Generalized inverses of increasing functions and Lebesgue decomposition

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Abstract

The reader should be aware of the explanatory nature of this article. Its main goal is to introduce to a broader vision of a topic than a more focused research paper, demonstrating some new results but mainly starting from some general consideration to build an overview of a theme with links to connected problems.

Our original question was related to the height of random growing trees. When investigating limit processes, we may consider some measures that are defined by increasing functions and their generalized inverses. And this leads to the analysis of Lebesgue decomposition of generalized inverses. Moreover, since the measures that motivated us initially are stochastic, there arises the idea of studying the continuity property of this transform in order to take limits.

When scaling growing processes like trees, time origin and scale can be replaced by another process. This leads us to a clock metaphor, to consider an increasing function as a clock reading from a given timeline. This is nothing more than an explanatory vision, not a mathematical concept, but this is the nature of this paper. So we consider an increasing function as a time change between two timelines; it leads to the idea that an increasing function and its generalized inverse play symmetric roles. We then introduce a universal time that links symmetrically an increasing function and its generalized inverse. We show how both are smoothly defined from this universal time. This allows to describe the Lebesgue decomposition for both an increasing function and its generalized inverse.

1 Introduction

Real valued increasing functions can always be inverted, even when they are not bijective, leading to what is most often called a *generalized inverse*, but can also be called pseudo-inverse, quasi-inverse or quantile function. This applies obviously to monotonic functions, but we keep the focus on increasing functions for the sake of simplicity.

The history of the generalized inverse is difficult to draw since it is a well established auxiliary notion that was not considered for a dedicated study. The earliest appearance of the operation is due to Lebesgue (1910) in [13]. This notion has been commonly used in measure and probability theories, but also in other domains where it takes other names. However, with the use of more precise properties, there has been some papers dedicated to generalized inverses. The fuzzy logic domain has made precise definitions such as [12] citing [20] and even earlier work of [18] related to t-norms: in these papers the most general definitions are given at the price of potentially very irregular generalized inverses. In recent years there was a rising interest to clarify the definitions and the properties of generalized inverses as in [7, 10]. None of these papers try to link the measure defined by an increasing function and the one defined by its generalized inverse, so that the result stated in this paper is new as far as we know, as the symmetric description we introduce for that purpose.

Now, we would like to give some motivation for this study. Consider an homogeneous discrete Markov growing tree $T(t)$ as in [9] but without deletion. The dynamics is quite simple: it starts with a unique root $T(0)$ as a node, and then every node gives birth to children with a Poisson law, independently of all other nodes. The number of nodes (i.e. the *size* $Y(t)$ of the tree) is a Yule process, known to grow exponentially (scaling by an exponential factor gives a limiting exponential distribution). The *height* $H(t)$ (i.e. the maximum distance of a node to the root) grows almost linearly. When considering limit (possibly stationary) processes, one can be tempted to look at what happens when the tree reaches the height h (at the stopping time $t_h = \inf_t\{H(t) \geq h\}$) and precisely at the height $H(t_h) = h$, discarding the part of the tree $T(t_h + t)$ whose height is less than h: it becomes a *forest* $T'_h(t)$ (i.e. a set of trees, starting with a single root at $t = 0$ that was the first node of the tree T to reach height h) but this is not a difficulty. This procedure is quite standard to try to build a stationary process (if T'_h is to converge when $h \to \infty$). One difficult part is to compute the time t_h when the tree T reaches height h. Note that this requires to inverse height and time: from $H(t)$, we want to compute $t(h) = t_h$. Both functions are increasing and are generalized inverses of each other, and also both can be seen as timelines.

Now, in our search for limit processes, we could also consider scaling in time and in space: then we have to go for continuous trees, as those beautifully described by Aldous [2, 1, 3]. These trees are linked to Brownian excursions and the construction makes use of the maximum of excursions. The same method can be applied for continuous tree construction directly to a Brownian motion using its minimum and maximum. Again, the generalized inverse appears: the maximum M_t of a Brownian motion B_t on $[0, t]$ is a continuous increasing process that is the generalized inverse of an increasing Lévy process. The intensity of the Lévy process is related to the negative excursions when the Brownian motion reaches its maximum, explaining the Markov property of this generalized inverse. The maximum M_t is not Markovian but can be considered as kind of semi-Markov process, where the remaining time to spend in a state only depends of the time already spent in this state (except at the transition times where the dynamics is more complex). Here the generalized inverse gives a Markovian description for a non-Markov process.

Generalized inverses of increasing functions (hence positive measures) therefore look like an interesting tool to use for some problems related to random processes. There are also other works relating these two measures. In measure theory, the generalized inverse is linked to change-of-variables formulæ for Lebesgue–Stieltjes integrals as in [8]; in probability theory it is linked to the distribution function of a real-valued random variable: the generalized inverse appears naturally to transform a uniform random variable into a random variable with a given distribution function and this technique is widely applied to simulation; for the same reason the generalized inverse is sometimes called *quantile function* and the previous operation a quantile transformation [19]: it is used in statistics (and applied e.g. to insurance and finance). So several measures can be related by using generalized inverses, but, to our knowledge, there is no publication describing this operation without conditions. And so we would like to relate the distribution described by an increasing function (and its Lebesgue decomposition [14]) with the distribution described by its generalized inverse. At the same time, we believe that the concepts built for that purpose can be useful for further analysis, like continuity properties of the transform at a functional level, which is necessary for random measures.

Since we would like this article to be expository, we emphasize the concepts more than the results so the structure of the article is as follows. Section 2 sets up a few basic definitions and properties of increasing functions and generalized inverses. Then we present our clock metaphor in Section 3 where we build a symmetric representation of the pair made by an increasing function and its generalized inverse. Section 4 further refines this representation in order to state a smooth relationship linking the two measures defined by such a pair of functions. Section 5 finally links precisely the two Lebesgue decompositions with Section 6 presenting our conclusion.

2 Generalized inverses

This section presents a few basic concepts and properties on generalized inverses that are needed for further analysis. We also intentionally state and prove a few inequalities between an increasing function and its generalized inverse since even if these inequalities are basic, one can easily find erroneous inequalities.

We consider a function f that is a mapping from $T_1 \subset \mathbb{R}$ onto $T_2 \subset \mathbb{R}$ where T_1 and T_2 are finite closed intervals i.e. $T_1 = [a_1, b_1]$ with $-\infty < a_1 < b_1 < \infty$ and $T_2 = [a_2, b_2]$ with $-\infty < a_2 < b_2 < \infty$.

Throughout the paper we will use the term *increasing* for non-decreasing functions f:

$$x > y \implies f(x) \geq f(y) \tag{2.1}$$

If the right inequality is strict in (2.1) we say f is *strictly increasing*. Properties of increasing functions and their generalized inverses can be lengthy if we consider all the special cases at the endpoints, therefore in the sequel, we will skip these parts of the proofs (e.g. right below, skip the discussion of the existence of right [resp. left] limits at the upper [resp. lower] endpoint of our sets).

Left and right limits always exist for increasing functions and are denoted by

$$f(x-) \overset{\text{def}}{=} \lim_{z\uparrow x} f(z) = \sup_{z<x} f(z), \tag{2.2}$$

$$f(x+) \overset{\text{def}}{=} \lim_{z\downarrow x} f(z) = \inf_{z>x} f(z). \tag{2.3}$$

By definition, f is *right-continuous* [resp. *left-continuous*] at x when $f(x) = f(x+)$ [resp. $f(x) = f(x-)$]. And f is *continuous* at x when it is right and left continuous at x. A *càdlàg function* is a function that is right continuous with left limits. Note that, in the following, we will consider only càdlàg increasing functions.

