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FINITE DEFORMATION CONSTITUTIVE RELATIONS INCLUDING DUCTILE FRACTURE DAMAGE

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Constitutive relations are developed for finite deformation of plastically dilatant materials. These relations, which model the ductile fracture of metals, are derived from the macroscopic viewpoint, in the framework of generalized standard materials. An exponential dependence of ductile fracture damage on stress triaxiality is demonstrated and the occurrence of material instability, $\dot{\sigma}\dot{\varepsilon}^{P} < 0$, is shown by various examples. In finite element applications to cracked specimens, stable crack growth takes place naturally by localization of deformation without it being necessary to postulate a local fracture criterion nor to release the nodes.

1. Introduction

The examination of a crack tip or the minimum section of a tensile specimen points out that ductile fracture in metals involves considerable damage, via the nucleation and growth of voids, which should be taken into account in the mathematical modelling in order to predict the conditions at fracture and also to characterize the stress–strain relations before the ultimate stage of damage. A damage function may be used with usual stress-strain relations to define the conditions at fracture, but this is only an approximation, a reasonable one for fatigue and creep damage, but not for ductile fracture. Usual theories of plasticity imply plastic incompressibility, which is inconsistent with the dilatancy evident in ductile fracture mechanisms.

For the construction of a new plasticity theory including ductile fracture damage, which reduces to usual theories of plasticity if damage is negligeable, two approaches may be taken: microscopic and macroscopic.

In the first approach, taken by Gurson (1977), the macroscopic constitutive relations are constructed from microscopic components of material: matrix, particles, voids, and their individual and interactive behavior. The transition from nonhomogeneous microscopic to homogeneous macroscopic material is the main difficulty here, requiring a number of simplifying assumptions. Moreover the microscopic components are not well known. McClintock (1968), Rice and Tracey (1969) have developed models of the growth of voids, that give an exponential dependence on stress triaxiality, σ_m/σ_0, where $\sigma_m = \sigma_{kk}/3$ is the hydrostatic tension and σ_0 the yield stress. The growth rate of a single spherical void in a rigid-plastic material is approximated by Rice and Tracey as,

$$\frac{\dot{R}_0}{R_0} = 0.283\dot{\varepsilon}^{\mathrm{P}}_{\mathrm{eq}} \exp\left(\frac{3\sigma_m}{2\sigma_0}\right), \tag{1}$$

where R_0 is the void radius and $\dot{\varepsilon}^{\mathrm{P}}_{\mathrm{eq}}$ the remote equivalent strain rate. But as far as we know, there are no acknowledged models of void interaction and coalescence, that apply to the last stages of the ductile fracture of metal.

The second approach, used here, is macroscopic. It requires no description of micromechanisms and will be considered applicable to the whole process of damage, on condition that the predictions of the model are consistent with the current understanding of ductile fracture. The theory is motivated by the microscopic models but is not deduced from them.

We suppose that the thermodynamical state of the material, the hardening and also damage, are characterized by internal variables in the framework of generalized standard materials (GSM), i.e. existence of a quasi-potential of dissipation and normality rule for the plastic strain rate *and* the internal variables rates, as proposed by Nguyen (1973). The characterization of damage by internal variables is a natural consequence of the choice of GSM. The same hypothesis was formulated quite independently by Kachanov (1958) and Rabotnov (1968) for creep damage and developed recently by Lemaitre and Chaboche (1978), who establish creep, fatigue, and creep-fatigue cumulation models.

Ductile rupture is generally preceded by large plastic deformations Lautridou and Pineau (1978), by recrystallization technique, have measured a strain of 100% at the tip of a blunted crack in A508 Cl 3 steel. Hence constitutive relations are established for *finite deformations*.

First an account of GSM is given for finite deformation. Then application to ductile fracture damage is performed. Finally, the initiation and stable growth of a crack in a bend specimen are analyzed by a finite element model which takes into account some finite deformation effects.

2. Generalized standard materials (GSM) in finite deformation

Consider the configurations (o) of a macroscopic element of material at time 0, in an unstressed state, and (a) at time t, under stress σ. Introduce (Mandel, 1973) an actual intermediate relaxed configuration (κ), by supposing the element of material unloaded at time t[1]. Let G be the gradient of the

[1] *Virtual* unloading according to the elastic properties, in order to avoid plastic deformations in the opposite direction of the deformation experienced during loading.

total transformation (o)→(a), P the gradient of the plastic transformation (o)→(κ) and E the gradient of the elastic transformation (κ)→(a). As $G=EP$, the velocity gradient tensor grad $\vec{v}=(\mathrm{D}G/\mathrm{D}t)G^{-1}$ is,

$$\operatorname{grad} \vec{v}=\mathfrak{D}+\omega=\frac{\mathrm{D}E}{\mathrm{D}t}E^{-1}+E\frac{\mathrm{D}P}{\mathrm{D}t}P^{-1}E^{-1}, \tag{2}$$

where the symmetric and antisymmetric parts are the deformation rate tensor, $\mathfrak{D}$, and spin tensor, ω. The objective derivative $\mathrm{D}/\mathrm{D}t$ may be a convected derivative, the Jaumann derivative, or the derivative introduced by Mandel (1973) relative to the rotation of the director frame which specifies the physical orientation of the macroelement.

Consider only isothermal quasistatic transformations. The thermodynamic state of the macroelement is defined by $\Delta^e=(E^tE-1)/2$, the Green deformation tensor between (κ) and (a), and α_i, scalar or generally tensorial variables that characterize damage and strain-hardening of the material; including the orientation of the director frame (Mandel, 1973).

The local form of the second principle of thermodynamics is,

$$\Phi=\operatorname{tr}\left(\frac{\sigma\mathfrak{D}}{\rho}\right)-\frac{\mathrm{D}\varphi}{\mathrm{D}t}\geqslant 0, \tag{3}$$

where $\varphi(\Delta^e, \alpha_i)$ is the specific free energy. Equations (2) and (3) give,

$$\Phi=\operatorname{tr}\left(E^{-1}\frac{\sigma}{\rho}E\frac{\mathrm{D}P}{\mathrm{D}t}P^{-1}\right)-\operatorname{tr}\left(\frac{\partial\varphi}{\partial\Delta^e}-\frac{\pi}{\rho_\kappa}\right)\frac{\mathrm{D}\Delta^e}{\mathrm{D}t}-\frac{\partial\varphi}{\partial\alpha_i}\frac{\mathrm{D}\alpha_i}{\mathrm{D}t}\geqslant 0, \tag{4}$$

where $\rho=1/\det G$ and $\rho_\kappa=1/\det P$ are the densities in the configurations (a) and (κ) ($\rho=1$ in the configuration (o)) and π is the Kirchhoff stress tensor relative to the configuration (κ),

$$\frac{\sigma}{\rho}=E\frac{\pi}{\rho\kappa}E^t. \tag{5}$$

In the (virtual) reversible elastic transformation from (κ) to (a), the dissipated power Φ is zero, and $\mathrm{D}P/\mathrm{D}t=\mathrm{D}\alpha_i/\mathrm{D}t=0$, so,

$$\frac{\pi}{\rho_\kappa}=\frac{\partial\varphi}{\partial\Delta^e}. \tag{6}$$

The generalized forces,

$$\Sigma=\left\{E^t\frac{\sigma}{\rho}E^{t-1}\right\},\qquad A_i=-\frac{\partial\varphi}{\partial\alpha_i}, \tag{7}$$

are the work conjugates of the plastic deformation rate, $\mathfrak{D}^{\mathrm{P}}=\{(\mathrm{D}P/\mathrm{D}t)P^{-1}\}$, and the internal variables rates, $\mathrm{D}\alpha_i/\mathrm{D}t$ (Nguyen, 1973).

The resulting inequality is,

$$\Phi=\mathrm{tr}(\Sigma\mathfrak{D}^{\mathrm{P}})+A_i\frac{\mathrm{D}\alpha_i}{\mathrm{D}t}\geqslant 0. \tag{8}$$

The constitutive relations of the material express the rates in terms of the generalized forces. Normal dissipativity is assumed, i.e. (Moreau, 1970): there exists a convex quasi-potential of dissipation $\Psi(\Sigma, A_i)$ and the conjugated rates are given by the normality rule.

Considering nonviscous plastic materials only, we further assume that the yield surface is defined by the single differentiable plastic potential, $F(\Sigma, A_i)$. It implies[2] that

$$\mathfrak{D}^{\mathrm{P}}=\lambda\frac{\partial F}{\partial\Sigma}, \qquad \frac{\mathrm{D}\alpha_i}{\mathrm{D}t}=\lambda\frac{\partial F}{\partial A_i}, \tag{9}$$

where $\lambda\geqslant 0$ if $F=\dot{F}=0$, otherwise $\lambda=0$; i.e., the rates are oriented in the direction of the external normal to the yield surface.

These postulates characterize GSM. In the usual theories of plasticity, only the first equation (9) applies[3].

