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Abstract

We extend the theory of Kazantzis-Kravaris/Luenberger (KKL) observers. These observers consist in immersing the system into a linear stable filter of the output with sufficiently large dimension and appropriate structure. After discussing the uniqueness of such an immersion, we provide two main results about its existence. The first one extends a known existence result by generalizing the structure of the target linear filter and reducing its dimension. While this approach relies on a generic choice of a sufficiently large number of distinct eigenvalues in the filter, we then propose a second existence result in the novel symmetric case where instead, the target filter is a cascade of a sufficiently large number of one-dimensional filters sharing the same eigenvalue. Finally, we propose a new cascaded procedure for the design of KKL observers. This method can be used in two ways: either to pre-filter a noisy output before using it in the observer, or to simplify the construction of the observer when the system can be written as the cascade of a nonlinear system and a linear one.

Key words: observers, nonlinear systems, KKL observers, Luenberger observers

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1 Introduction

The synthesis of observers is a standard problem in control and automation. Over the last four decades, many methods have been developed allowing the design of these estimation algorithms. The interested reader can refer to [6] which is a survey on the various methods allowing to design such algorithms for nonlinear dynamical systems. Among these listed methods, the *Kazantzis-Kravaris/Luenberger* (KKL) approach or *Nonlinear Luenberger approach* initially developped in [28,17,18,4] is one of the most powerful one from a theoretical point of view. Indeed, the so-called *backward-distinguishability* assumption guaranteeing its existence is very weak and does not require any particular normal form.

When D. Luenberger published his first results concerning the design of observers for linear systems in [20], his idea was to look for a linear change of coordinates T transforming the linear plant dynamics

$$\dot{x} = Fx\,, \qquad y = Hx\,,$$

with state x in $\mathbb{R}^n$, output y in $\mathbb{R}$, and F and H matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{1\times n}$ respectively, into a form

$$\dot{z} = Az + By \tag{1}$$

with z in $\mathbb{R}^n$ and A Hurwitz, for which a trivial observer is simply made of a copy of its dynamics. Indeed, since Tx verifies (1), the estimation error $e = z - Tx$ for any solution z of (1) evolves along the contracting dynamics $\dot{e} = Ae$, so that any solution z converges to Tx. It follows that an estimate $\hat{x}$ of x can be obtained from z by inverting the transformation T. Lunberger proved that when the pair (F, H) is observable, this is always possible for any Hurwitz matrix A in $\mathbb{R}^{n\times n}$ with no common eigenvalues with F, and any vector B in $\mathbb{R}^n$ such that the pair (A, B) is controllable. This is based on the fact that the Sylvester equation

$$TF = AT + BH \tag{2}$$

ensuring that Tx follows (1) admits in this case a solution that is unique and invertible.

Some researchers have then tried to reproduce Luenberger's methodology on nonlinear systems in the form

$$\dot{x} = f(x) \quad , \quad y = h(x)\ , \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are two sufficiently smooth functions. Following [28,17,18,4], a *nonlinear Luenberger* observer or *Kazantzis-*

Kravaris/Luenberger (KKL) observer is a dynamical system of the form

$$\dot{z} \;=\; A\,z \;+\; By \quad , \quad \hat{x} \;=\; T^{\text{inv}}(z)\;, \tag{4}$$

with state z in $\mathbb{R}^m$ (or $\mathbb{C}^m$), a Hurwitz matrix A in $\mathbb{R}^{m\times m}$ (or $\mathbb{C}^{m\times m}$), a vector B in $\mathbb{R}^m$ such that the pair (A,B) is controllable and $T^{\text{inv}}:\mathbb{R}^m\to\mathbb{R}^n$ a continuous map.
Given $\mathcal{X}$ an open and bounded subset of $\mathbb{R}^n$ containing the system trajectories of interest, following [28,17,18,4], the motivation for this structure is to design the mapping T^{inv} as a continuous left inverse [1] of a C^1 mapping [2] $T:\mathcal{X}\to\mathbb{R}^m$, i.e., verifying

$$T^{\text{inv}}(T(x)) = x\ ,\ \forall x\in\mathcal{X}\ , \tag{5}$$

for some T satisfying

$$\frac{\partial T}{\partial x}(x)f(x) = AT(x) + Bh(x)\ ,\ \forall x\in\mathcal{X}\ . \tag{6}$$