To ensure correctness of the statements at the endpoints of intervals T_1 and T_2, we assume $f(a_1) = a_2$ and $f(b_1) = b_2$, possibly shifting the endpoints of T_2. In line with the càdlàg property, we take the convention $f(a_1-) = a_2$ and $f(b_1+) = b_2$. Now, if we consider a map f defined on (semi-)infinite intervals, most results can be translated by compactification: $g = \tanh \circ f \circ \tanh^{-1}$ is a map from $T_1' = [\tanh(a_1), \tanh(b_1)]$ into $T_2' = [\tanh(a_2), \tanh(b_2)]$ using the convention $\tanh(-\infty) = -1$ and $\tanh(\infty) = 1$ and by extending f at infinity whenever the corresponding endpoints a_1 or b_1 are: $f(-\infty) = \lim_{-\infty} f$ and $f(\infty) = \lim_{\infty} f$; since f is increasing, these limits always exist (they may be infinite); T_1' and T_2' are finite intervals, $g(a_1) = a_2$ and $g(b_1) = b_2$. The properties of g can be translated to f by the relation $f = \tanh^{-1} \circ g \circ \tanh$, but this requires some care at the endpoints when they go to infinity, using the continuity (by construction) of g at endpoints. This procedure allows also to deal with (possibly) unbounded f on a finite open interval, such as $\tanh^{-1}$ on $(-1, 1)$ that can be extended to $[-1, 1]$.

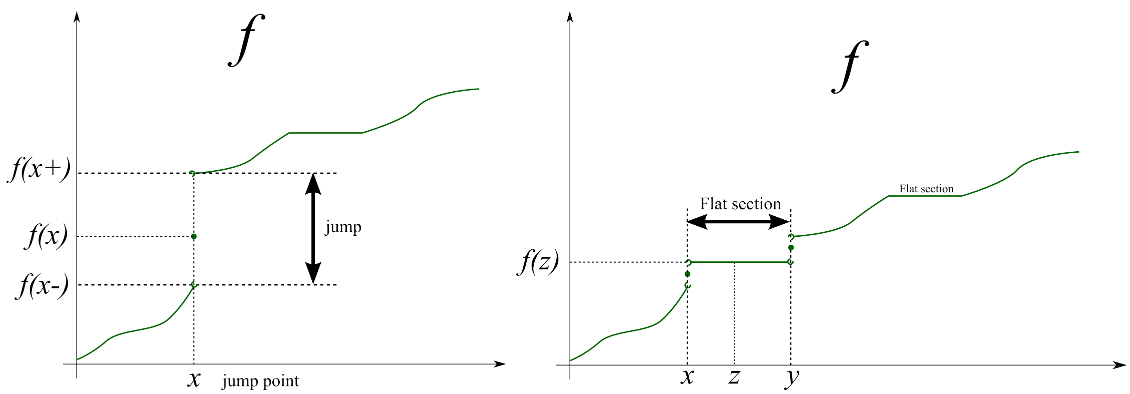

Figure 1: The ambiguity at jumps. Left: Jumps of an increasing function with an illustration of "free" value at a jump. Right: Flat sections of an increasing function and the related concepts: interior (x, y), value of the flat section $f(z)$ and "free" value at the boundaries x and y.

A function f has a *jump* at x when $f(x-) < f(x+)$. This means the value $f(x)$ can be "anywhere" between $f(x-)$ and $f(x+)$ as illustrated by Figure 1. This "ambiguity" plays a role in the definition of generalized inverses since a flat section is transformed into a jump (and conversely) but the choice of the value at the jump is arbitrary; only the left and right limits are non-ambiguously defined. Note that two increasing functions f and g having the same left and right limits (i.e. $f(x-) = g(x-)$ and $f(x+) = g(x+) \quad \forall x \in T_1$) are almost everywhere equal with respect to Lebesgue measure since there is at most a countable number of jumps (where left and right limits differ); the minimal such function is left-continuous and the maximal one is right-continuous. More precisely, this class of functions is equal to the set of all functions greater than the (unique) left-continuous representative and less than its unique right-continuous representative.

This discussion is important to understand that there is an arbitrary decision to make when one defines the generalized inverse: since only left and right limits are unambiguously defined, all generalized inverses are almost surely equal but one can choose any representative in each class. We refer to [6] for more details.

To make this arbitrary decision explicit, consider an increasing function $f : T_1 \to T_2$. Then to say that g is a *generalized inverse* for f means that $g : T_2 \to T_1$ and for each $y \in T_2$, $f(g(y)-) \leq y \leq f(g(y)+)$ (remember by convention $f(a_1-) = a_2$ and $f(b_1+) = b_2$). A generalized inverse for f is automatically increasing. Let $f^\wedge$ and $f^\vee$ be the functions on T_2 defined by $f^\wedge(y) = \sup^{T_1}\{f < y\}$ and $f^\vee(y) = \inf^{T_1}\{f > y\}$, where the notation $\{f < y\}$ is an abbreviation for the set $\{x \in T_1 : f(x) > y\}$, and where $\sup^{T_1}$ and $\inf^{T_1}$ denote the supremum and infimum in T_1 , so that $\sup^{T_1} \emptyset = a_1$ and $\inf^{T_1} \emptyset = b_1$. Then a function g on T_2 is a generalized inverse for f if, and only if, $f^\wedge \leq g \leq f^\vee$. In particular $f^\wedge$ and $f^\vee$ are, respectively, the smallest and the largest generalized inverses for f.

In the next sections, all increasing functions will be chosen to be right-continuous (see Figure 2): this is a consistent choice and it is convenient for our probabilistic background. It goes with the following definition.

Definition 2.1 (Generalized inverse) *Let $f : T_1 \to T_2$ be an increasing function. The generalized inverse $f^\vee$ is defined by*

$$f^\vee(y) \stackrel{\text{def}}{=} \inf{}^{T_1}\{f > y\} \tag{2.4}$$

For the sake of clarity, we use $f^\vee : T_2 \to T_1$ for the generalized inverse, and f^{-1} for the inverse image (as acting on sets).

The generalized inverse has some interesting properties. Lots of them are already demonstrated but we give a fairly exhaustive list hereafter for the sake of completeness and to help following proofs.

Proposition 2.2 (generalized inverse properties) *The generalized inverse $f^\vee$ of an increasing function f has following properties:*

1. *The generalized inverse $f^\vee$ can be alternatively defined by:*

$$f^\vee(y) = \sup{}^{T_1}\{f \leq y\} \tag{2.5}$$

2. *f is right-continuous if, and only if, $\{f \geq y\}$ is closed for all $y \in T_2$;*

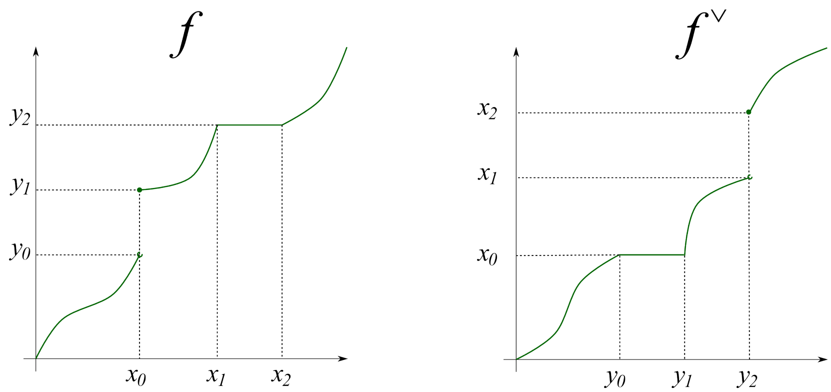

Figure 2: The right-continuous generalized inverse.