The configuration (κ) may be chosen so that the elastic transformation is a pure deformation, $E=E^{\mathrm{t}}$. Since elastic strains in metals are small, they may be neglected compared to unity. Then $E=1+\Delta^{\mathrm{e}}=1+\varepsilon^{\mathrm{e}}$ and $\rho=\rho_\kappa$; $\sigma=\pi$, $\Sigma=\sigma/\rho$. With these hypotheses we have,

$$\mathfrak{D}=\frac{\mathrm{D}\varepsilon^{\mathrm{e}}}{\mathrm{D}t}+\mathfrak{D}^{\mathrm{P}}, \tag{10}$$

and the constitutive relations become,

$$\frac{\sigma}{\rho}=\frac{\partial\varphi}{\partial\varepsilon^{\mathrm{e}}}, \qquad A_i=-\frac{\partial\varphi}{\partial\alpha_i}, \tag{11}$$

$$\mathfrak{D}^{\mathrm{P}}=\lambda\frac{\partial F}{\partial(\sigma/\rho)}, \qquad \frac{\mathrm{D}\alpha_i}{\mathrm{D}t}=\lambda\frac{\partial F}{\partial A_i}, \tag{12}$$

where $\lambda\geqslant 0$ if $F(\sigma/\rho, A_i)=\dot{F}(\sigma/\rho, A_i)=0$, otherwise $\lambda=0$.

3. Application to ductile fracture damage

The hardening of the metal is supposed to be isotropic, characterized by a single scalar internal variable $\alpha_1=\alpha$. This is a reasonable assumption as chiefly monotically increasing loadings are considered in ductile fracture,

[2]As a general rule, multiple potentials may be considered (Nguyen, 1973). $\Psi(\Sigma, A_i)$ is the indicatory function of the convex bounded by the yield surface.

[3]In the general anisotropic case, a constitutive equation for the plastic spin tensor ω^{P} has to be added to Eqs. (9) (Mandel, 1973). It defines the orientation of the director frame.

with no noticeable reverse plastic deformation. However, the generalization to anisotropic hardening may be made, in the same way as in the theories of plasticity without damage.

The ductile fracture damage is also suppose to be isotropic, characterized by the scalar $\alpha_2 = \beta$. This is a simplifying hypothesis, because an initially spherical void usually grows into an ellipsoid, the characterization of which requires a symmetric tensorial internal variable $\beta_{ij} = \beta_{ji}$. On the other hand, the isotropy is derived from Rice and Tracey's (1969) results of symmetric void growth under large stress triaxiality, σ_m/σ_0.

The following form of the specific free energy is considered:

$$\varphi(\varepsilon^e, \alpha, \beta) = \frac{1}{2}\varepsilon^e L \varepsilon^e + \varphi_1(\alpha) + \varphi_2(\beta). \tag{13}$$

The first term is the elastic recoverable energy. $\varphi_1 + \varphi_2$ is the "locked" free energy, related to dislocations, residual stresses, voids, etc. The split of the free energy into three terms means that the elastic moduli tensor L does not depend on hardening and damage. Only for very large plastic strains, due to high hardening, which creates a texture in the metal, or high damage, the elastic moduli may be altered. The isotropic part of this alteration is certainly taken into account, according to the elastic constitutive relation (11): $\sigma/\rho = \partial\varphi/\partial\varepsilon^e$, that gives $\sigma = \rho L \varepsilon^e$. The density ρ is related to the damage, so are the apparent elastic moduli, $L_a = \rho L$. In fact, measurements of the variation of density and Young's modulus are performed as an indirect assessment of damage (Lemaitre and Chaboche, 1978); an isotropic damage D has been first introduced in the constitutive relations by Kachanov (1958) and Rabotnov (1968) with the notion of effective stress $\sigma_a = \sigma/(1-D)$, that gives $L_a = (1-D)L$. In the case of ductile fracture damage, and with the present formulation[4], it leads to the definition of damage $\beta = D = 1 - \rho$.

Finally, the split of the terms $\varphi_1(\alpha)$ and $\varphi_2(\beta)$ in the specific free energy means that the texture created by hardening and the porosity resulting from damage do not affect the characteristics of each other.

The Von Mises form of the plastic potential is,

$$F\left(\frac{\sigma}{\rho}, A\right) = \left[J_2\left(\frac{\sigma}{\rho}\right)\right]^{1/2} + \frac{A}{\sqrt{3}}, \tag{14}$$

where $J_2(T)$ is the second invariant of the deviatoric part of a second-order tensor T. The generalized force, $A = -\mathrm{d}\varphi_1(\alpha)/\mathrm{d}\alpha$, characterizes the hardening curve of the metal.

[4]Lemaître and Chaboche (1978) consider the *volumic* free energy $\rho\varphi$ instead of the specific free energy φ. The damage D only appears in the elastic term in the form,

$$\rho\varphi = \tfrac{1}{2}(1-D)\varepsilon^e L \varepsilon^e + \rho\varphi_1(\alpha_i),$$

where D has to be independent of ρ.

Damage is introduced with a third term, depending only on the first invariant, σ_m, of the stress tensor, according to theoretical and experimental results on ductile fracture up to date (Rice and Tracey, 1969; Hancock and Mackenzie, 1976; Auger and François, 1977),

$$F\left(\frac{\sigma}{\rho}, A, B\right)=\left[J_2\left(\frac{\sigma}{\rho}\right)\right]^{1/2}+\frac{A}{\sqrt{3}}+Bg\left(\frac{\sigma_m}{\rho}\right). \tag{15}$$

The generalized force B is the work conjugate of β. Let d^P and s be the deviatoric parts, $\mathcal{D}^P_m$ and σ_m the spherically symmetric parts of $\mathcal{D}^P$ and σ, respectively. The constitutive relations (11) and (12) give,

$$\sigma=\rho L\varepsilon^e, \tag{16}$$

$$A=-\varphi_1'(\alpha), \qquad B=-\varphi_2'(\beta), \tag{17}$$

$$d^P=\lambda\frac{s}{2[J_2(\sigma)]^{1/2}}, \qquad \mathcal{D}^P_m=\lambda\frac{B}{3}g'\left(\frac{\sigma_m}{\rho}\right), \tag{18}$$

$$\dot{\alpha}=\frac{\lambda}{\sqrt{3}}, \qquad \dot{\beta}=\lambda g\left(\frac{\sigma_m}{\rho}\right). \tag{19}$$

The equivalent deformation rate is usually defined as,

$$\mathcal{D}^P_{eq}=\left[\tfrac{4}{3}J_2(\mathcal{D}^P)\right]^{1/2}=\left[\tfrac{2}{3}d^P_{ij}d^P_{ij}\right]^{1/2}. \tag{20}$$

Taking the second invariant of both sides of the first equation (18), it follows that

$$\mathcal{D}^P_{eq}=\frac{\lambda}{\sqrt{3}}=\dot{\alpha}, \tag{21}$$

$$d^P=\frac{3\mathcal{D}^P_{eq}}{\sigma_{eq}}s, \tag{22}$$

where $\sigma_{eq}=[3J_2(\sigma)]^{1/2}$ is the equivalent stress. In the absence of damage, the hardening curve is $\sigma_{eq}=\varphi_1'(\alpha)$, where $\alpha=\int \mathcal{D}^P_{eq}\,dt$.

Equation (22) is the usual constitutive relation with the Von Mises yield criterion. In addition, the second equations in (18) and (19) characterize the volumetric plastic deformation rate and the development of damage.

It may be argued that damage coincides with incipient plastic deformation, as the same threshold $F=0$ (15) is considered for plasticity and damage. It is a reasonable approximation when the stress triaxiality, σ_m/σ_0, is large (Beremin, 1979). In any case damage indeed starts with the formation and piling up of dislocation loops at the matrix–particle interface, at incipient plastic strains, and not only with the subsequent nucleation of voids (Henry and Hortsmann, 1979). Besides, if the stress triaxiality is low, the plastic potential (15) will reduce to the usual form (14), as will be demonstrated below.

With the hypotheses of isotropic hardening and damage, from an isotropic unstressed initial configuration[5], the orientation of the director frame, which is then corotational with the macroelement, does not intervene any longer in the constitutive relations (16)–(19) (Mandel, 1973), and the Jaumann and Mandel derivatives are identical. As α and β are scalars, it seems that these derivatives, and the spin tensor $\omega=\omega^{\mathrm{P}}$, do not enter the constitutive relations. Actually, the time derivative of the elastic relation $\sigma=\rho L\varepsilon^{\mathrm{e}}$ is required, and according to Eq. (10),

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\sigma}{\rho}\right)=L(\mathcal{D}-\mathcal{D}^{\mathrm{P}}); \tag{23}$$

The spin tensor appears in the derivative,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\sigma}{\rho}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{\rho}\right)-\frac{\omega\sigma}{\rho}+\frac{\sigma\omega}{\rho}. \tag{24}$$

The function $g(\sigma_{\mathrm{m}}/\rho)$ has not been explicitly defined as yet.