Indeed, (6) is a direct extension of the Sylvester equation (2) and says that along trajectories $t\mapsto x(t)$ of system (3) remaining in $\mathcal{X}$, $T(x)$ is solution to (1) with $y=h(x)$. Then, any other solution z to (1) with $y=h(x)$ converges to $T(x)$, so that $T^{\text{inv}}(z)$ asymptotically provides an estimate of x thanks to (5) by (uniform) continuity of T^{inv}. Hence the observer given by (4) converges asymptotically. This is summed up in the following theorem which is a direct consequence of [4, Theorem 2.2] in the case where $\mathcal{X}$ is bounded.

Theorem 1.1 ([4, Theorem 2.2]) *Assume there exist an integer m, a Hurwitz matrix A in $\mathbb{R}^{m\times m}$, a vector B in $\mathbb{R}^m$ and an injective function T : $\text{cl}(\mathcal{X})\to\mathbb{R}^m$ satisfying* (6)*. Under these conditions, there exists a continuous function $T^{\text{inv}}:\mathbb{R}^m\to\mathbb{R}^n$ verifying* (5)*. Besides, for any such map T^{inv}, for any x in $\mathcal{X}$ such that the solution $t\mapsto X(x,t)$ to* (3) *is defined on* $[0,+\infty)$ *with values in $\mathcal{X}$, and for any z in $\mathbb{R}^m$, the (unique) solution $t\mapsto(X(x,t),Z(x,z,t))$ of*

$$\dot{x} \;=\; f(x) \quad , \qquad \dot{z} \;=\; Az + Bh(x) \tag{7}$$

is defined on $[0,+\infty)$ *and verifies*

$$\lim_{t\to+\infty}\left|\hat{X}(x,z,t) - X(x,t)\right| = 0$$

where $\hat{X}(x,z,t) = T^{\text{inv}}(Z(x,z,t))$.

The proof of Theorem 1.1 is postponed to appendix A. From there, several questions can be raised:

[1] In practice, $T^{\text{inv}}(z) = \text{argmin}_{x\in\text{cl}(\mathcal{X})}|z - T(x)|$ can be employed even though it is not continuous.

[2] As shown in [4], we do not need T to be C^1 as long as the Lie derivative of T along f exists.

(1) For which choice of m, A and B does an injective solution T to (6) exist?
(2) Is this solution unique?
(3) How to construct such a solution?

The existence question was first considered in [28], [17] and [19] in the analytic context and around an equilibrium point. Then, the localness was dropped following another perspective in [18] where a global existence result was proposed based on a strong observability assumption which unfortunately did not provide an indication on the necessary dimension of the pair (A, B). This problem was solved in [4] by proving the existence of the injective map T under a weak *backward-distinguishability* condition, for A complex diagonal of dimension $n+1$, with a generic choice of $n+1$ *distinct* complex eigenvalues. Those results have then been extended to non autonomous systems [5], discrete-time autonomous systems [10], and to the problem of *functional* observer design when the full state is not observable and only a function of the state needs to be estimated [30].

In terms of design, an explicit expression of the map T can sometimes be found in particular contexts such as parameter identification [2], state/parameter estimation for electrical machines [16,7]. On the other hand, when an expression for T, or its left-inverse T^{inv} is not available, approximation approaches have been proposed as in [21]. More recently, numerical methods based on neural networks are being developed to learn a model of the maps T and T^{inv} based on the generation of a data set of points (x, z) approximating $(x, T(x))$ through backward and forward integration of the dynamics [27,11].

Organization of the paper. In this note, we give several further answers to the questions (1) and (2) above. We start by discussing uniqueness in Section 2. Then, two novel existence results are provided in Section 3: one refining [4] with almost any choice of controllable pair (A, B) having A real diagonalizable of dimension $2n+1$, and the second in the analytic context for a different structure of the pair (A, B) with only one eigenvalue of sufficiently large multiplicity. Then, in Section 4, we introduce a cascaded procedure which facilitates the synthesis of such an observer for certain cascaded nonlinear systems and which allows the use of a filtered version of y in the observer.