3. *$f^\vee$ is increasing, has left limits and is right continuous (càdlàg);*

4. *the following implications hold for all $x \in T_1$ and $y \in T_2$:*

$$
\begin{aligned}
y > f(x) &\implies f^\vee(y-) > x, && (2.6)\\
y \geq f(x-) &\implies f^\vee(y) \geq x, && (2.7)\\
y \leq f(x) &\implies f^\vee(y-) \leq x, && (2.8)\\
y < f(x) &\implies f^\vee(y) \leq x, && (2.9)\\
y < f(x-) &\implies f^\vee(y) < x, && (2.10)
\end{aligned}
$$

5. *$(f^\vee)^\vee = f$;*

6. *$f^\vee(f(x)) \geq x \quad \forall x \in T_1$ and $f(f^\vee(y)) \geq y \quad \forall y \in T_2$;*

7. *f is strictly increasing if, and only if, $f^\vee$ is continuous. Similarly $f^\vee$ is strictly increasing if, and only if, f is continuous.*

Proof : For the first claim, let's consider $y \in T_2$. Define $x_2 = \inf^{T_1}\{f > y\}$ and $x_1 = \sup^{T_1}\{f \leq y\}$: $x_1 \leq x_2$ since f is increasing. Now, by definition of x_2, for all $x < x_2$, $f(x) \leq y$ which leads to $x_1 \geq x_2$ by applying the definition of x_1. Therefore $x_1 = x_2$ and $f^\vee(y) = \inf^{T_1}\{f > y\} = \sup^{T_1}\{f \leq y\}$.

For the second statement, consider the set $A = \{f \geq y\}$ and $x_0 = \inf^{T_1} A$. Note that for any $z \in A$, $z' \geq z$ implies $z' \in A$. Therefore the set is an interval whose upper endpoint is b_1. To prove that A is closed, we need only to prove it includes its lower endpoint x_0. By definition of A, $f(x_0+) \geq y$. If f is right continuous, $f(x_0+) = f(x_0) \geq y$ hence $x_0 \in A$ and A is closed. Hence the right-continuity of f implies the closedness of $\{f \geq y\}$.

Reciprocally assume f is not right-continuous: then there exists z such that $f(z) < f(z+)$. Take $y = f(z+)$ and define A and x_0 as above. Since $f(z) < y$ and f is increasing, $x_0 \geq z$. $f(z+) = y$ implies $f(x) \geq y$ for all $x > z$ therefore $x_0 \leq z$ by definition of A. This proves $z = x_0$. However $z = x_0 \notin A$ because $f(z) < f(z+)$ hence the set A is not closed. By contraposition this proves the closedness of $\{f \geq y\}$ implies the right-continuity of f and the equivalence is proved.

For the third point, note first that the sets $E_y \stackrel{\text{def}}{=} \{f > y\}$, for $y \in T_2$, are decreasing, therefore the infimum $f^\vee(y) = \inf{}^{T_1} E_y$ is increasing. We then have to prove $f^\vee$ is càdlàg. Left and right limits always exist for increasing functions. Since the sets E_y are open and decreasing for $y \in T_2$, we have

$$\bigcup_{z>y} E_z = \bigcup_{z>y} \{x \in T_1 : f(x) > z > y\} = \{x \in T_1 : f(x) > y\} = E_y$$

from which we get

$$f^\vee(y+) = \inf_{z>y} f^\vee(z) = \inf_{z>y} \inf{}^{T_1} E_z = \inf{}^{T_1} \bigcup_{z>y} E_z = \inf{}^{T_1} E_y = f^\vee(y)$$

so that $f^\vee$ is right continuous.

For the fourth point, the implications (2.6)–(2.10) are ordered with respect to the left condition. We will demonstrate first only (2.6), (2.9) and (2.10). The demonstrations of (2.7) and (2.8) are postponed after demonstration of the fifth point.

Equation (2.4) yields directly (2.9).

For (2.6), let $x_n = \min(x + 1/n, b_1)$ and take z such that $f(x) < z < y$. Since f is right-continuous, $f(x_n) \downarrow f(x)$ and there exists n such that $f(x) \leq f(x_n) < z < y$. Now by definition $f^\vee(z) = \sup^{T_1}\{f \leq z\} \geq x_n > x$. Since $f^\vee(z) \leq f^\vee(y-)$, we have $f^\vee(y-) > x$.

For (2.10), assume $y < f(x-)$. Then take $x_n = \max(a_1, x - 1/n)$. By definition of $f(x-)$, and since $y < f(x-)$, there exists[1] n such that $y < f(x_n) < f(x-)$, hence $f^\vee(y) \leq x_n < x$ is a direct consequence of Equation (2.5). Hence (2.10) is demonstrated.

For the fifth point $(f^\vee)^\vee = f$, following the proof above, we need to use only (2.6) and (2.10). Take $x \in T_1$ and apply Definition 2.1:

$$\left(f^\vee\right)^\vee(x) = \inf{}^{T_2}\{f^\vee > x\} = \sup{}^{T_2}\{f^\vee \leq x\} \tag{2.11}$$

The contrapositive of (2.10) yields $f^\vee(z) > x \implies f(x) \leq z$. Therefore the second term in (2.11) is greater than $f(x)$ and $(f^\vee)^\vee(x) \geq f(x)$. Since $f^\vee(y) \geq f^\vee(y-)$, from (2.6) we derive the softer implication $y > f(x) \implies f^\vee(y) > x$. The contrapositive yields $f^\vee(z) \leq x \implies z \leq f(x)$. Applying this to the third term of (2.11) gives us $(f^\vee)^\vee(x) \leq f(x)$. Therefore $(f^\vee)^\vee(x) = f(x)$.

Let's achieve now the demonstration of the Implications (2.7) and (2.8). The equivalences below start with (2.6) which is demonstrated. Then we take the contrapositive and the last

[1] As we mentioned earlier, for the sake of brevity, endpoints of T_1 are not considered in the proofs but do not pose a serious problem; here for example if $x = a_1$ this sequence is constant and cannot be used for the proof, but since $f(a_1-) = a_2$ by convention, there is no $y < a_2$ and the implication still holds.

step is to apply the second implication to $f^\vee$ (and exchange according to this operation x and y) and simultaneously use the fifth point (i.e. $(f^\vee)^\vee = f$). The third implication is (2.7), which proves it. This reads:

$$\begin{aligned}\Big(y > f(x) \implies f^\vee(y-) > x\Big) \quad &\iff \quad \Big(x \geq f^\vee(y-) \implies f(x) \geq y\Big)\\ &\iff \quad \Big(y \geq f(x-) \implies f^\vee(y) \geq x\Big)\end{aligned}$$

And we have a similar way of deriving an equivalence between Implications (2.8) and (2.10).

The sixth point, inequality $f^\vee(f(x)) \geq x$, is a direct consequence of Implication (2.7) by taking $y = f(x)$. The second inequality derives from the first one and the identity $(f^\vee)^\vee = f$.

Let us demonstrate the seventh point by contrapositive. Assume f is discontinuous at point x. Then there exists infinitely many y such that $f(x-) \leq y < f(x)$. Applying Implication (2.7) to the left inequality yields $f^\vee(y) \geq x$ and applying Implication (2.9) to the right inequality yields $f^\vee(y) \leq x$. Therefore $f^\vee(y) = x$ for all those y and $f^\vee$ is not strictly increasing.

Assume now that $f^\vee$ is not strictly increasing. This means there exists $x \in T_1$ and $y_1 < y_2$ such that $f^\vee(y_1) = f^\vee(y_2) = x$. Therefore $f^\vee$ is constant on $[y_1, y_2]$ and $f^\vee(y_2-) = x$. Taking Implication (2.8) applied to $f^\vee$ instead of f, and using $(f^\vee)^\vee = f$, we get $x \leq f^\vee(y_1) \implies f(x-) \leq y_1$. Similarly transforming Implication (2.7) yields $x \geq f^\vee(y_2-) \implies f(x) \geq y_2$. Therefore $f(x-) \leq y_1 < y_2 \leq f(x)$ and f is discontinuous. ■

What we learn from these properties is that f and $f^\vee$ play symmetric roles at a functional level ($(f^\vee)^\vee = f$) but it is not true pointwise ($f^\vee(f(x)) \geq x$). Moreover jumps are transformed into flat sections and conversely: a smooth f may lead to an irregular $f^\vee$ (and conversely); Therefore, the Lebesgue decomposition of the measure $\mathrm{d}f^\vee$ must be related to f in a rather irregular way. So we would like to solve these two questions in the next section: Can we find a representation where f and $f^\vee$ play fully symmetric roles? Can this representation present enough regularities?