In ductile fracture, the damage parameter β is directly related to the change of density of the metal; thus $\beta=\beta(\rho)$. Expressing $\mathcal{D}_{\mathrm{m}}^{\mathrm{P}}$ and $\dot{\beta}=\beta'(\rho)\dot{\rho}$ in the mass conservation law $\dot{\rho}+3\rho\mathcal{D}_{\mathrm{m}}^{\mathrm{P}}=0$[6], according to (18) and (19), we find,

$$\frac{g'(\sigma_{\mathrm{m}}/\rho)}{g(\sigma_{\mathrm{m}}/\rho)}=-\frac{1}{\rho\beta'(\rho)B}, \tag{25}$$

where $B=-\varphi_2'(\beta(\rho))$ is a function of ρ only. Therefore, the two sides of (25) are constant of dimension $1/\sigma$, say C/σ_0, where σ_0 is the yield stress. The integration of the left-hand side gives,

$$g\left(\frac{\sigma_{\mathrm{m}}}{\rho}\right)\equiv D\exp\left(\frac{C\sigma_{\mathrm{m}}}{\rho\sigma_0}\right), \tag{26}$$

where D is the constant of integration; C and D are supposed to be positive, in order that damage increases with stress triaxiality and that $\dot{\beta}>0$; damage is irreversible, since λ cannot be negative.

The general hypotheses—generalized standard materials—and plastic potential depending only on the two first invariants of the stress tensor—yield an exponential dependence of damage on stress triaxiality. This result is similar to that obtained by McClintock (1968), Rice and Tracey (1969).

[5] The elastic moduli tensor is also isotropic, $L_{ijhk}=\lambda^*\delta_{ij}\delta_{hk}+\mu(\delta_{ih}\delta_{jk}+\delta_{ik}\delta_{jh})$.

[6] The elastic deformations are small, so $\rho\simeq\rho_\kappa$ and div $\vec{v}=3\mathcal{D}_{\mathrm{m}}\simeq3\mathcal{D}_{\mathrm{m}}^{\mathrm{P}}$. As for an exact measure of damage, ρ_κ should be considered instead of ρ, i.e. the elastic volumetric deformation is excluded.

Still there is a difference, the consequence of which will be shown; the density ρ appears in the exponential. This is due to the fact that, in the change of configuration, σ is a material 2—contravariant tensorial density. In usual plasticity theories, ρ is assumed to be constant (unity), but for ductile fracture damage this is not a good assumption.

The functions $\beta(\rho)$ and $\varphi_2(\beta)$ are related by

$$C\rho\beta'(\rho)\,\varphi_2'(\beta(\rho))=\sigma_0. \tag{27}$$

One of these two functions, just as the constants C and D, must be chosen to match the theoretical and experimental results for given materials and micromechanisms. Simple choices are:

$$\beta=1-\rho\Rightarrow\varphi_2(\beta)=\frac{\sigma_0}{C}\ln(1-\beta), \tag{28}$$

$$\beta=\frac{1}{\rho}-1\Rightarrow\varphi_2(\beta)=-\frac{\sigma_0}{C}\ln(1+\beta), \tag{29}$$

$$\beta=f-f_0\Rightarrow\varphi_2(\beta)=\frac{\sigma_0}{C}\ln(1-f_0-\beta). \tag{30}$$

In the latter example (not really distinct from $\beta=1-\rho$), f is the void volume fraction, including the particles, the volume fraction of which is f_0. As the metal of the matrix is supposed to be incompressible in the plastic deformation, according to (18) without damage ($g\equiv 0$), the relation between f and ρ is,

$$\rho=\frac{1-f}{1-f_0}. \tag{31}$$

An alternative choice for $\beta(\rho)$ is suggested by the Rice and Tracey formula (1) for the growth of a spherical void in case of high stress triaxiality,

$$\frac{\dot{R}_0}{R_0}=0.283\mathfrak{D}^{\mathrm{P}}_{\mathrm{eq}}\exp\left(\frac{3\sigma_{\mathrm{m}}}{2\sigma_0}\right), \tag{32}$$

(with the present notations). This formula is only valid for incipient void growth ($R=R_0$ and $\rho=1$) but it may be extended to the whole damage process. As $\lambda=\mathfrak{D}^{\mathrm{P}}_{\mathrm{eq}}\sqrt{3}$ and

$$\frac{\dot{f}}{f(1-f)}=3\frac{\dot{R}}{R}, \tag{33}$$

(19) and (32) give, if $C=3/2$,

$$\frac{\dot{f}}{f(1-f)\dot{\beta}}=\frac{0.283\sqrt{3}}{D}. \tag{34}$$

It thus follows that,

$$C=3/2, \qquad D=0.49, \tag{35}$$

$$\beta=\ln\frac{f(1-f_0)}{f_0(1-f)}=\ln\left(1+\frac{1-\rho}{\rho f_0}\right), \tag{36}$$

$$\varphi_2(\beta)=\frac{\sigma_0}{C}\ln\frac{1-f}{1-f_0}=-\frac{\sigma_0}{C}\ln(1-f_0+f_0\exp\beta), \tag{37}$$

$$B(\beta)=-\varphi_2'(\beta)=\frac{\sigma_0 f}{C}=\frac{\sigma_0}{C}\frac{f_0\exp\beta}{1-f_0+f_0\exp\beta}. \tag{38}$$

The yield criterion, $F=0$, Eq. (15), becomes,

$$\frac{\sigma_{\mathrm{eq}}}{\rho}+0.57\sigma_0 f\exp\left(\frac{3\sigma_{\mathrm{m}}}{2\rho\sigma_0}\right)-\varphi_1'(\alpha)=0. \tag{39}$$

This can be compared with the Gurson approximation (1977) resulting from his microscopic analysis, with the matrix material idealized as rigid—perfectly-plastic,

$$\sigma_{\mathrm{eq}}^2+2\sigma_0^2 f\cosh\left(\frac{3\sigma_{\mathrm{m}}}{2\sigma_0}\right)-(1+f^2)\sigma_0^2=0. \tag{40}$$

If the stress triaxiality $\sigma_{\mathrm{m}}/\sigma_0$ is large, 2 cosh = exp. As the Gurson analysis is similar to that of Rice and Tracey, the correspondence between (39) and (40) is not unexpected[7].

4. Infinitesimal strain approximation

Assume both elastic and plastic deformations are small. The constitutive relations become,

$$\sigma=L\varepsilon^{\mathrm{e}}, \tag{41}$$

$$\dot{e}^{\mathrm{P}}=\lambda\frac{\sqrt{3}\,s}{2\sigma_{\mathrm{eq}}}, \qquad \dot{\varepsilon}_{\mathrm{m}}^{\mathrm{p}}=\lambda\frac{DC}{3\sigma_0}B(\beta)\exp\left(\frac{C\sigma_{\mathrm{m}}}{\sigma_0}\right), \tag{42}$$

$$\dot{\alpha}=\frac{\lambda}{\sqrt{3}}, \qquad \dot{\beta}=\lambda D\exp\left(\frac{C\sigma_{\mathrm{m}}}{\sigma_0}\right). \tag{43}$$

[7] The consideration of σ_{eq}^2 instead of σ_{eq} in the plastic potential would make it difficult to give a physical meaning to the hardening variable α. With the present formulation, $\alpha=\int \dot{\varepsilon}_{\mathrm{eq}}^{\mathrm{P}}\,\mathrm{d}t$. For large $\sigma_{\mathrm{m}}/\sigma_0$ and incipient damage (small f and f exp), an approximate form of Eq. (40) is, $\sigma_{\mathrm{eq}}+0.5\sigma_0 f\exp(3\sigma_{\mathrm{m}}/2\sigma_0)-\sigma_0=0$, which is similar (39).

Since $\mathcal{D}^{\mathrm{P}}$ is now the time derivative, $\dot{\varepsilon}^{\mathrm{P}}=\mathrm{d}\varepsilon^{\mathrm{P}}/\mathrm{d}t$, of plastic strain tensor, ε^{P}, with deviatoric and spherical symmetric parts denoted by e^{P} and $\varepsilon_{\mathrm{m}}^{\mathrm{P}}=\varepsilon_{\mathrm{kk}}^{\mathrm{P}}/3$, respectively.