Notations. A map $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{K}$ if it is continuous, increasing and such that $\rho(0) = 0$. For a differential equation $\dot{x} = f(x)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz, we denote by $X(x, t)$ the value at time t of the solution initialized at $x \in \mathbb{R}^n$ at time 0, and by $(\sigma^-(x), \sigma^+(x))$ its maximal domain of definition. Given a subset $\mathcal{X} \subseteq \mathbb{R}^n$ and a positive real number δ, $\mathcal{X} + \delta$ is the open set defined as

$$\mathcal{X} + \delta = \{x \in \mathbb{R}^n \mid \exists x_{\mathcal{X}} \in \mathcal{X}, |x - x_{\mathcal{X}}| < \delta\} . \tag{8}$$

The real and imaginary parts of a complex number are denoted by Re and Im respectively, and

$$\mathbb{R}_\rho = \{\lambda \in \mathbb{R} : \lambda < -\rho\} \quad , \quad \mathbb{C}_\rho = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < -\rho\} \ . \tag{9}$$

We say that a map $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is in $C^\infty(\mathbb{R}; C^1(\mathbb{R}^n; \mathbb{R}))$ if $\lambda \mapsto g(\lambda, x)$ is C^∞ for all $x \in \mathbb{R}^n$ and $x \mapsto \frac{\partial^k g}{\partial \lambda^k}(\lambda, x)$ is C^1 for all $\lambda \in \mathbb{R}$ and all $k \in \mathbb{N}$.

2 Remarks on the uniqueness of the map T

Typical KKL theorems as in [4] or in the next section, provide the existence of an injective solution T to the partial differential equation (PDE) (6). However, we might find other solutions of this PDE (via exact computations or a numerical approach [27,11]) and it is legitimate to wonder how much any such maps differ.

Proposition 2.1 *Let $\mathcal{O}$ be a subset of $\mathbb{R}^n$ that is backward invariant*[3] *by f and consider A a Hurwitz matrix in $\mathbb{R}^{m\times m}$ and B a vector in $\mathbb{R}^m$. Let $T_a : \mathcal{O} \mapsto \mathbb{R}^m$ and $T_b : \mathcal{O} \mapsto \mathbb{R}^m$ be two C^1 solutions of*

$$\frac{\partial T}{\partial x}(x) f(x) = AT(x) + Bh(x) \ , \ \forall x \in \mathcal{O}.$$

If either

(i) T_a *and* T_b *are bounded on* $\mathcal{O}$,

(ii) or there exist positive constants κ_f, ρ_f, κ_a, q_a, ρ_a, κ_b, q_b *and* ρ_b *such that for all* $x \in \mathcal{O}$,

$$|f(x)| \leqslant \kappa_f |x| + \rho_f, \quad |T_a(x)| \leqslant \kappa_a |x|^{q_a} + \rho_a, \quad |T_b(x)| \leqslant \kappa_b |x|^{q_b} + \rho_b$$

with $\kappa_f q_a < |\mathrm{Re}(\lambda_m(A))|$ *and* $\kappa_f q_b < |\mathrm{Re}(\lambda_m(A))|$, *where* $\lambda_m(A)$ *is the slowest eigenvalue of* A,

(iii) or there exist positive constants κ_a, ρ_a, κ_b *and* ρ_b *such that for all* $x \in \mathcal{O}$,

$$\left|\frac{\partial T_a}{\partial x}(x) f(x)\right| \leqslant \kappa_a |T_a(x)| + \rho_a \ \text{and} \ \left|\frac{\partial T_b}{\partial x}(x) f(x)\right| \leqslant \kappa_b |T_b(x)| + \rho_b$$

with $\kappa_a < |\mathrm{Re}(\lambda_m(A))|$ *and* $\kappa_b < |\mathrm{Re}(\lambda_m(A))|$,

then $T_a(x) = T_b(x)$ *for all* x *in* $\mathcal{O}$.

In particular, if the system admits a compact backward-invariant set $\mathcal{C}$, then any C^1 solution to the PDE is unique on $\mathcal{C}$.

[3] That is, for all $x \in \mathcal{O}$, $\sigma^-(x) = -\infty$ and $X(x, t) \in \mathcal{O}$ for all $t \leqslant 0$.