3 A symmetric representation

We remind the reader that, guided by our analysis of growing trees, we would like to view T_1 and T_2 as two timelines to consider an increasing function f as the act of reading a clock on T_2 from our own timeline T_1 equipped with its own clock: when our own clock is at time t_1, we read the time t_2 displayed by the second clock and we define the function f by $f(t_1) = t_2$. It is quite intuitive that the symmetric operation would build the generalized inverse as $t_2 = f^\vee(t_1)$. The difficulty lies in frozen times: when f is constant, this means time is frozen on T_2 with respect to T_1. Reciprocally, there is a jump for f when time is frozen on T_1 with respect to T_2. This section presents a symmetric representation of an increasing function and its generalized inverse with names for concepts derived from this clock metaphor.

We call a pair $(t_1, t_2) \in T_1 \times T_2$ an event. The idea is that it could be either of the form $f(t_1) = t_2$ or $t_1 = f^\vee(t_2)$. The fact that clocks never decrease means that events can be totally ordered by the product order in $T_1 \times T_2$ defined by:

$$(t_1, t_2) \leq (t_1', t_2') \iff t_1 \leq t_1' \text{ and } t_2 \leq t_2' \tag{3.1}$$

We use the strict inequality $(t_1, t_2) < (t_1', t_2')$ as usual for $(t_1, t_2) \leq (t_1', t_2')$ and $(t_1, t_2) \neq (t_1', t_2')$ (it means we could have $t_1 = t_1'$ or $t_2 = t_2'$, but not both at the same time).

Now we introduce a new set — in fact a new timeline — that we call a *graph of events.*

Definition 3.1 (Graph of events) *A set* Γ *is said to be a graph of events if* $\Gamma \subset T_1 \times T_2$ *is closed, totally ordered with respect to the product order and if its projections* τ_1 *and* τ_2 *cover* T_1 *and* T_2*:* $\tau_1(\Gamma) = T_1$ *and* $\tau_2(\Gamma) = T_2$.

Note that Γ is closed and bounded, hence compact. This definition imposes $(a_1, a_2) \in \Gamma$ and $(b_1, b_2) \in \Gamma$: since Γ is totally ordered and compact, we can pick its minimal element (a, a'). Then $(t_1, t_2) \in \Gamma \implies (t_1, t_2) \geq (a, a')$ and $\min \tau_1(\Gamma) = a$ hence $a = a_1$, and similarly $a' = a_2$. The same reasoning holds for upper endpoints.

Now, we can derive symmetrically a pair of functions from this graph of events:

Definition 3.2 *Let* Γ *be a graph of events according to Definition 3.1. Then we define a pair of functions* (f_1, f_2) *with* $f_1 : T_1 \to T_2$ *and* $f_2 : T_2 \to T_1$ *by*

$$f_1(t_1) \stackrel{\text{def}}{=} \sup{}^{T_2} \tau_2\Big(\tau_1^{-1}\big(\{t_1\}\big) \cap \Gamma\Big) = \sup{}^{T_2}\Big\{t_2 \in T_2 : (t_1, t_2) \in \Gamma\Big\} \tag{3.2}$$

$$f_2(t_2) \stackrel{\text{def}}{=} \sup{}^{T_1} \tau_1\Big(\tau_2^{-1}\big(\{t_2\}\big) \cap \Gamma\Big) = \sup{}^{T_1}\Big\{t_1 \in T_1 : (t_1, t_2) \in \Gamma\Big\} \tag{3.3}$$

This definition makes sense because by Definition 3.1 $\tau_1^{-1}\big(\{t_1\}\big) \cap \Gamma$ is not empty. And the lowest upper bound of any set within T_2 lies within T_2 because it is compact. Denote by t_2^* this lowest upper bound: this means $f_1(t_1) = t_2^*$. Since Γ is compact, it means $(t_1, t_2^*) \in \Gamma$. Hence $\big(t_1, f_1(t_1)\big) \in \Gamma$ as expected, and symmetrically for f_2.

Proposition 3.3 *The functions* f_1 *and* f_2 *defined as of Definition 3.2 are increasing and càdlàg.*

Proof : Since Γ is totally ordered for the product order by Definition 3.1, this implies straightforwardly that f_1 is increasing. By symmetry all properties for f_1 also hold for f_2 hence both functions are increasing.

Now we have to prove the right-continuity of f_1. For this, consider any decreasing sequence $t_1^n \in T_1$ converging toward $t_1 \in T_1$. The sequence $f_1(t_1^n) \in T_2$ is a decreasing sequence such that $f_1(t_1^n) \geq f_1(t_1)$. This sequence converges and since T_2 is compact, the limit value belongs to T_2: $t_2' = \lim_n f_1(t_1^n) \in T_2$ and we have $t_2' \geq f_1(t_1)$. Now the whole sequence $(t_1^n, f_1(t_1^n)) \in \Gamma$ is converging and since Γ is compact, the limit point also belongs to Γ: $(t_1, t_2') \in \Gamma$. The definition of f_1 in Equation (3.2) implies that $t_2' \leq f_1(t_1)$. Since we also had $t_2' \geq f_1(t_1)$, this means $t_2' = f_1(t_1)$. Which proves that $\lim_n f_1(t_1^n) = f_1(t_1)$ hence f_1 is right-continuous. By swapping the times, this also proves f_2 is right-continuous. Since both functions are increasing, they always admit left limits hence both f_1 and f_2 are càdlàg. ■

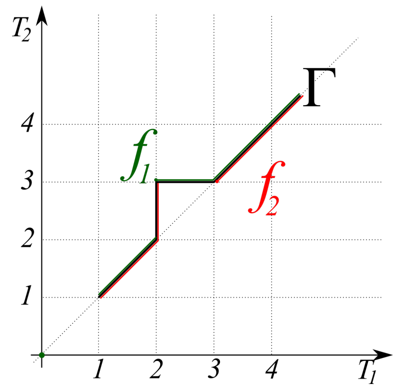

Figure 3: The graph of events Γ and the pair of functions f_1 and f_2 derived from Definition 3.2.

We now show that this symmetrical construction indeed builds a pair of generalized inverses.

Proposition 3.4 *The functions f_1 and f_2 derived from a graph of events Γ as in Definition 3.2 are generalized inverses of each other.*

Proof : Consider $y \in T_2$ and $x \in T_1$ such that $f_1(x) > y$. Since $\big(f_2(y), y\big) \in \Gamma$ and $\big(x, f_1(x)\big) \in \Gamma$, since Γ is totally ordered for the product order, then $f_1(x) > y$, implies $\big(x, f_1(x)\big) > \big(f_2(y), y\big)$, which in turn implies $x \geq f_2(y)$. Hence we have demonstrated that, for any $y \in T_2$,

$$f_1(x) > y \implies x \geq f_2(y) \tag{3.4}$$

and we can derive, taking the infimum of all such x and applying the definition of $f_1^\vee$:

$$\inf{}^{T_1}\{f_1 > y\} = f_1^\vee(y) \geq f_2(y), \quad \forall y \in T_2$$

Now, assume $f_1^\vee(y) > f_2(y)$. Then there exists x such that $f_2(y) < x < f_1^\vee(y)$. The left inequality, combined with the maximality of $f_2(y)$, implies $f_1(x) \geq y$. The right inequality $x < f_1^\vee(y)$ implies $f_1(x) \leq y$ by using Implication (2.9) and $(f_1^\vee)^\vee = f_1$. Therefore necessarily $f_1(x) = y$ and $\big(x, y\big) \in \Gamma$ as well as $\big(f_2(y), y\big) \in \Gamma$. Now the maximality of $f_2(y)$ implies $f_2(y) \geq x$ which is a contradiction, hence our hypothesis is false and $f_1^\vee(y) \leq f_2(y)$. Combining with the previous bound, we get $f_1^\vee(y) = f_2(y)$ for all $y \in T_2$. By symmetry, $f_2^\vee = f_1$ and the proof is concluded. ■

Since usually one starts with an increasing function rather than with a graph of events, we would like to conclude this section by building the graph of events that defines f.

