcal{R}}_{\mathrm{M}}, \rho\mathbf{v}_h\nabla\mathbf{w}_h\right)_K + \left(2\mu\boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_h)\right)_\Omega \\ \qquad -\left(p_h, \nabla\cdot\mathbf{w}_h\right)_\Omega - \sum_{K\in\mathcal{T}_h} \left(\tau_{\mathrm{C}}\mathcal{R}_{\mathrm{C}}, \nabla\cdot\mathbf{w}_h\right)_K = \left(\mathbf{f}, \mathbf{w}_h\right)_\Omega, \quad \forall \mathbf{w}_h \in V_{h,0} \\ \left(\nabla\cdot\mathbf{v}_h, q_h\right)_\Omega - \sum_{K\in\mathcal{T}_h} \left(\tau_K \boldsymbol{\mathcal{R}}_{\mathrm{M}}, \nabla q_h\right)_K = 0, \quad \forall q_h \in Q_h \end{array} \right. \tag{16}$$

Finally, substituing the residuals of the momentum equation and developping all the additional terms, we obtain a modified coarse scale formulation expressed exclusively in terms of coarse scale variables. The new modified problem for linear tetrahedral elements can now be decomposed into four main term: the first one is the standard Galerkin contribution, the second and the third terms take into account the influence of the fine-scale velocity on the finite element components and the last term models the influence of the fine-scale pressure onto the large-scale problem. We finally get:

$$
\begin{aligned}
&\rho\left(\partial_t \mathbf{v}_h + \mathbf{v}_h.\nabla \mathbf{v}_h, \mathbf{w}_h\right)_\Omega + \left(2\mu\boldsymbol{\varepsilon}(\mathbf{v}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_h)\right)_\Omega - \left(p_h, \nabla.\mathbf{w}_h\right)_\Omega + \left(\nabla.\mathbf{v}_h, q_h\right)_\Omega - \left(\mathbf{f}, \mathbf{w}_h\right)_\Omega \\
&+ \sum_{K\in\mathcal{T}_h} \tau_K \left(\rho(\partial_t \mathbf{v}_h + \mathbf{v}_h.\nabla \mathbf{v}_h) + \nabla p_h - \mathbf{f}, \rho \mathbf{v}_h \nabla \mathbf{w}_h\right)_K \\
&+ \sum_{K\in\mathcal{T}_h} \tau_K \left(\rho(\partial_t \mathbf{v}_h + \mathbf{v}_h.\nabla \mathbf{v}_h) + \nabla p_h - \mathbf{f}, \nabla q_h\right)_K \\
&+ \sum_{K\in\mathcal{T}_h} \left(\tau_{\mathrm{C}} \nabla\cdot\mathbf{v}_h, \nabla\cdot\mathbf{w}_h\right)_K = 0 \quad \forall \mathbf{w}_h \in V_{h,0}\ , \quad \forall q_h \in Q_h
\end{aligned}
\tag{17}
$$

Compared to the standard Galerkin method, the proposed stable formulation involves additional integrals that are evaluated element-wise. These additional terms represent the stabilizing effect of the sub-grid scales and are introduced in a consistent way in the Galerkin formulation. They make it possible to avoid instabilities caused by both dominant convection terms and incompatible approximation spaces.

3. A posteriori error estimation on solution's subscales

In this paper, the first objective is to compute an a posteriori subscales error estimator for the incompressible Navier-Stokes equation. To do so, we will use two different approach. The first one is based on the computation of the stabilizing parameters proposed in the previous section. The second one however, is based on the building of high order bubble functions on the element.

3.1. Computation of the error estimator with the stabilizing parameters approach

The stabilizing parameter approach for the a posteriori error estimator computation consists in using the assumptions made in Section 2.2.2. In fact, the application of the VMS approach leads to an approximation of the sub-grid variables $\mathbf{v}'$ and p' using an explicit expression taking into account the residuals and the stabilizing parameters (see Eq. 10). Therefore, it is possible to compute an element-wise expression of the sub-grid variables.

Considering $\mathbf{v}'$, for all $K \in \mathcal{T}_h$, we have:

$$\begin{aligned}
&\mathbf{v}'|_K \approx \tau_K \boldsymbol{\mathcal{R}}_{\mathrm{M}} \\
&\mathbf{v}'|_K \approx \tau_K \times (\mathbf{f} - \rho\partial_t\mathbf{v}_h - \rho\mathbf{v}_h \cdot \nabla\mathbf{v}_h - \nabla p_h) \\
&\mathbf{v}'|_K \approx \left[\left(\tfrac{2\rho\|\mathbf{v}_h\|_K}{h_K}\right)^2 + \left(\tfrac{4\mu}{h_K^2}\right)^2\right]^{-\frac{1}{2}} \times \ (\mathbf{f} - \rho\partial_t\mathbf{v}_h - \rho\mathbf{v}_h \cdot \nabla\mathbf{v}_h - \nabla p_h)
\end{aligned} \tag{18}$$