Proof : Since $\mathcal{O}$ is backward invariant, we have for all x in $\mathcal{O}$ and all $t \leqslant 0$,

$$\frac{d}{dt}T_a(X(x,t)) = AT_a(X(x,t)) + Bh(X(x,t)) \ ,$$

and,

$$\frac{d}{dt}T_b(X(x,t)) = AT_b(X(x,t)) + Bh(X(x,t)) \ ,$$

which implies

$$T_a(X(x,t)) - T_b(X(x,t)) = \exp(At)[T_a(x) - T_b(x)] \ , \ \forall t \leqslant 0 \ , \forall x \in \mathcal{O} \ .$$

Hence,

$$T_a(x) - T_b(x) = \exp(-At)[T_a(X(x,t)) - T_b(X(x,t))] \ , \ \forall t \leqslant 0 \ , \forall x \in \mathcal{O} \ .$$

Now make t go to $-\infty$.

In case *(i)*, we get that $\exp(-At)[T_a(X(x,t)) - T_b(X(x,t))]$ tends towards 0 by boundedness of T_a and T_b since A is Hurwitz.

In case *(ii)*, we get that $|X(x,t)| \leqslant M \exp(-\kappa_f t)$ for all $t \leqslant 0$ for some $M > 0$, hence

$$|T_a(X(x,t)) - T_b(X(x,t))| \leqslant \kappa_a M^{q_a} \exp(-q_a \kappa_f t) + \rho_a + \kappa_b M^{q_b} \exp(-q_b \kappa_f t) + \rho_b \ .$$

Since $\kappa_f q_a < |\operatorname{Re}\lambda_m(A)|$ and $\kappa_f q_b < |\operatorname{Re}\lambda_m(A)|$, $\exp(-At)[T_a(X(x,t)) - T_b(X(x,t))]$ tends towards 0.

In case *(iii)*, we get by Grönwall's inequality that

$$|T_a(X(x,t))| \leqslant (|T_a(x)| + \rho_a|t|)\exp(-\kappa_a t) \quad \text{and} \quad |T_b(X(x,t))| \leqslant (|T_b(x)| + \rho_b|t|)\exp(-\kappa_b t).$$

Hence

$$\begin{aligned}|T_a(X(x,t)) - T_b(X(x,t))| \leqslant (|T_a(x)| + \rho_a|t|)\exp(-\kappa_a t) \\ + (|T_b(x)| + \rho_b|t|)\exp(-\kappa_b t).\end{aligned}$$

Since $\kappa_a < |\operatorname{Re}\lambda_m(A)|$ and $\kappa_b < |\operatorname{Re}\lambda_m(A)|$, $\exp(-At)[T_a(X(x,t)) - T_b(X(x,t))]$ tends towards 0.

Thus, in any case, $T_a(x) = T_b(x)$ for all x in $\mathcal{O}$. □

If multiple solutions to the PDE (6) exist, the injectivity of one solution may not imply the injectivity of all solutions. In the following we give sufficient conditions for a particular bounded injective solution denoted T to exist. If another solution T_a is found by other means on a backward invariant set of the system and if this map T_a is bounded on that set, then it is actually unique and coincides with the theoretical injective solution. Otherwise, injectivity of T_a is not ensured a priori, and must be checked on each individual example.

Actually, given x in $\mathcal{X}$ such that $X(x,t)$ belongs to the bounded set $\mathcal{X}$ for all $t \geqslant 0$, the ω-limit set of x

$$\omega(x) = \cap_{t\geqslant 0} \operatorname{cl}(\cup_{s\geqslant t}\{X(x,s)\}$$

is a compact backward and forward invariant set. Hence, with the former proposition T and T_a coincide on this set. In other words, T_a coincides on the set $\omega(x)$ with an injective map. Besides, because T and T_a are both solutions to the PDE, the state z of the KKL observer converges both towards $T_a(X(x,t))$ and $T(X(x,t))$, which tend to each other. However, this does not mean that $X(x,t)$ can be uniquely determined asymptotically from the knowledge of T_a, since there could be other $x' \in \mathcal{X}$ such that $T(x^*) = T_a(x^*) = T_a(x')$ for $x^* \in \omega(x)$.