Definition 3.5 *Let $f : T_1 \to T_2$ be an increasing function such that $f(a_1) = a_2$ and $f(b_1) = b_2$. We define the completed graph $\Gamma(f)$ as*

$$\Gamma(f) \stackrel{\text{def}}{=} \big\{(x, y) \in T_1 \times T_2 : f(x-) \leq y \leq f(x+)\big\} \tag{3.5}$$

Note that this definition holds for *any* increasing function f (not necessarily càdlàg). The endpoints condition means T_2 should be taken minimal.

Proposition 3.6 *Let $f : T_1 \to T_2$ be a càdlàg increasing function such that $f(a_1) = a_2$ and $f(b_1) = b_2$. Then the completed graph $\Gamma(f)$ is a graph of events and the functions f_1 and f_2 derived from it by Definition 3.2 verify $f_1 = f$ and $f_2 = f^\vee$.*

Proof : First we prove that $\Gamma(f)$ is totally ordered. Take 2 elements of Γ, $\gamma = (t_1, t_2)$ and $\gamma' = (t'_1, t'_2)$. Without loss of generality, we can assume $t_1 \leq t'_1$. If $t_1 < t'_1$, f is increasing so that $f(t_1+) \leq f(t'_1-)$ hence $t_2 \leq t'_2$ and $\gamma \leq \gamma'$ with respect to product order. If $t_1 = t'_1$, we have either $t_2 \leq t'_2$ or $t_2 \geq t'_2$, hence either $\gamma \leq \gamma'$ or $\gamma \geq \gamma'$. Therefore 2 elements of Γ are always comparable and the product order is a total order on Γ.

Second, we prove that the projections cover T_1 and T_2. By construction, $\tau_1(\Gamma) = T_1$. Take $y \in T_2$ and consider $x = f^\vee(y)$. Now we apply Implications (2.7) and (2.8) to $f^\vee$ and use $(f^\vee)^\vee = f$: using (2.7), $x \geq f^\vee(y-)$ implies $y \leq f(x) = f(x+)$ and using (2.8), $x \leq f^\vee(y)$ implies $f(x-) \leq y$. Therefore $(x, y) \in \Gamma$ and $y \in \tau_2(\Gamma)$. This is true for any $y \in T_2$, hence $\tau_2(\Gamma) = T_2$.

Third, we prove that Γ is closed. Consider a sequence $\gamma_n = (t^1_n, t^2_n) \in \Gamma$ converging to $\gamma = (t^1, t^2)$ (in $T_1 \times T_2$): $\lim_n t^1_n = t^1$ and $\lim_n t^2_n = t^2$. Since f is increasing, left and right limits verify $f(t^1-) \leq \liminf_n f(t^1_n-) \leq \limsup_n f(t^1_n+) \leq f(t^1+)$. Since $\gamma_n \in \Gamma$, $f(t^1_n-) \leq t^2_n \leq f(t^1_n+)$ and $\liminf_n f(t^1_n-) \leq \lim_n t^2_n \leq \limsup_n f(t^1_n+)$. Therefore $f(t^1-) \leq t^2 \leq f(t^1+)$ and $\gamma \in \Gamma$. This proves Γ is closed.

Consider f_1 and f_2 derived from $\Gamma(f)$ by Definition 3.2. Let $x \in T_1$: the set $\tau_1^{-1}\big(\{x\}\big) \cap \Gamma(f) = [f(x-), f(x+)]$ by definition of $\Gamma(f)$. Therefore $f_1(x) = f(x+) = f(x)$ because f is càdlàg. And $f_2 = f_1^\vee = f^\vee$. ∎

This graph is unique in the sense that there is no other graph of events such that $f_1 = f$ but for the sake of brevity, we omit a demonstration of this property. It is linked to the fact that $\Gamma(f)$ can be also defined as the union of all points $(x, f(x))$ for $x \in T_1$ and $(f^\vee(y), y)$ for $y \in T_2$, i.e. by all events, as we called them in the introduction of this section. And uniqueness means any graph of events is made only of events. Uniqueness also implies that, starting from a graph of events Γ and deriving (f_1, f_2), we have $\Gamma(f_1) = \Gamma$. $\Gamma(f^\vee)$ is also well defined but it is a subset of $T_2 \times T_1$ and the symmetric of $\Gamma(f) \subset T_1 \times T_2$.

To conclude, Proposition 3.6 shows that we lose no generality by starting all theorems with a graph of events so that all statements are symmetric.

4 A smooth representation

In the previous section, we have built the graph of events $\Gamma \subset T_1 \times T_2$ seen as a symmetric way to define a pair of increasing functions (f_1, f_2) that are generalized inverses of each other. Now, we can go further into the metaphor: we could very well create a clock on Γ, each event defining an instant on Γ. Then, reading the clocks T_1 and T_2 from this time —

that we call universal time — would be simple since events contain all the information on both clocks T_1 and T_2. And clearly this clock would never stop, and this means smoothness.

In this section we introduce the concept of *universal time*, extending times T_1 and T_2, which is defined as the graph of events $\Gamma \subset \mathbb{R}^2$ itself equipped with the right topology and therefore seen as a closed interval within $\mathbb{R}$. Furthermore, the times T_1 and T_2 can be seen as projections of this universal time, leading to a symmetric decomposition of a pair of generalized inverses using these projections, which are by nature very regular (1-Lipschitz).

Consider the natural distance d for Γ, as the trace of L_1 norm of $\mathbb{R}^2$:

$$d(\gamma, \gamma') = |t_2' - t_2| + |t_1' - t_1| = \|\gamma' - \gamma\|_{T_1 \times T_2}$$

for any $\gamma = (t_1, t_2)$ and $\gamma' = (t_1', t_2')$.

We define the mapping $h : \mathbb{R}^2 \to \mathbb{R}$ as $h(\gamma) = t_1 + t_2$. Since h is continuous and Γ is closed in $\mathbb{R}^2$, then $T = h(\Gamma) \subset \mathbb{R}$ is closed. Now we consider h as a mapping from Γ equipped with distance d. By construction, and since Γ is totally ordered (this is used in the left-hand side), we have

$$|h(\gamma') - h(\gamma)| = \|\gamma' - \gamma\|_{T_1 \times T_2} = d(\gamma, \gamma') \tag{4.1}$$

and h is an isometry, therefore a bijection. Note that Γ being totally ordered, any $\gamma \neq \gamma'$ can be compared and either $\gamma < \gamma'$ or $\gamma' < \gamma$; moreover it is straightforward to see that this order is preserved by h: $\gamma < \gamma' \iff h(\gamma) < h(\gamma')$. What is to be proved now is that T is the closed interval $[a_1 + a_2, b_1 + b_2]$.

The 2 endpoints of T are $a = a_1 + a_2$ and $b = b_1 + b_2$ by construction and belong to T: this is due to Definition 3.1 implying $(a_1, a_2) \in \Gamma$ and $(b_1, b_2) \in \Gamma$. Let us prove there is no hole in T. Assume there exists $a < x_0 < b$ such that $x_0 \notin T$. Since T is closed, we can consider the maximal open interval (x, x') containing x_0, hence non-empty: $x, x' \in T$ and for any $x'' \in (x, x')$, $x'' \notin T$. Now, since $x, x' \in T$, there exist $\gamma < \gamma' \in \Gamma$ such that $h(\gamma) = x$ and $h(\gamma') = x'$. Write $\gamma = (t_1, t_2)$ and $\gamma' = (t_1', t_2')$ and consider $t_1'' = (t_1 + t_1')/2$: by Definition 3.2 we can define $t_2'' = f_1(t_1'')$ such that $\gamma'' = (t_1'', t_2'') \in \Gamma$. Since $\gamma < \gamma'' < \gamma'$, we have $x = h(\gamma) < h(\gamma'') < h(\gamma') = x'$ with $h(\gamma'') \in T$: this is a contradiction, hence our assumption was false and all $a < x_0 < b$ belong to T, hence T is a closed interval.