The plastic potential is,

$$F(\sigma,\alpha,\beta)=\sigma_{eq}-\varphi_1'(\alpha)+DB(\beta)\exp\left(\frac{C\sigma_{\mathrm{m}}}{\sigma_0}\right). \tag{44}$$

The plastic multiplier λ is expressed in terms of the stress rate $\dot{\sigma}$,

$$\lambda=\frac{1}{H}\frac{\partial F}{\partial\sigma}\dot{\sigma}, \tag{45}$$

where

$$H=\frac{\partial F}{\partial A_i}\frac{\partial^2\varphi}{\partial\alpha_i\partial\alpha_j}\frac{\partial F}{\partial A_j}. \tag{46}$$

The elastic energy on the one hand, hardening ($\alpha_1=\alpha$) and damage ($\alpha_2=\beta$) on the other hand, are supposed to be uncoupled in the free energy (13), $\partial^2\varphi/\partial\alpha_i\partial\varepsilon^{\mathrm{e}}=0$. Thus,

$$\dot{A}_i=\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\partial\varphi}{\partial\alpha_i}\right)=-\frac{\partial^2\varphi}{\partial\alpha_i\partial\alpha_j}\dot{\alpha}_j=-\frac{\partial^2\varphi}{\partial\alpha_i\partial\alpha_j}\lambda\frac{\partial F}{\partial A_j}. \tag{47}$$

In plastic loading, $F=\dot{F}=0$, and hence,

$$\frac{\partial F}{\partial\sigma}\dot{\sigma}+\frac{\partial F}{\partial A_i}\dot{A}_i=0. \tag{48}$$

Equations (47) and (48) can then be combined to give (45). Equation (45) yields,

$$\dot{\sigma}\dot{\varepsilon}^{\mathrm{P}}=\lambda\frac{\partial F}{\partial\sigma}\dot{\sigma}=H\lambda^2. \tag{49}$$

If the matrix $Z_{ij}=\partial^2\varphi/\partial\alpha_i\partial\alpha_j$ is positive, then $H\geqslant 0$ and the Drucker inequality holds (Nguyen, 1973; Nguyen and Bui, 1974),

$$\dot{\sigma}\dot{\varepsilon}^{\mathrm{P}}\geqslant 0, \tag{50}$$

the equality implying $\dot{\varepsilon}^{\mathrm{P}}=0$. In the present theory,

$$Z=\begin{bmatrix}\varphi_1''(\alpha) & 0\\ 0 & \varphi_2''(\beta)\end{bmatrix}. \tag{51}$$

Assume the hardening of the matrix material is positive, then $\varphi_1''(\alpha)>0$.

If the damage is defined by $\beta=1/\rho-1$, (29), then $\varphi_2''(\beta)=\sigma_0/C(1+\beta)^2$ is also positive. In that case, as stated by Rousselier (1979), instability (in Drucker's sense), and therefore rupture, is impossible. It is still necessary to postulate a fracture criterion. In the next section it will be demonstrated that instability takes place if the infinitesimal strain hypothesis is relaxed.

The positiveness of Z is not a general rule. The choice of $\beta=1-\rho$, (28), or $\beta=f-f_0$, (30), yields $\varphi_2''(\beta)=-\sigma_0/C(1-\beta)^2$ or $\varphi_2''(\beta)=-\sigma_0/C(1-f_0-\beta)^2$, both negative. Equation (36), suggested by the Rice and Tracey formula (1), gives,

$$\varphi_2''(\beta)=-\frac{\sigma_0}{C}\frac{(1-f_0)f_0\exp\beta}{(1-f_0+f_0\exp\beta)^2}, \tag{52}$$

which is also negative. In infinitesimal strain numerical applications, the damage β should be defined so as to avoid a positive $\varphi_2''(\beta)$ and allow instability of the material.

5. Instability and fracture: example

Consider the homogeneous transformation without rotation of an element of material subjected to a triaxial stress state,

$$\begin{aligned}&\sigma_{22}=\sigma_{33}=k\sigma_{11}, \qquad k \text{ constant}<1,\\ &\sigma_{12}=\sigma_{23}=\sigma_{31}=0.\end{aligned} \tag{53}$$

The plastic potential and the constitutive relations for $\dot\alpha$ and $\mathfrak{D}_m^P=-\dot\rho/3\rho$ give,

$$\frac{1-k}{\sqrt3}\frac{\sigma_{11}}{\rho}-\frac{\varphi_1'(\alpha)}{\sqrt3}+DB(\beta)\exp\left[\frac{C(1+2k)}{3\sigma_0}\frac{\sigma_{11}}{\rho}\right]=0, \tag{54}$$

$$\dot\rho=-\dot\alpha\frac{DC\sqrt3}{\sigma_0}\rho B(\beta)\exp\left[\frac{C(1+2k)}{3\sigma_0}\frac{\sigma_{11}}{\rho}\right]=0. \tag{55}$$

According to Eq. (27), $B(\beta)=-\sigma_0/C\rho\beta'(\rho)$, and the combination of the two latter equations gives,

$$\frac{\dot\rho}{\rho\dot\alpha}-\frac{3(1-k)}{1+2k}\ln\left[\frac{\beta'(\rho)}{D\sqrt3}\frac{\dot\rho}{\dot\alpha}\right]+C\frac{\varphi_1'(\alpha)}{\sigma_0}=0, \tag{56}$$

$$(1-k)\frac{\sigma_{11}}{\rho\sigma_0}-\frac{D\sqrt3}{C\rho\beta'(\rho)}\exp\left[\frac{(1+2k)C}{3}\frac{\sigma_{11}}{\rho\sigma_0}\right]-\frac{\varphi_1'(\alpha)}{\sigma_0}=0. \tag{57}$$

Equation (56) is an algebraic equation for $\dot\rho/\dot\alpha$. From an actual state $(\sigma_{11},\rho,\alpha)$, its numerical solution, considering an increment $\Delta\alpha$, yields the new values $\alpha+\Delta\alpha$, $\rho+(\dot\rho/\dot\alpha)\Delta\alpha$. The new stress σ_{11} is given by the numerical solution of the algebraic Eq. (57).

The strain rates are,

$$\begin{aligned}&d_{11}^P=-2d_{22}^P=-2d_{33}^P=\mathfrak{D}_{eq}^P=\dot\alpha,\\ &d_{12}^P=d_{23}^P=d_{31}^P=0, \qquad \mathfrak{D}_m^P=-\frac{\dot\rho}{3\rho}.\end{aligned} \tag{58}$$

We shall consider the "logarithmic plastic deformation tensor", $\hat{\varepsilon}^{P}=\int \mathcal{D}^{P}\, dt$, the first component of which is,

$$\hat{\varepsilon}^{P}_{11}=\alpha-\tfrac{1}{3}\ln \rho . \tag{59}$$

The results are shown in Figs. 1–8. In Fig. 1 a power-hardening law $\varphi_1'(\alpha)=\sigma_0(1+\sqrt{\alpha})$ is considered. The damage is defined by $\beta=1/\rho-1$, (29), or by $\beta=f-f_0$, (30). In both cases, after a certain amount of plastic deformation depending on stress triaxiality, *instability takes place*: $\dot{\sigma}\dot{\varepsilon}^{P}<0$. The damage-related softening overcomes the strain-hardening of the matrix material.

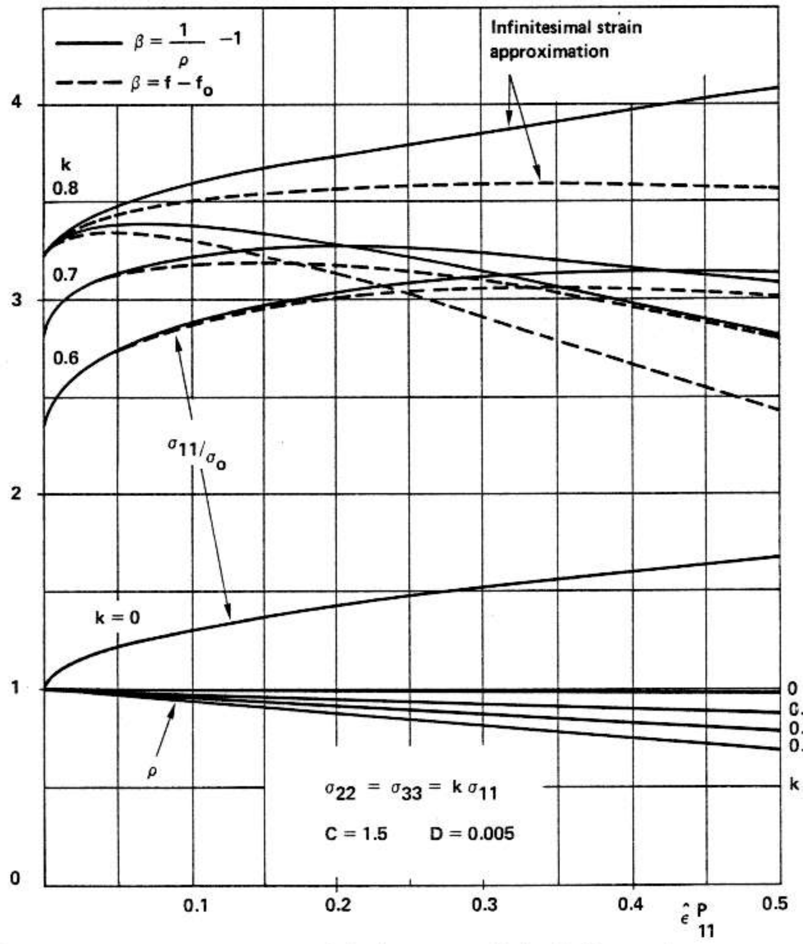

Fig. 1. Stress–strain and density curves with ductile fracture damage.