Note however that because the observer is supposed to estimate trajectories remaining in $\mathcal{X}$ for all positive times, the map f may be modified outside of $\mathcal{X}$ as long as the observability properties given below are preserved. It is thus usually possible to replace f by modified dynamics $\dot{x} = \chi(x)f(x)$, which admit a backward invariant compact set (by making f vanish outside of a larger open set containing $\operatorname{cl}(\mathcal{X})$ and ensuring observability, for instance as in (48) below) Once this regularization has been done, any solution T_a to the PDE found on that set is unique and thus injective on $\mathcal{X}$ if the required observability properties are preserved. For instance, in a numerical KKL design [27,11], where T_a is learned on a compact set, a trick to ensure injectivity is to apply the learning procedure to the modified f on the whole backward invariant set. This has the additional advantage to make the solutions well-defined and bounded in backward-time, which is crucial in the learning procedure.

Example 2.2 *Consider the trivial one-dimensional example*

$$\dot{x} = -ax \quad , \quad y = x$$

with $a > 0$*. This example falls into the original linear Luenberger context [20] where an injective solution to PDE* (6) *is known to exist with dimension* $m = 1$*. Taking* $A = -\lambda$ *and* $B = 1$ *with* $\lambda > 0$ *and* $\lambda \neq a$*, the map* T_λ^0 *defined by* $T_\lambda^0(x) = \frac{1}{\lambda - a}x$ *verifies the PDE*

$$\frac{\partial T_\lambda}{\partial x}(x)f(x) = -\lambda T_\lambda(x) + h(x)$$

everywhere. Clearly, T_λ^0 *is injective. However, for any real number* α*,*

$$T_\lambda(x) = \alpha \operatorname{sign}(x)|x|^{\frac{\lambda}{a}} + \frac{1}{\lambda - a}x$$

is also a C^1 solution to the PDE everywhere and clearly T_λ can be non injective for some values of α. Note that in this example, there is no backward-invariant set apart from $\{0\}$, where the maps T_λ indeed agree.

3 Remarks on the existence of an injective map T

3.1 Existence result based on A diagonalizable with $2n+1$ eigenvalues

As shown in [4], one of the main interests of the KKL observer is that its existence is guaranteed under a very weak observability assumption. Indeed, assume that for any $x \in \mathcal{X}$, the past output path $t \mapsto h(X(x,t))$ of (3) restricted to the time in which the trajectory remains in a certain set determines x uniquely. Then, from [4], it is sufficient to choose $m = 2n+2$ and A the real representation of a diagonal Hurwitz complex matrix in $\mathbb{C}^{n+1}$ to get the existence of an injective map T solving (6). The specific observability condition made is the following.

Assumption 3.1 (($\mathcal{O}, \delta_d$)-backward distinguishability) *There exists an open bounded set $\mathcal{O}$ of $\mathbb{R}^n$ containing $\mathrm{cl}(\mathcal{X})$ and a positive real number δ_d such that, for each pair of distinct points x_a and x_b in $\mathcal{O}$, there exists a negative time t in $\left(\max\left\{\sigma^-_{\mathcal{O}+\delta_d}(x_a), \sigma^-_{\mathcal{O}+\delta_d}(x_b)\right\}, 0\right]$ such that :*

$$h(X(x_a,t)) \neq h(X(x_b,t)) \ .$$

This distinguishability assumption says that the present state x can be distinguished from other states in $\mathcal{O}$ by looking at the past output path restricted to the time in which the solution remains in $\mathcal{O}+\delta_d$.

One of the results obtained in [4] can be reformulated as follows.