Since $T \subset \mathbb{R}$ is a closed interval, define μ as the uniform measure (i.e. Lebesgue's measure) on it and the measure μ_Γ on Γ:

$$\mu_\Gamma = \mu \circ h$$

Note that μ_Γ is also the uniform measure on Γ for the distance d.

The 2 projections τ_1 [resp. τ_2] from Γ onto T_1 [resp. T_2] verify $h = \tau_1 + \tau_2$, hence we have

$$\mu_\Gamma = \mathrm{d}h = \mathrm{d}\tau_1 + \mathrm{d}\tau_2$$

Here we have to clarify the expressions $\mathrm{d}\tau_1$ (and $\mathrm{d}\tau_2$ and $\mathrm{d}h$). We have defined càdlàg increasing functions, and the related concepts, on $\mathbb{R}$, not from or onto Γ which is a subset of $\mathbb{R}^2$, as $\tau_1 : \Gamma \to T_1$. In order to extend the definitions, we have to use $T \subset \mathbb{R}$, that is a closed interval, isomorphic by h to Γ, hence Γ can be considered as a real closed interval. Therefore a function like $\tau_1 : \Gamma \to T_1$ is increasing because $\gamma < \gamma' \implies \tau_1(\gamma) \leq \tau_1(\gamma')$ but we could also define it purely with real numbers by stating it is increasing because $h(\gamma) < h(\gamma') \implies \tau_1(\gamma) \leq \tau_1(\gamma')$ which is equivalent since $h(\gamma) < h(\gamma') \iff \gamma < \gamma'$.

Similarly, we can extend the definition of the generalized inverse by extending Equation (2.4) to get $\tau_1^\vee : T_1 \to \Gamma$ or by extending it directly using the isomorphism h:

$$\tau_1^\vee(x) \overset{\text{def}}{=} \inf{}^\Gamma \big\{\tau_1 > x\big\} = h^{-1}\Big(\inf{}^T \big\{\tau_1 \circ h^{-1} > x\big\}\Big) = h^{-1} \circ (\tau_1 \circ h^{-1})^\vee(x)$$

Note that the second term of the above equalities shows that h acts like an identity and there is no dependence on h. We can further extend the differentiation of càdlàg increasing functions to produce measures, like $\mathrm{d}\tau_1$. Consider the function $\tau_1 \circ h^{-1} : T \to T_1$: it is clearly a real increasing càdlàg function (more precisely 1-Lipschitz) so that the positive measure $\mathrm{d}\big(\tau_1 \circ h^{-1}\big)$ is well defined. Now we can translate this measure into a measure on Γ that we define as $\mathrm{d}\tau_1$:

$$\mathrm{d}\tau_1 \overset{\text{def}}{=} \mathrm{d}\big(\tau_1 \circ h^{-1}\big) \circ h$$

The same process defines $\mathrm{d}\tau_2$ and $\mathrm{d}h$ on Γ. All these measures are very regular, since they are defined by isomorphisms and 1-Lipschitz functions. Note that, again, $\mathrm{d}\tau_1$ and $\mathrm{d}\tau_2$ do not depend upon h since it is also perfectly defined by

$$\mathrm{d}\tau_i\big((\gamma, \gamma']\big) \overset{\text{def}}{=} \tau_i(\gamma') - \tau_i(\gamma) \qquad i \in \{1, 2\} \quad \forall \gamma < \gamma' \tag{4.2}$$

The previous demonstrations prove the following proposition.

Proposition 4.1 (Universal time decomposition) *Let Γ be a graph of events in $T_1 \times T_2$. Then Γ is isomorphic to a closed interval and the corresponding uniform measure μ_Γ can be decomposed as*

$$\mu_\Gamma = \mathrm{d}\tau_1 + \mathrm{d}\tau_2 \tag{4.3}$$

where τ_1 [resp. τ_2] is the canonical projection from Γ onto T_1 [resp. T_2] and $\mathrm{d}\tau_1$, $\mathrm{d}\tau_2$ are defined by (4.2).

Note that this property can also be expressed in terms of Radon-Nikodym derivatives:

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}\mu_\Gamma} + \frac{\mathrm{d}\tau_2}{\mathrm{d}\mu_\Gamma} = 1 \tag{4.4}$$

Definition 3.2 together with Proposition 3.4 demonstrate that the 2 projections τ_1 and τ_2 are sufficient to define an increasing function and its generalized inverse from a graph of events. More precisely we have the following decomposition.

Proposition 4.2 (Increasing functions decomposition) *Let Γ be a graph of events in $T_1 \times T_2$, τ_1 [resp. τ_2] the canonical projection from Γ onto T_1 [resp. T_2] and (f_1, f_2) the pair* *of symmetrically generalized inverses that it defines by Definition 3.2. Then we have the following decomposition*

$$f_1 = \tau_2 \circ \tau_1^{\vee} \tag{4.5}$$
$$f_2 = \tau_1 \circ \tau_2^{\vee} \tag{4.6}$$

This proposition is a straightforward consequence of the expression of the generalized inverses of the projections:

$$\tau_1^{\vee}(x) = \big(x, f_1(x)\big) \tag{4.7}$$
$$\tau_2^{\vee}(y) = \big(f_2(y), y\big) \tag{4.8}$$

To prove Equation (4.7), consider the definition

$$\tau_1^{\vee}(x) \stackrel{\text{def}}{=} \inf{}^{\Gamma}\big\{\tau_1 > x\big\}$$

Clearly, $\big(x, f_1(x)\big) \leq \tau_1^{\vee}(x)$. Now, take $x_n \downarrow x$. Since f_1 is càdlàg, $f_1(x_n) \downarrow f_1(x)$ and $\big(x_n, f_1(x_n)\big) \downarrow \big(x, f_1(x)\big)$, so that $\big(x, f_1(x)\big) \geq \tau_1^{\vee}(x)$. Hence Equation (4.7) is proved, and Equation (4.8) by symmetry.

The beauty of this decomposition — besides its symmetry — is that τ_1 and τ_2 are very regular increasing functions since they are 1-Lipschitz and their Lebesgue decomposition have only absolutely continuous parts. Therefore we can deduce that the singular part (including jumps) in an increasing function comes from the generalized inverse $\tau_1^{\vee}$ (or $\tau_2^{\vee}$). This is the focus of the next section.

5 Lebesgue decomposition of generalized inverses

In the previous sections, we have shown that an increasing function and its generalized inverse can be considered as a pair of symmetric functions and that these functions can be smoothly expressed from an intermediary set, the universal time. It leads to a symmetric decomposition of the pair of generalized inverses using smooth functions that are projections from the universal time. Now we would like to understand how these tools may apply to the measures defined by those increasing functions. In particular, we would like to link the irregularities (the singular parts of the measures) to this decomposition.

From our clock metaphor, we believe the irregularities are due to frozen times. However, while this is quite simple for jumps (in one-to-one correspondence with flat sections), it proves to be more difficult for the singular continuous part of Lebesgue decomposition, and it requires a precise analysis.

First, we define our measures. Let Γ be a graph of events in $T_1 \times T_2$ and (f_1, f_2) the pair of symmetrically generalized inverses that it defines by Definition 3.2 and follow the notation of Proposition 4.2. We denote by μ_1 [resp. μ_2] the uniform measure on T_1 [resp. T_2]. First, we would like to note that

$$\mu_1 = \mathrm{d}\tau_1 \circ \tau_1^{-1} \tag{5.1}$$

and similarly for μ_2. Note that, as defined earlier, τ_1^{-1} is the inverse image, as acting on sets, not a proper inverse function. This comes directly from Equation (4.2). Now, let's turn to the measure $\mathrm{d}f_1$. First, we need the following technical formula, valid for all $t_1 < t_1' \in T_1$:

$$\tau_1^{-1}\big((t_1, t_1']\big) = \big(\tau_1^{\vee}(t_1), \tau_1^{\vee}(t_1')\big]$$

Consider the right endpoint t_1': $\tau_1^{\vee}(t_1') = \big(t_1', f_1(t_1')\big)$ therefore $\tau_1^{\vee}(t_1') \in \tau_1^{-1}\big((t_1, t_1']\big)$. But the maximality stated in Definition 3.2 implies that for all $\gamma \in \Gamma$ such that $\gamma > \tau_1^{\vee}(t_1')$, $\tau_1(\gamma) > t_1'$ hence the lowest upper bound of $\tau_1^{-1}\big((t_1, t_1']\big)$ is $\tau_1^{\vee}(t_1')$ and is included.