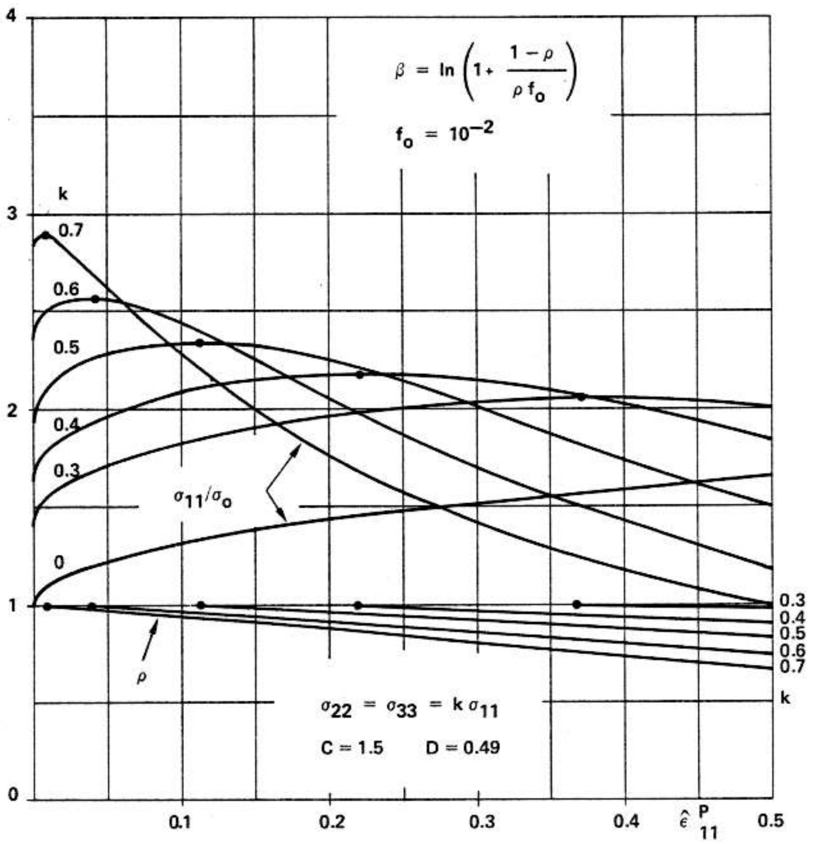

Fig. 2. Stress–strain and density curves with ductile fracture damage.

If the infinitesimal strain constitutive relations, (41)–(44), are used instead of the finite strain ones, instability is not revealed if $\beta = 1/\rho - 1$, even with a very high stress triaxiality ($k = 0.8$). It was demonstrated in Section 4 that instability was impossible indeed in that case. If $\beta = f - f_0$, instability is still possible but is dramatically postponed (Fig. 1).

In Figs. 2 to 5 damage defined by $B = \sigma_0 f/C$ ((35) to (38)) is considered, as suggested by the Rice and Tracey formula, with the power-hardening law of Fig. 1. The effects of various stress triaxialities and various initial void volume fractions f_0 are investigated. As expected, the smaller the volume fraction f_0 the higher the stress triaxiality and plastic deformation required to yield instability. This is emphasized in Fig. 6, where the plastic strain at instability is plotted versus $\sigma_m/\sigma_{eq} \equiv (1+2k)/3(1-k)$ (that has to be distinguished from σ_m/σ_0). The dependence on f_0 is slightly stronger than

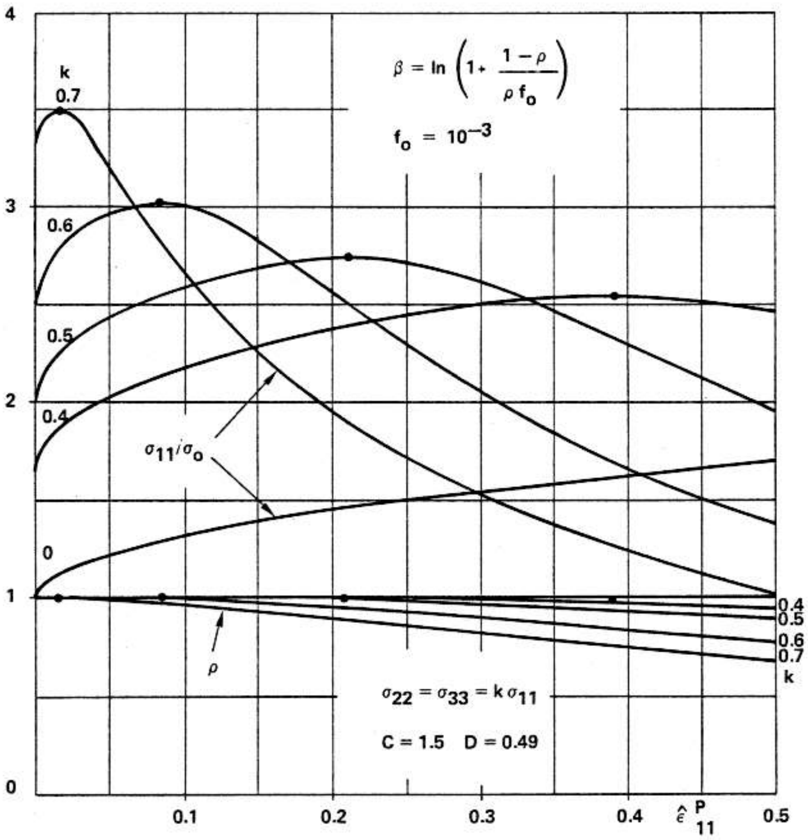

Fig. 3. Stress–strain and density curves with ductile fracture damage.

the squaring of the initial void fraction required to double the critical strain obtained by McClintock (1968).

With $B=\sigma_0 f/C$, the infinitesimal strain curves, shown in Fig. 5, do not differ from the finite strain ones until instability. But the final decohesion resulting from these curves is entirely different as shown in Fig. 7 with a reduced strain scale (in Fig. 7, the power-hardening law is changed for an exponential one: $\varphi_1'(\alpha)=\sigma_0[2-\exp(-20\alpha)]$).

The finite strain curves of Fig. 7 clearly show that the present theory, with damage depending exponentially on σ_m/ρ, is perfectly fitted to the modelling of instability and decohesion in ductile fracture. *This is not the case with an exponential dependence on σ_m alone.*

Finally, it should be noticed that the use of fracture criteria, here $\beta=\beta_c$ or $\rho=\rho_c$ or $f=f_c$, is not consistent with the results of the present analysis. The

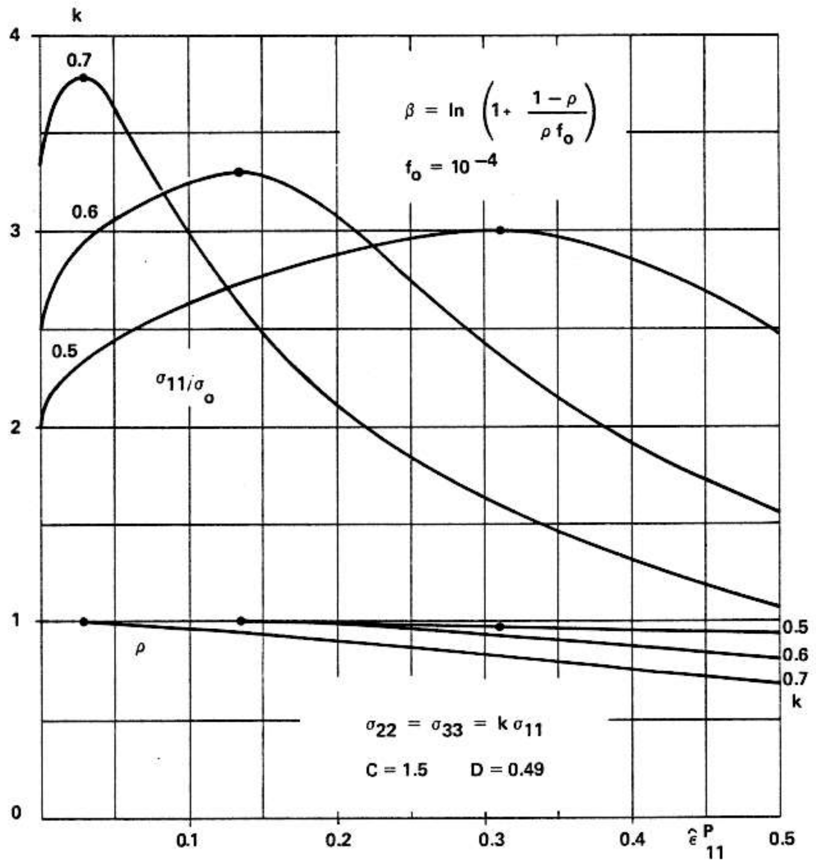

Fig. 4. Stress–strain and density curves with ductile fracture damage.

critical values of β, ρ or f, corresponding to material instability, depend on the mean stress σ_m; see Fig. 8. This is in agreement with the experimental results of Beremin (1979) where a decrease of the critical void growth at instability with increasing stress triaxiality is obtained.