For here, we can compute 3 different type of norms for $\mathbf{v}'|_K$. We denote these 3 norms L^r with $r = 1, 2, ..., \infty$. In practice, as referred in [5], it is recommend to take $r = 1, 2$. We thus get:

$$\begin{aligned}
&\|\mathbf{v}'\|_{L^r(K)} \approx \left\| \ \left[\left(\tfrac{2\rho\|\mathbf{v}_h\|_K}{h_K}\right)^2 + \left(\tfrac{4\mu}{h_K^2}\right)^2\right]^{-\frac{1}{2}} \times \ (\mathbf{f} - \rho\partial_t\mathbf{v}_h - \rho\mathbf{v}_h \cdot \nabla\mathbf{v}_h - \nabla p_h) \ \right\|_{L^r(K)} \\
&\|\mathbf{v}'\|_{L^r(K)} \approx \left[\left(\tfrac{2\rho\|\mathbf{v}_h\|_K}{h_K}\right)^2 + \left(\tfrac{4\mu}{h_K^2}\right)^2\right]^{-\frac{1}{2}} \times \ \| \ (\mathbf{f} - \rho\partial_t\mathbf{v}_h - \rho\mathbf{v}_h \cdot \nabla\mathbf{v}_h - \nabla p_h) \ \|_{L^r(K)}
\end{aligned} \tag{19}$$

If we take, for example $r = 2$, we get:

$$\|\mathbf{v}'\|_{L^r(K)} \approx \left[\left(\frac{2\rho\|\mathbf{v}_h\|_K}{h_K}\right)^2 + \left(\frac{4\mu}{h_K^2}\right)^2\right]^{-\frac{1}{2}} \times \sqrt{|K|} \times \sqrt{\sum_{1 \leqslant i \leqslant N_{interp}} \|(\mathbf{f} - \rho\partial_t\mathbf{v}_h - \rho\mathbf{v}_h \cdot \nabla\mathbf{v}_h - \nabla p_h)(\mathbf{x}_i} \tag{20}$$

where $(\mathbf{x}_i)_{1 \leqslant i \leqslant N_{interp}}$ are the interpolation points of the element and N_{interp} is the number of interpolation points.

In the same way, considering p' we have:

$$\begin{aligned}
&p' \approx \tau_{\mathrm{C}} \boldsymbol{\mathcal{R}}_C \\
&p' \approx \tau_{\mathrm{C}} \times (-\nabla \cdot \mathbf{v}_h) \\
&p' \approx \left[\left(\tfrac{\mu}{\rho}\right)^2 + \left(\tfrac{c_2\|\mathbf{v}_h\|_K}{c_1 h_K}\right)^2\right]^{\frac{1}{2}} \times (-\nabla \cdot \mathbf{v}_h)
\end{aligned} \tag{21}$$

Taking the L^r norm, we get:

$$\begin{aligned}
&\|p'\|_{L^r(K)} \approx \left\| \left[\left(\tfrac{\mu}{\rho}\right)^2 + \left(\tfrac{c_2\|\mathbf{v}_h\|_K}{c_1 h_K}\right)^2\right]^{\frac{1}{2}} \times (-\nabla \cdot \mathbf{v}_h) \right\|_{L^r(K)} \\
&\|p'\|_{L^r(K)} \approx \left[\left(\tfrac{\mu}{\rho}\right)^2 + \left(\tfrac{c_2\|\mathbf{v}_h\|_K}{c_1 h_K}\right)^2\right]^{\frac{1}{2}} \times \|(-\nabla \cdot \mathbf{v}_h)\|_{L^r(K)}
\end{aligned} \tag{22}$$

In the same way, taking for example $r = 2$, we get:

$$\|p'\|_{L^r(K)} \approx \left[\left(\frac{\mu}{\rho}\right)^2 + \left(\frac{c_2\|\mathbf{v}_h\|_K}{c_1 h_K}\right)^2\right]^{\frac{1}{2}} \times \sqrt{|K|} \times \sqrt{\sum_{1\leqslant i\leqslant N_{interp}} \|(-\nabla\cdot\mathbf{v}_h)(\mathbf{x}_i)\|^2} \tag{23}$$

where $(\mathbf{x}_i)_{1\leqslant i\leqslant N_{interp}}$ are the interpolation points of the element and N_{interp} is the number of interpolation points.

Finally, we obtain two subscales error estimators $\|\mathbf{v}'\|_{L^r(K)}\|$ and $p'\|_{L^r(K)}$ in the L^r norm for the subscales variables $\mathbf{v}'$ and p' respectively. These error estimators are computed thanks to the stabilizing parameters τ_K and τ_{C}. Furthermore, they are both computed *element-wise* and can be used as such in mesh adaptation.