For the left endpoint, one verifies that $\tau_1^{\vee}(t_1) = \big(t_1, f_1(t_1)\big) \notin \tau_1^{-1}\big((t_1, t_1']\big)$. Then, considering a sequence $x_n \in (t_1, t_1']$ decreasing toward t_1, using the càdlàg property of $\tau_1^{\vee}$, $\tau_1^{\vee}(x_n) \downarrow \tau_1^{\vee}(t_1)$ and $\tau_1^{\vee}(t_1)$ is the left endpoint but is excluded.

Using this property, we can compute

$$\begin{aligned} \mathrm{d}f_1\big((t_1, t_1']\big) &= f_1(t_1') - f_1(t_1) = \tau_2 \circ \tau_1^{\vee}(t_1') - \tau_2 \circ \tau_1^{\vee}(t_1) \\ &= \mathrm{d}\tau_2\Big(\big(\tau_1^{\vee}(t_1'), \tau_1^{\vee}(t_1)\big]\Big) = \mathrm{d}\tau_2\Big(\tau_1^{-1}\big((t_1, t_1']\big)\Big) \end{aligned}$$

This proves that the decomposition of Proposition 4.2 is consistent for our purpose:

Proposition 5.1 (Measure decomposition) *Using the notation of Proposition 4.2, the measures associated with f_1, and f_2 can be decomposed into*

$$\mathrm{d}f_1 = \mathrm{d}\tau_2 \circ \tau_1^{-1} \tag{5.2}$$

$$\mathrm{d}f_2 = \mathrm{d}\tau_1 \circ \tau_2^{-1} \tag{5.3}$$

This relates, as we mentioned before, the regularity (or singularity) of $\mathrm{d}f_1$ with respect to μ_1 to τ_1^{-1}. And by considering Equations (5.1) and (5.2) one would like to write the Radon-Nikodym densities as $\mathrm{d}f_1/\mathrm{d}\mu_1 = \mathrm{d}\tau_2/\mathrm{d}\tau_1$; Even if this is neither elegant nor correct, it gives the idea: singularities on $\mathrm{d}f_1$ (w.r.t. μ_1) happen when times increases along T_2 but is stopped along T_1. Extending classical sets defined for the proof of Radon-Nikodym theorem (see, e.g. [16, 14, 15, 17]), we can define:

Definition 5.2 *Using the notation of Proposition 4.1 and Equation (4.4), a graph of events Γ can be partitioned into 3 sets*

$$\Gamma = F_1 \biguplus R \biguplus F_2 \tag{5.4}$$

with

$$F_1 = \left\{\frac{\mathrm{d}\tau_1}{\mathrm{d}\mu_\Gamma} = 1\right\} = \left\{\frac{\mathrm{d}\tau_2}{\mathrm{d}\mu_\Gamma} = 0\right\} \tag{5.5}$$

$$R = \left\{0 < \frac{\mathrm{d}\tau_1}{\mathrm{d}\mu_\Gamma} < 1\right\} = \left\{0 < \frac{\mathrm{d}\tau_2}{\mathrm{d}\mu_\Gamma} < 1\right\} \tag{5.6}$$

$$F_2 = \left\{\frac{\mathrm{d}\tau_1}{\mathrm{d}\mu_\Gamma} = 0\right\} = \left\{\frac{\mathrm{d}\tau_2}{\mathrm{d}\mu_\Gamma} = 1\right\} \tag{5.7}$$

Note that these sets are defined μ_Γ-almost surely, since the Radon-Nikodym derivatives are so defined. To be sure of the equalities, we have to exhibit two Radon-Nikodym derivatives $\mathrm{d}\tau_1/\mathrm{d}\mu_\Gamma$ and $\mathrm{d}\tau_2/\mathrm{d}\mu_\Gamma$ such that Equations (5.5)–(5.7) hold. Take any Radon-Nikodym derivatives $\mathrm{d}\tau_1/\mathrm{d}\mu_\Gamma$. By definition, $0 \le \mathrm{d}\tau_1/\mathrm{d}\mu_\Gamma \le 1$ μ_Γ-almost surely. Hence we can build a regularization of this function such that these inequalities hold in all points of Γ, e.g. $\min(\max(\mathrm{d}\tau_1/\mathrm{d}\mu_\Gamma, 0), 1)$. By defining $\mathrm{d}\tau_2/\mathrm{d}\mu_\Gamma = 1 - \mathrm{d}\tau_1/\mathrm{d}\mu_\Gamma$, we have build these two representatives that verify Equations (5.5)–(5.7). Since the sets $\{0\}$, $(0,1)$ and $\{1\}$ form a partition of $[0,1]$, the 3 sets F_1, R and F_2 define also a partition, i.e. Equation (5.4).

Now we have clearly the following properties for these sets:

$$\mathrm{d}\tau_2(F_1) = 0 \tag{5.8}$$
$$\mathrm{d}\tau_1(F_2) = 0 \tag{5.9}$$
$$\mathrm{d}\tau_1(A) > 0 \iff \mathrm{d}\tau_2(A) > 0, \qquad \forall A \subset R \tag{5.10}$$

Now, we can use the partition (5.4) in order to define a decomposition for $\mathrm{d}f_1$

$$\nu_1^s \stackrel{\text{def}}{=} \left(\mathrm{d}\tau_2 \mathbb{1}_{\{F_2\}}\right) \circ \tau_1^{-1} \tag{5.11}$$
$$\nu_1^r \stackrel{\text{def}}{=} \left(\mathrm{d}\tau_2 \mathbb{1}_{\{R\}}\right) \circ \tau_1^{-1} \tag{5.12}$$

Since $\mathrm{d}\tau_2 \mathbb{1}_{\{F_1\}} = 0$, we clearly have $\mathrm{d}f_1 = \nu_1^s + \nu_1^r$ where ν_1^s is the singular part of f_1 with respect to μ_1 and ν_1^r is the regular part. Now we can further decompose the singular part into a jump part and the purely singular part of Lebesgue decomposition.

By projection onto T_1 and T_2, one can define from F_1, R and F_2 new sets as follows. $O_1' = \tau_1(F_1)$ is a generalization of the flat sections for f_1 since $\mathrm{d}f_1 = 0$ on F_1. $S_1' = \tau_1(F_2)$ is a singular part for f_1 since $\mu_1(S_1') = 0$ but $\mathrm{d}f_1$ may increase. And $R_1 = \tau_1(R)$ is a regular part. We can refine the set $O_1' = O_1 \biguplus K_1$ into flat sections O_1 (the interior of O_1') and a purely singular set K_1 (possibly of positive measure but with no interior) where $\mathrm{d}f_1 = 0$, i.e. f_1 is flat. And $S_1' = J_1 \biguplus S_1$ where J_1 is made of the jump points of f_1 and S_1 supports the purely singular part of f_1 (but may be not closed, hence is not generally the support). One defines similarly J_2, S_2, O_2, K_2 and R_2 on T_2.

Note that O_1', S_1' and R_1 do not form a partition of T_1 since τ_1 is not one-to-one. To build it as a partition is feasible, by considering $S_1' \setminus O_1'$ and $R_1 \setminus (O_1' \cup S_1')$. Now, if we want O_1 to really be the flat sections (i.e. the maximal open sets where $\mathrm{d}f_1 = 0$), we need, before the previous operation, to consider a careful regularization of the Radon-Nikodym derivatives in order to avoid some "holes" in O_1 (we could even have a countable number of such holes, leading to an empty O_1 even though there are flat sections).