6. Finite element analysis of a cracked specimen

The analysis of ductile fracture and stable crack growth in a three-point bend specimen has been performed with a 2D infinitesimal strain finite element model (plane strain). As pointed out in Sections 4 and 5, the infinitesimal strain approximation is a serious limitation of the present

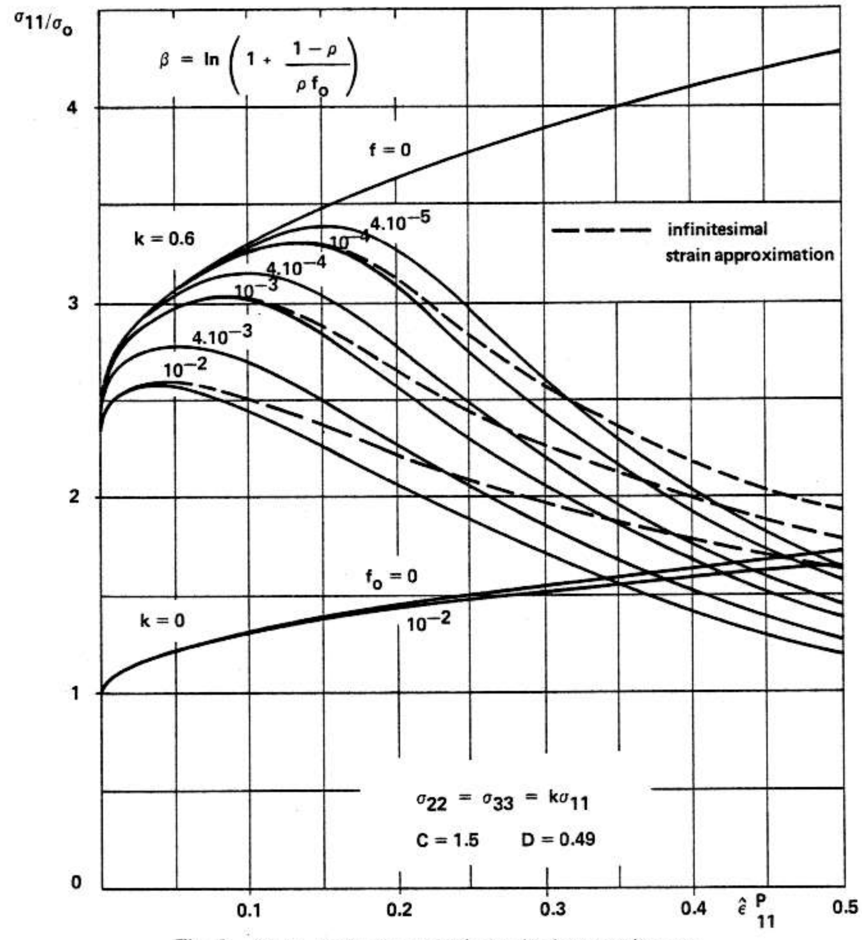

Fig. 5. Stress–strain curves with ductile fracture damage.

theory. That is why Eq. (41)–(44) are modified in order to *include some finite deformation effect*: namely, the ratio σ_m/ρ is substituted for σ_m into all the exponentials, according to the finite deformation relation (26), and σ_{eq}/ρ for σ_{eq} into the plastic potential (44).

A constant stiffness method is used. It requires the iterative computation of the "initial stress' $S = L\Delta\varepsilon^P$ that appears in the linear elastic relation between the increments of stress and strain: $\Delta\sigma = L\Delta\varepsilon - S$, which is the approximate form of Eq. (23). The initial stress, S, is computed with an implicit algorithm (Nguyen, 1973); see Appendix. This algorithm eliminates the systematic numerical errors usually found in the explicit method and is

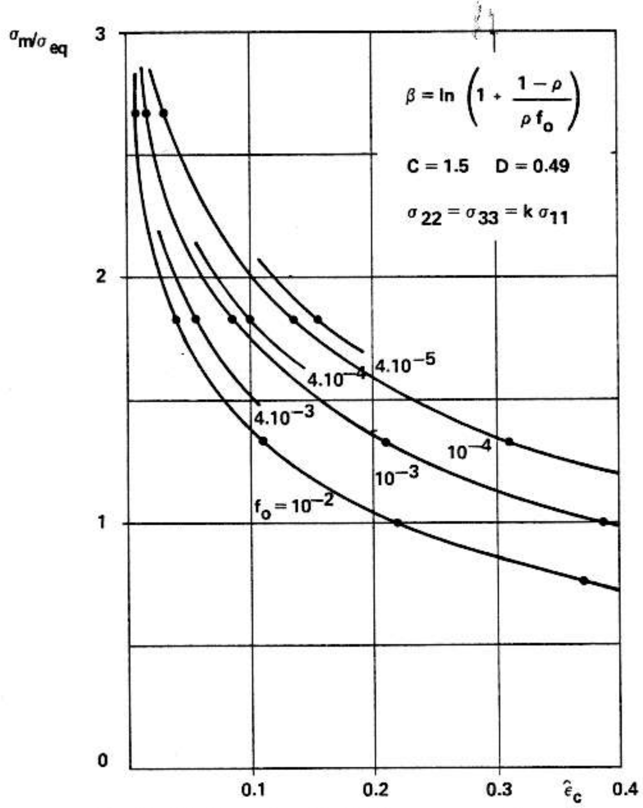

Fig. 6. Plastic strain at instability versus σ_m/σ_{eq}

particularly suited to the present analysis where material instability $\dot{\sigma}\dot{\varepsilon}^{P} < 0$ takes place.

The geometry of the specimen is specified in Fig. 9. The increments of the displacement $u_1 = d$ of node A are prescribed; the corresponding load is $F_1 = P/2B$. The specimen is discretized into 377 constant strain triangular elements, 420 nodes and 840 degrees of liberty. The size of the finite elements at the crack tip is shown in Fig. 10.

Note that at the tip of a crack, where steep gradients of stresses and especially strains take place, the critical conditions for instability shall be achieved over some characteristic length l_c, related to interparticle spacing. Otherwise void coalescence and material decohesion will not occur. In the numerical modelling, with constant strain elements, l_c is the length of the finite elements at the crack tip. If the specimen size or geometry are changed, the elements at the crack tip will keep the same absolute size.

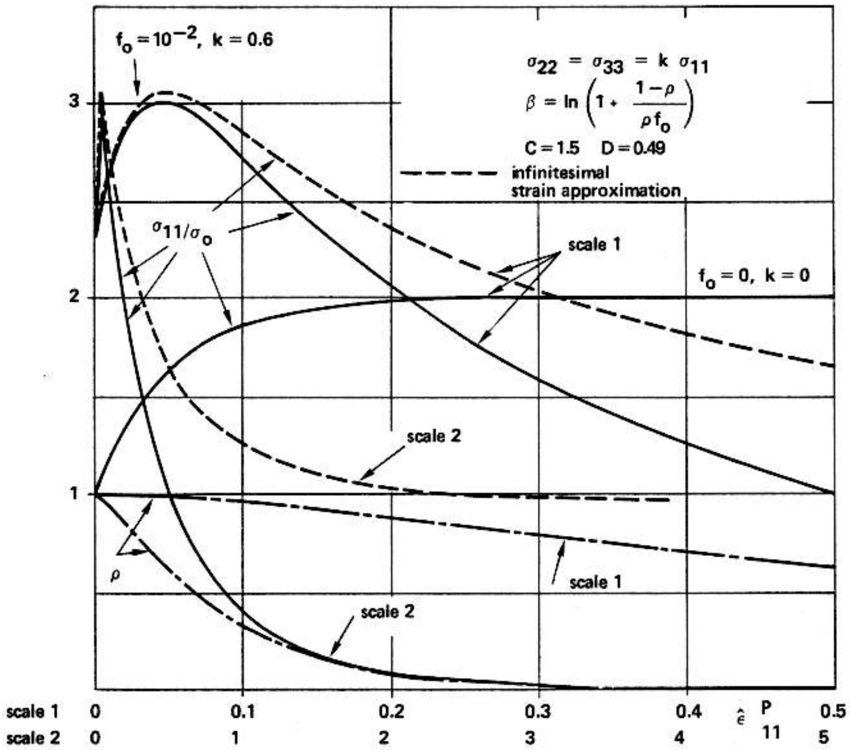

Fig. 7. Stress–strain and density curves with ductile fracture damage.

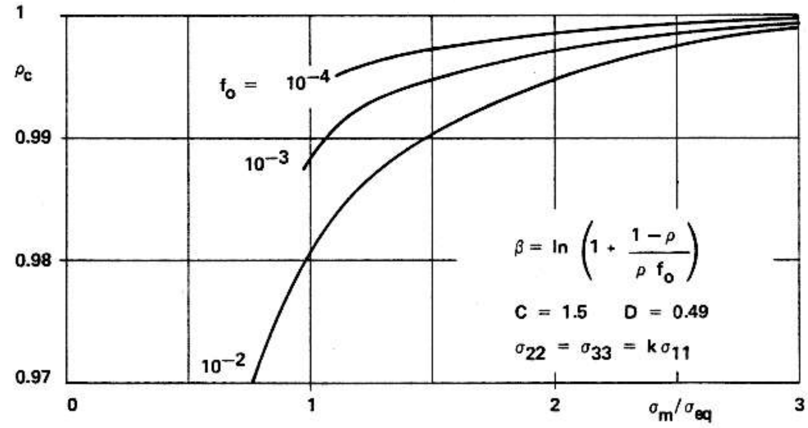

Fig. 8. Critical density at material instability as a function of σ_m/σ_{eq}.