Theorem 3.2 ([4]) *Assume System (3) satisfies Assumption 3.1. Then there exist a positive real number ρ and a zero Lebesgue measure subset $\mathcal{I}$ of $(\mathbb{C}_\rho)^{n+1}$ with $\mathbb{C}_\rho$ defined in (9) such that for each $(\lambda_1, \dots, \lambda_{n+1})$ in $(\mathbb{C}_\rho)^{n+1} \setminus \mathcal{I}$, there exists an injective map $T : \mathcal{O} \to \mathbb{C}^{n+1}$ verifying (6) with*

$$A = \mathrm{diag}\left(\lambda_1 \ \dots, \lambda_{n+1}\right) \ , \ B = \begin{bmatrix}1 \ \dots \ 1\end{bmatrix}^\top \ . \tag{10}$$

In [4], this result was not stated in this way. However, it is a direct consequence of the fact that we restrict our analysis to a bounded set $\mathcal{X}$ and that the output is one-dimensional. If the ouput is multi-dimensional then the same result holds but with the filter (1) applied to each output and thus T concatenating the solutions to (6) for each output.

Note that this observer can be realized in $\mathbb{R}^{2n+2}$ by picking

$$A_{\text{real}} = \text{diag}\left(\begin{bmatrix}\text{Re}(\lambda_i) & -\text{Im}(\lambda_i)\\ \text{Im}(\lambda_i) & \text{Re}(\lambda_i)\end{bmatrix}\right) \ , \ B_{\text{real}} = \begin{bmatrix} b_{\text{real}} \\ \vdots \\ b_{\text{real}}\end{bmatrix} \ , \ b_{\text{real}} = \begin{bmatrix}1\\0\end{bmatrix} \ . \qquad (11)$$

However, we see that the existence result imposes strong constraints on the matrices A and B. This is different from the result of Luenberger for linear systems for which no assumptions besides controllability and a spectrum different from F is required. The result we obtain in this paper is the following one.

Theorem 3.3 *Assume System (3) satisfies Assumption 3.1. Then, there exist a positive real number ρ and a zero Lebesgue measure subset $\mathcal{J}$ of $\mathbb{R}^{(2n+1)\times(2n+1)}\times\mathbb{R}^{2n+1}$ such that for any pair (A,B) in $(\mathbb{R}^{(2n+1)\times(2n+1)}\times\mathbb{R}^{2n+1})\setminus\mathcal{J}$ with $A+\rho I$ Hurwitz, there exists an injective map $T:\mathcal{O}\to\mathbb{R}^{2n+1}$ verifying (6).*

Remark 3.4 *Theorem 3.3 generalizes Theorem 3.2, in several directions :*

(1) The observer matrices do not need to be with complex eigenvalues.
(2) The dimension of the observer is $2n+1$, whereas the observer in Theorem 3.2 is of real dimension $2n+2$. This allows to recover some well known fact in observability theory that it is generically sufficient to extract $2n+1$ pieces of information from the output path to observe a state of dimension n (see for instance [1,31,14,12,29]).
(3) The matrices A and B do not need to have a particular structure unlike in (11). In particular, we show that they may be almost any controllable pair (A,B) with A diagonalizable. This “almost any” pair actually comes from an “almost any” choice of distinct $p_{\mathbb{C}}$ complex conjugate eigenvalues and $p_{\mathbb{R}}$ real eigenvalues in A such that $2p_{\mathbb{C}}+p_{\mathbb{R}}\geqslant 2n+1$. Indeed, we show that for any such $p_{\mathbb{C}}$ and $p_{\mathbb{R}}$, the set of eigenvalues in $\mathbb{C}_\rho^{p_{\mathbb{C}}}\times\mathbb{R}_\rho^{p_{\mathbb{R}}}$ which do not provide injectivity of T for (A,B) defined in (10) is of zero-Lebesgue measure in $\mathbb{C}_\rho^{p_{\mathbb{C}}}\times\mathbb{R}_\rho^{p_{\mathbb{R}}}$. This generalizes Theorem 3.2 where $p_{\mathbb{C}}$ is fixed to $n+1$ and $p_{\mathbb{R}}=0$. Then, in the case where $2p_{\mathbb{C}}+p_{\mathbb{R}}=2n+1$, we show that the set of matrices in $\mathbb{R}^{(2n+1)\times(2n+1)}$ having eigenvalues in the union of those zero-measure sets is of zero-measure in $\mathbb{R}^{(2n+1)\times(2n+1)}$. We refer the reader to the proof for more details on this genericity result.

The proof of this result is given in Section 5. In the following subsection, another existence result is given for some particular structures of matrices A and B which are not covered by Theorem 3.3.