3.2. Computation of the error estimator with the high order bubbles functions approach

Another way to compute the a posteriori error estimates for the incompressible Navier-Stokes equation is by using the error analysis developed by Hauke et al. in [5]. In this paper, Hauke et al. derive three different error estimators: the standard, the naive, and the upper bound. It is applied to multidimensional linear systems and in particular, to the incompressible Navier-Stokes equations. Therefore we will adopt the notation taking $\mathbf{Y} : \Omega \longrightarrow \mathbb{R}^{n_{eq}}$ the solution vector with n_{eq} the number of equations of the system (which coincides with the number of unknowns). In our case, in a 3D computation, we have:

$$\mathbf{Y}(\mathbf{x}) = \begin{pmatrix} v_x(\mathbf{x}) \\ v_y(\mathbf{x}) \\ v_z(\mathbf{x}) \\ p(\mathbf{x}) \end{pmatrix} \tag{24}$$

Using the variational multiscale method, as before, we introduce a sum decomposition of the exact solution $\mathbf{Y}(\mathbf{x})$ into the finite element solution (coarse scale) $\mathbf{Y}_h(\mathbf{x})$ and the error (or subscales) $\mathbf{Y}'(\mathbf{x})$. We do likewise for the weighting function. We have:

$$\mathbf{Y}(\mathbf{x}) = \mathbf{Y}_h(\mathbf{x}) + \mathbf{Y}'(\mathbf{x}) \tag{25}$$

Then, the method consists first in the splitting of the subscales error into two component stemming from the element interior residuals and the boundary element residuals, namely:

$$\mathbf{Y}'(\mathbf{x}) = \mathbf{Y}'_{int}(\mathbf{x}) + \mathbf{Y}'_{bnd}(\mathbf{x}) \tag{26}$$

Using the triangle inequality, we can write for each component [5]:

$$\|\mathbf{Y}'_i(\mathbf{x})\| \leqslant \|\mathbf{Y}'_{i,\ int}(\mathbf{x})\| + \|\mathbf{Y}'_{i,\ bnd}(\mathbf{x})\| \tag{27}$$

Using the same assumptions than in Section 2.2.1 for highly convective regimes, and considering the small scale variables as bubble functions vanishing on the element boundaries, we can write:

$$\|\mathbf{Y}'_{i,\ bnd}(\mathbf{x})\| \approx 0 \tag{28}$$

Therefore, we have:

$$\|\mathbf{Y}'_i(\mathbf{x})\| \leqslant \|\mathbf{Y}'_{i,\ int}(\mathbf{x})\| \tag{29}$$

As referred in [5] the naive approach concerning the a posteriori error estimates gives local efficiencies close to unity. Therefore, we choose this approach to estimate the error. According to Hauke et al., we have:

$$\|\mathbf{Y}'_{i,\ int}(\mathbf{x})\| \approx meas(K)^{1/r} \sum_j {\tau^{\mathbf{e}}}_{L_r,ij}\ \|\boldsymbol{\mathcal{R}}_j(\mathbf{x})\|_{L^\infty(K)} \tag{30}$$

with:

$$\boldsymbol{\mathcal{R}}(\mathbf{x}) = \begin{pmatrix} \boldsymbol{\mathcal{R}}_{\mathrm{M},x}(\mathbf{x}) \\ \boldsymbol{\mathcal{R}}_{\mathrm{M},y}(\mathbf{x}) \\ \boldsymbol{\mathcal{R}}_{\mathrm{M},z}(\mathbf{x}) \\ \boldsymbol{\mathcal{R}}_{C}(\mathbf{x}) \end{pmatrix} \tag{31}$$

and:

$${\tau^{\mathbf{e}}}_{L_r} = \frac{1}{meas(K)^{1/r}}\ \|\mathbf{B}^e_0(\mathbf{x})\|_{L_r(K)} \tag{32}$$

where $\mathbf{B}^e_0$ is a matrix residual-free bubble defined in [**? ? ?**].

For example, if we take $i = x$, we get:

$$\|v'_{x,\ int}(\mathbf{x})\| \approx meas(K)^{1/r} \times \big({\tau^{\mathbf{e}}}_{L_r,xx}\ \|\boldsymbol{\mathcal{R}}_{\mathrm{M},x}(\mathbf{x})\|_{L^\infty(K)} + {\tau^{\mathbf{e}}}_{L_r,xy}\ \|\boldsymbol{\mathcal{R}}_{\mathrm{M},y}(\mathbf{x})\|_{L^\infty(K)} + {\tau^{\mathbf{e}}}_{L_r,xz}\ \|\boldsymbol{\mathcal{R}}_{\mathrm{M},z}(\mathbf{x} \tag{33}$$

4. Mesh adaptation with the subscales error estimator

4.1. Principles of anisotropic mesh adaptation

4.2. Isotropic mesh adaptation with the subscales error estimator

4.3. Combination of subscales error estimator with anisotropic mesh adaptation

5. Numerical examples

5.1. Case 1: Driven flow cavity problem in 2D

5.2. Case 2: Driven flow cavity problem in 3D

6. Conclusions

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