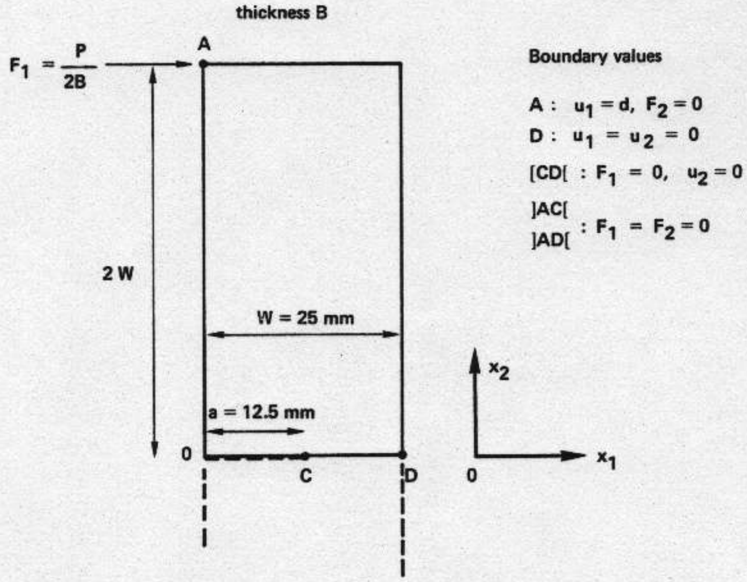

Fig. 9. Three-point bend specimen.

	d (mm)	P/B (N/mm)
– – –	0	0
━━━	0.548	686
───	0.748	652

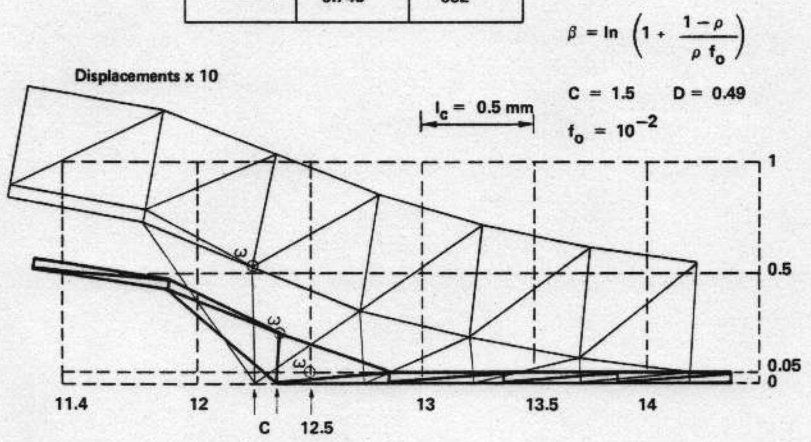

Fig. 10. Localization of deformation in the crack tip elements, by ductile fracture damage, resulting in stable crack growth.

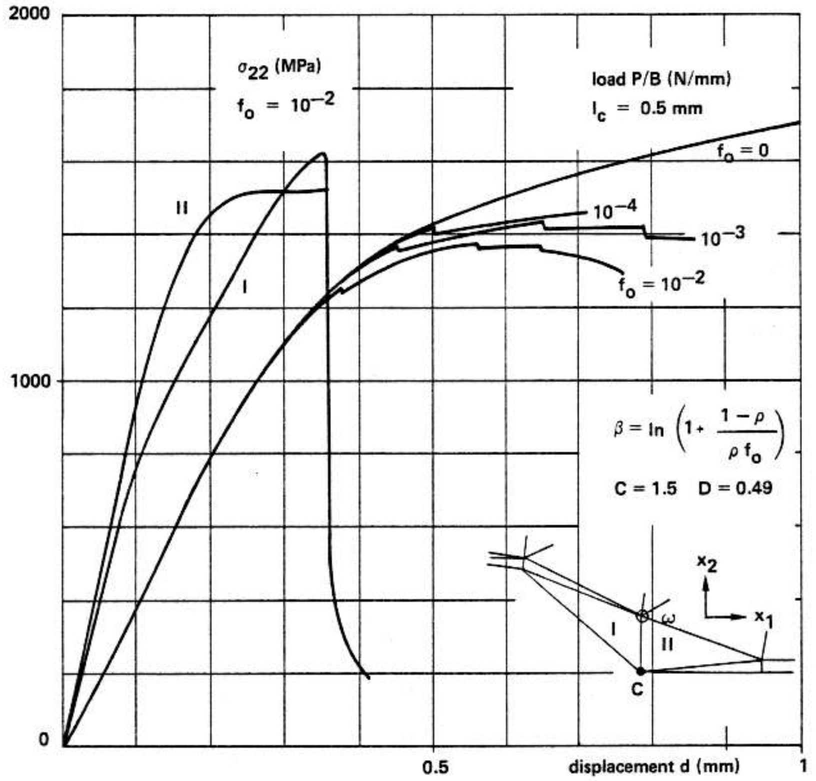

Fig. 11. Load-displacement curves of a three point bend specimen, with ductile fracture damage. Stress curves in two crack tip elements showing the local instability (crack tip geometry different from that of load curves).

Moreover, in order to model stable crack growth, the length of the elements on the crack prolongation are uniform.

The same exponential strain-hardening law as in Fig. 7 is considered; $\varphi_1'(\alpha)=\sigma_0[2-\exp(-20\alpha)]$ where $\sigma_0=500$ MPa. The elastic constants are $E=200$ GPa and $\nu=0.3$. According to the discussion in Sections 4 and 5, the definition of damage (35)–(38) is used. So, *the ductile fracture properties of the metal are defined by two parameters*: (1) l_c, related to the *interparticle spacing*; and (2) f_0, related to the *particle volume fraction*. These are required for the characterization of the particle size distribution[8].

The deformation of the crack tip zone is shown in Fig. 10. In the two most deformed elements, undergoing stress triaxialities σ_m/σ_0 of nearly 3, the damage increases rapidly. At some point, corresponding to the maximum of

[8]Note that in a tensile specimen, with no strain gradients, the ductility is dependent on the volume fraction f_0 only, as observed by Edelson and Baldwin (1962).

the local stress–strain curves, as in Fig. 7, the deformation of these elements increases abruptly, and the stresses decrease according to curves I and II of Fig. 11. This quasi-rupture of the two elements may be identified with the coalescence stage of the ductile fracture process. The node ω (Fig. 10) is no longer bound to node C (initial crack tip) nor to the symmetric node ω'. So *stable crack growth occurs naturally, by localization of deformation, resulting from the constitutive relations only, without it being necessary to define a critical state nor to release the nodes* as in usual models. A further stage of stable crack growth (four nodes, $d = 0.748$ mm) is shown in Fig. 10; the strains in the most deformed elements exceed unity.

As constant strain elements are used, the localization of deformation takes place over the whole element. That is why very thin elements are needed along the crack path. This drawback should be avoided with nonlinear elements that would give a better localization of deformation and allow crack propagation in any direction. Note that the blunting of the initial crack tip, that may be of importance for the initiation of crack growth, is not modelled by the simple constant strain elements of Fig. 10.

The load-displacement curves are given in Fig. 11 for various initial void volume fractions $f_0 = 10^{-2}$, 10^{-3}, 10^{-4} and 0 (no damage). In spite of strain hardening, up to $2\sigma_0$ at the most, the load rapidly stops increasing when ductile fracture and stable crack growth take place. After the maximum load the ductile tearing of the specimen goes on under decreasing load, in agreement with the acknowledged experimental evidence.

7. Conclusions

Constitutive relations have been developed for finite transformation of plastic dilatant materials. The dilatancy is related to the growth of voids nucleated at particles present in the matrix material. These constitutive relations model the ductile fracture of metals. They have been derived, in a macroscopic approach, from the internal variables hypothesis within the framework of generalized standard materials.

An exponential dependence of ductile fracture damage on stress triaxiality is demonstrated, and the occurrence of material instability $\dot{\sigma}\dot{\varepsilon}^{P} < 0$ shown for various examples. The finite transformation effects are also discussed.

Finite element analyses of a cracked specimen have been performed. Stable crack growth occurs naturally by localization of deformation, ~~and~~ is which in better agreement with the physical behavior of metals, and the method is simpler, and probably cheaper than usual node relaxation models. An advantage of the present approach is that nonsymmetrical complex loading conditions (the angled crack extension) and 3D-problems could be handled, without special finite element modelling in the crack tip zone.

In further applications nonlinear elements must be considered, as constant strain triangular elements are not suitable for localization of deformation. The necessity of modelling the initial crack tip blunting and of a complete finite transformation formulation, according to the equations of Section 3, should also be investigated.

As for the parameters involved in the theory, C and D are derived from microscopic models, and could be checked by experiments. The testing and computing of circumferentially notched tensile specimens give f_0, which should be consistent with the observed inclusions and/or precipitates volume fractions, according to the micromechanisms involved. The characteristic length l_c that intervenes in crack problems, though relevant to microstructural aspects (a few interparticle spacings), must be regarded as essentially an empirically obtained quantity. Its determination requires experiments on appropriate specimen geometries, which can be modelled with 2D-finite element analyses, like side grooved CT specimens or circumferentially cracked tensile specimens. Such an experimental program, in collaboration with the French research group Beremin, is in progress on A508 C1 3 steel.