What is interesting, is that these sets are associated with measures by the generalized inverses: F_1 and O_1' are related to the singular parts of $\mathrm{d}f_2$; F_2 and S_1' which define the singular part of $\mathrm{d}f_1$ are associated to O_2'. Finally R_1 and R_2 are associated and correspond to the regular parts. It shows that we cannot directly establish the correspondence between measures, but we rather need to associate sets and measures. This was also clearly emphasized by previous studies that demonstrate that the jumps J_2 of f_2 (i.e. a discrete measure) are in one-to-one correspondence with the flat sections O_1 of f_1 (i.e. a set).

Proposition 5.3 (Lebesgue decomposition of generalized inverses) *Let T_1 and T_2 be 2 intervals and Γ be a graph of events in $T_1 \times T_2$. Using the notation of Proposition 5.1 and the partition of Definition 5.2, one can define by (5.11)–(5.12) the singular measure ν_1^s and the regular measure ν_1^r (and similarly for f_2) so that 2 increasing functions f_1 and f_2 satisfy:*

$$\mathrm{d}f_1 = \mathrm{d}\tau_2 \circ \tau_1^{-1} = \nu_1^s + \nu_1^r \tag{5.13}$$

$$\mathrm{d}f_2 = \mathrm{d}\tau_1 \circ \tau_2^{-1} = \nu_2^s + \nu_2^r \tag{5.14}$$

Moreover, T_1 can be partitioned into a countable set J_1 of jumps (for f_1), a singular set S_1 of null measure for the purely singular part of $\mathrm{d}f_1$, flat sections O_1 (a disjoint union of open intervals), a purely singular set K_1 with empty interior (but may have positive measure) on which $\mathrm{d}f_1 = 0$ and a regular set R_1 such that $\nu_1^r \mathbb{1}_{\{R_1\}} = \nu_1^r$. Similarly for T_2.

From this decomposition, we can also get the following results. Note that, except for the jumps sets J_i and flat sections O_i, these sets are not uniquely defined (since the Radon-Nikodym derivatives are not uniquely defined).

- The jumps of f_1 are in bijection with flat sections O_2 of f_2 and symmetrically;
- The purely singular part of f_1 is associated to the purely singular set K_2 and symmetrically;
- The strictly increasing absolutely continuous part of f_1 on R_1 is associated to the strictly increasing absolutely continuous part of f_2 on R_2.

Note that there exists another approach to build these sets, without using the Radon-Nikodym theorem: it is another proof to the Lebesgue decomposition due to James K. Brooks [4, 5]. For the sake of beauty we give it here, with simplified notation, but it translates easily to our decomposition. Take μ_1 and μ_2 to be finite measures. The collection of measurable sets A with $\mu_2(A) = 0$ is closed under countable unions, so it has an element F_1 of largest μ_1-measure. Then on $X \setminus F_1$, we have $\mu_1 \ll \mu_2$. The collection of measurable sets B with $\mu_1(B) = 0$ is closed under countable unions, so it has an element F_2 of largest μ_2-measure. Then on $X \setminus F_2$, we have $\mu_2 \ll \mu_1$. Now $(\mu_1 + \mu_2)(F_1 \cap F_2) = 0$, so we may take F_1 and F_2 to be disjoint. Let $R = X \setminus (F_1 \cup F_2)$. Then for each measurable set $C \subset R$, we have $\mu_1(C) > 0 \iff \mu_2(C) > 0$. And we have defined the 3 sets F_1, F_2 and R of Proposition 4.2.

The novelty of this result, besides the decomposition of Proposition 4.2, lies in the fact that we have clearly related the singular part of Lebesgue decomposition with *frozen times* for the generalized inverse: one can interpret it by saying that the singular increases are related to universal times when the clock T_1 is stopped while T_2 goes on ticking. But it is much more complex than the jumps and the flat sections (that was known) since it leads to the definition of a rather singular set (of positive measure but no interior) where f_1 can be considered as flat.

This result extends two previous known facts. We already mentioned the first: jumps are in correspondence with flat sections. The second property is the characterization of real increasing diffeomorphisms; Consider an increasing bijection f (hence $f^\vee$ is also an increasing bijection): f and $f^\vee$ are diffeomorphisms if, and only if, f and $f^\vee$ are differentiable and their derivatives are strictly positive everywhere; this translates to the condition $\Gamma = R$ in our settings. Note that our result shows that we can extend this result the following way: if f_1 is purely singular and its support covers T_1 (i.e. $J_1 = O_1 = R_1 = \emptyset$ and K_1 is equal to T_1 modulo null set S_1), then f_2 is also purely singular and its support covers T_2; moreover the 2 functions are bijective (with null derivatives almost everywhere, hence not a diffeomorphism).

6 Conclusion

In this article, we have shown the symmetrical relationship between an increasing function and its generalized inverse and how we can exploit it to relate the measures defined by such functions. After carefully defining the objects and basic properties in Section 2, we describe both functions in Section 3 as a pair symmetrically defined from a new set that we call universal time. Using the intermediary functions (projections, hence smooth), we build in Section 4 a symmetrical representation of both measures. This leads finally in Section 5 to a description of the relationship between the Lebesgue decomposition of both measures.

This article shows also a metaphor guiding our construction. Though it is not necessary, we believe it can help the reader to follow the progress of the construction. Moreover, this metaphor stems from related topics, like the link between trees and random walks. This could also be an opportunity for the reader to get to these topics or have a renewed interest into them. Indeed, the purpose of this article is not only to give an introduction to a point of analysis, but much more to open perspectives.

We would like to mention a few potential topics to go further. A first point could be to look at the question of several timelines, not only 2. Equations (4.5)–(4.6) could be clearly generalized for n indexes, using an n-dimensional universal time. In our example of Aldous' continuum random tree, this could link the height of several branches. Maybe we could even relate the height of all branches if n could be infinite.

Another topic could be to refine the description of the relationship between measures as in Proposition 5.3. If we have a converging sequence of increasing functions, how would the measures defined by the generalized inverse converge? How would this convergence appear on the universal time? Since we can shift time and space, and since functions are càdlàg, is it linked with Skorokhod metric?

We could also consider convergence of discrete structures (e.g. discrete trees) and so study increasing functions on more general sets than a closed interval, e.g. closed sets since they encompass both discrete sets (as $\mathbb{Z}$) and closed intervals. When we do so, a new phenomenon occurs: their may be "gaps" in the timelines, where time necessarily "jumps" (not relatively to another timeline as the frozen times of an increasing function we have considered in this paper, but the clock itself jumps, as read from its own timeline). The method developed in this paper can be generalized to such times, but there are more ambiguities in the definition of generalized inverses: $\Gamma(f)$ may not be unique and this cannot be solved by regularization, like chosing a càdlàg representative. This difficulty is solved if one starts with the graph of events, hence its centrality in this article. These ambiguities can be related to ordering questions in discrete events systems, and such an extension could give a better insight of combinations of discrete and continuous dynamical systems.

Finally we mention a last topic, taken from our introduction. Some processes — as the maximum of the Brownian motion — can be turned into a Markov process by considering their generalized inverse. Is this Markov property readable on the universal time, and the loss of this property would be an "accident" of the projections? Conversely, can we start from a Markov process (in time) to define for the generalized inverse a Markov property in space that would define extensions of semi-Markov processes? How would that relate to space and time Markov processes, such as the discrete growing trees that motivated us initially?

These questions do not have the same level of complexity or interest. They are formulated in the hope of stimulating the imagination of the reader at the end of this expository article, the same way I have been challenged some time ago by Guy Fayolle during my PhD and after. This paper represents a modest contribution to thank Guy for his vision of probability theory and mathematics that he shared with me. And even more, I would like to express my gratitude to Guy for his contagious cheerful energy.

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