Appendix. Implicit algorithm for the constitutive relations with ductile rupture damage

In the numerical solution of the elastic-plastic problem, for the increment $t-t+\Delta t$, the initial stress $S=L\Delta\varepsilon^{\mathrm{P}}$ has to be expressed as a function of $\sigma(t)$, $\alpha_i(t)$, and $\Delta\varepsilon$. The following form of the constitutive relations is used:

$$\Delta s=2\mu(\Delta e-\Delta e^{\mathrm{P}}), \tag{A.1}$$

$$\Delta\sigma_{\mathrm{m}}=(3\lambda^*+2\mu)(\Delta\varepsilon_{\mathrm{m}}-\Delta\varepsilon_{\mathrm{m}}^{\mathrm{P}}), \tag{A.2}$$

$$\Delta e^{\mathrm{P}}=\lambda\frac{s+\Delta s}{2\sqrt{J_2(s+\Delta s)}}, \tag{A.3}$$

$$\Delta\varepsilon_{\mathrm{m}}^{\mathrm{P}}=\lambda\frac{DC}{3\sigma_0}B(\beta+\Delta\beta)\exp\left(\frac{C}{\rho\sigma_0}(\sigma_{\mathrm{m}}+\Delta\sigma_{\mathrm{m}})\right), \tag{A.4}$$

$$\Delta\alpha=\frac{\lambda}{\sqrt{3}}, \tag{A.5}$$

$$\Delta\beta=\lambda D\exp\left(\frac{C}{\rho\sigma_0}(\sigma_{\mathrm{m}}+\Delta\sigma_{\mathrm{m}})\right), \tag{A.6}$$

$$\sqrt{3J_2\left(\frac{s+\Delta s}{\rho}\right)}+\sqrt{3}\,DB(\beta+\Delta\beta)\exp\left(\frac{C}{\rho\sigma_0}(\sigma_{\mathrm{m}}+\Delta\sigma_{\mathrm{m}})\right)-\varphi_1'(\alpha+\Delta\alpha)=0. \tag{A.7}$$

These relations are written at time $t+\Delta t$ instead of t in the explicit algorithm, and the plastic potential $F=0$, (A.7), is used instead of Eq. (45). If Δs and $\Delta\sigma_m$ are eliminated between (A.1) and (A.3), (A.2) and (A.4) respectively,

$$\Delta\varepsilon_m^P = \Delta\alpha \frac{DC}{\sigma_0\sqrt{3}} B(\beta+\Delta\beta) \times \exp\left\{\frac{C}{\rho\sigma_0}\left[\sigma_m + (3\lambda^* + 2\mu)\left(\Delta\varepsilon_m - \Delta\varepsilon_m^P\right)\right]\right\} \tag{A.8}$$

$$s+\Delta s = s + 2\mu(\Delta e - \Delta e^P) = s + 2\mu\Delta e - \mu\Delta\alpha\sqrt{3}\,\frac{s+\Delta s}{\sqrt{J_2(s+\Delta s)}},$$

then,

$$\left(1 + \frac{\mu\Delta\alpha\sqrt{3}}{\sqrt{J_2(s+\Delta s)}}\right)(s+\Delta s) = s + 2\mu\Delta e. \tag{A.9}$$

Taking the second invariant of this equation, we obtain,

$$\sqrt{J_2(s+\Delta s)} + \mu\Delta\alpha\sqrt{3} = \frac{\gamma}{\sqrt{3}}, \tag{A.10}$$

where

$$\gamma^2 = \frac{3}{2}(s+2\mu\Delta e)(s+2\mu\Delta e). \tag{A.11}$$

Equation (A.9) gives,

$$\frac{\gamma(s+\Delta s)}{\sqrt{3J_2(s+\delta s)}} = s + 2\mu\Delta e.$$

Finally,

$$\Delta e^P = \frac{3\Delta\alpha}{2\gamma}(s+2\mu\Delta e). \tag{A.12}$$

Equations (A.7) and (A.10) give,

$$\frac{\gamma - 3\mu\Delta\alpha}{\rho} - \varphi_1'(\alpha+\Delta\alpha) + \sqrt{3}\,DB(\beta+\Delta\beta) \exp\left\{\frac{C}{\rho\sigma_0}\left[\sigma_m + (3\lambda^* + 2\mu)\left(\Delta\varepsilon_m - \Delta\varepsilon_m^P\right)\right]\right\} = 0. \tag{A.13}$$

The exponential is eliminated between (A.8) and (A.13),

$$\Delta\varepsilon_m^P = \frac{C\Delta\alpha}{3\sigma_0}\left(\varphi_1'(\alpha+\Delta\alpha) + \frac{3\mu\Delta\alpha - \gamma}{\rho}\right), \tag{A.14}$$

and (A.6) is written as

$$\Delta\beta = D\Delta\alpha\sqrt{3}\ \exp\left\{\frac{C}{\rho\sigma_0}\left[\sigma_m + (3\lambda^* + 2\mu)\left(\Delta\varepsilon_m - \Delta\varepsilon_m^P\right)\right]\right\}. \quad (A.15)$$

The substitution of (A.14) for $\Delta\varepsilon_m^P$ into (A.13) and (A.15) and of (A.15) for $\Delta\beta$ into (A.13) yields a *simple algebraic equation for* $\Delta\alpha$ *alone*. The numerical solution of this equation gives $\Delta\alpha$ as a function of σ, α, β and $\Delta\varepsilon$. The initial stress $S = L\Delta\varepsilon^P = (3\lambda^* + 2\mu)\Delta\varepsilon_m^P + 2\mu\Delta e^P$ is then given by (A.12) and (A.14).

Note: With the infinitesimal strain hypothesis, $\rho \equiv 1$. In the above equations the density ρ is given by (A.15) and $\beta + \Delta\beta = \beta(\rho)$ (see Section 3).

References

Auger, J. P. and François, D. (1977), "Variation of Fracture Toughness of a 7075 Aluminum Alloy with Hydrostatic Pressure and Relationship with Tensile Ductility," *Int. J. Fract.*, **13**, 321.

Beremin (1979), *Rapport d'Activité 1979*, Framatome, Paris, TM/C DC/79.095.

Edelson, B. I. and Baldwin, W. M., Jr. (1962), "The Effect of Second Phases on the Mechanical Properties of Alloys," *Trans. Am. Soc. Metals*, **55**, 230.

Gurson, A. L. (1977), "Continuum Theory of Ductile Rupture by Void Nucleation and Growth: Part I—Yield Criteria and Flow Rules for Porous Ductile Media," *J. Engg. Mat. Tech.*, **99**, 2.

Hancock, J. W. and Mackenzie, A. C. (1976), "On the Mechanisms of Ductile Failure in High-strength Steels Subjected to Multi-axial Stress-states," *J. Mech. Phys. Solids*, **24**, 147.

Henry, G. and Hortsmann, D. (1979), *De Ferri Metallographia*, V, Verlag Stahleisen, Düsseldorf.

Kachanov, L. M. (1958), "Time of the Rupture Process under Creep Conditions," *Izv. Akad. Nauk. S.S.R. Otd. Tekh. Nauk.*, **8**, 26.

Lautridou, J. C. and Pineau, A. (1978), private communication, published in Beremin (1979); see above.

Lemaitre, J. and Chaboche, J. L. (1978), "Aspect Phénomènologique de la Rupture par Endommagement," *J. Mécanique Appl.*, **2**, 317.

McClintock, F. A. (1968), "A Criterion for Ductile Fracture by the Growth of Holes," *J. Appl. Mech.*, **35**, 363.

Mandel, J. (1973), "Equations Constitutives et Directeurs dans les Matériaux Plastiques et Viscoplastiques," *Int. J. Solids Struct.*, **9**, 725.

Moreau, J. J. (1970), "Sur les Lois de Frottement de Plasticité et de Viscosité," *C.R. Acad. Sci.*, série A, **271**, 608.

Nguyen, Q. S. (1973), "Contribution à la Théorie Macroscopique de l'Élastoplasticité avec Écrouissage," Thèse Doctorat-ès-Sciences, Paris, CNRS A.O. 9317.

Nguyen, Q. S. and Bui, H. D. (1974), "Sur les Matétiaux Élastoplastiques à Écrouissage Postif et Négatif," *J. Mécanique*, **13**, 321.

Rabotnov, Y. N. (1968), "Creep Rupture," *Proc. XII Int. Cong. Appl. Mech.*, Springer Verlag, Stanford, 342.

Rice, J. R. and Tracey, D. M. (1969), "On the Ductile Enlargement of Voids in Triaxial Stress Fields," *J. Mech. Phys. Solids*, **17**, 201.

Rousselier, G. (1979a), "Contribution à l'Étude de la Rupture des Métaux dans le Domaine de l'Élastoplasticité," Thèse Doctorat-ès-Sciences, Paris, CNRS.

Rousselier, G. (1979b), "Numerical Treatment of Crack Growth Problems," *Advances in Elasto-Plastic Fracture Mechanics*, L. H. Larsson (ed.), Applied Science Publishers, London, 165.