3.2 Existence result for A triangular with single eigenvalue

In this section, inspired by [8], we consider the case in which the pair (A, B) in $\mathbb{R}^{m\times m} \times \mathbb{R}^m$ is in the form

$$A_{\lambda,m} = \begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 1 & \lambda & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda \end{pmatrix}, \qquad B_m = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \tag{12}$$

for some negative real number λ. This corresponds to a chain of filters, successively filtering m times the output with the same eigenvalue λ. In other words, instead of parallelizing the filters with different eigenvalues, this choice rather exploits the *depth* of the filter.

This case is not covered by Theorem 3.3 because its proof relies on diagonalizable matrices A with a generic choice of *distinct* eigenvalues. Instead, the choice of (12) is parameterized by a single real parameter λ, which is typically in the zero-measure set of Theorem 3.3. It thus requires another type of analysis which leads to the following result in the analytic context.

Theorem 3.5 *Assume that $\mathcal{O}$ is a backward invariant open subset such that Assumption 3.1 holds. Let Θ be a non-empty open subset of $\mathbb{R}_{<0}$. Assume there exists a C^∞ map $T_0 : \Theta \times \mathcal{O} \mapsto \mathbb{R}$, such that for each λ in Θ, $x \mapsto T_0(\lambda, x)$ is an analytic bounded function on $\mathcal{O}$ which satisfies*

$$\frac{\partial T_0}{\partial x}(\lambda, x) f(x) = \lambda T_0(\lambda, x) + h(x) \ . \tag{13}$$

Assume moreover that h is bounded on $\mathcal{O}$. Then, for each λ in Θ, for any compact subset $\mathcal{C} \subset \mathcal{O}$, there exists $m^\star \in \mathbb{N}$ such that for all $m \geqslant m^\star$, the (unique) solution $T_{\lambda,m}$ of (6) with $(A, B) = (A_{\lambda,m}, B_m) \in \mathbb{R}^{m\times m} \times \mathbb{R}^m$ given in (12) is injective on $\mathcal{C}$.

Proof : First of all, the set $\mathcal{O}$ being backward invariant for the flow and the mapping h being bounded in $\mathcal{O}$, this implies that the function

$$S(\lambda, x) = \int_{-\infty}^{0} \exp(-\lambda s) h(X(x, s)) ds \tag{14}$$

is well defined on $\mathbb{R}_{<0} \times \mathcal{O}$ and such that for all $\lambda < 0$, $S(\lambda, \cdot)$ is bounded and solution to (13) on $\mathcal{O}$. Moreover, for all $x \in \mathcal{O}$, $S(\cdot, x)$ is analytic on $\mathbb{R}_{<0}$. With Proposition 2.1, it implies that $T_0 = S$ on $\Theta \times \mathcal{O}$ and therefore, for all $\lambda \in \Theta$, $S(\lambda, \cdot)$ is analytic on $\mathcal{O}$. For m in $\mathbb{N}$, let

$$T_{\lambda,m}(x) = (T_0(\lambda, x), \dots, T_{m-1}(\lambda, x)) \tag{15a}$$

where

$$T_i(\lambda, x) = \frac{\partial^i T_0}{\partial \lambda^i}(\lambda, x) \ , \ i = \{0, \dots, m-1\} \ . \tag{15b}$$

Since T_0 is C^∞, for all $(\lambda, x) \in \Theta \times \mathcal{O}$,

$$\frac{\partial T_1}{\partial x}(\lambda, x) f(x) = \frac{\partial^2 T_0}{\partial \lambda \partial x}(\lambda, x) f(x) = \frac{\partial}{\partial \lambda}(\lambda T_0(\lambda, x) + h(x)) = \lambda T_1(\lambda, x) + T_0(\lambda, x)$$

and iteratively for all i

$$\frac{\partial T_i}{\partial x}(\lambda, x) f(x) = \lambda T_i(\lambda, x) + T_{i-1}(\lambda, x) \ ,$$

and consequently, $T_{\lambda,m}$ is the (unique) bounded solution of (6) with $(A, B) = (A_{\lambda,m}, B_m) \in \mathbb{R}^{m \times m} \times \mathbb{R}^m$ given in (12).

Let $g : \mathbb{R}_{<0} \times \mathcal{O} \times \mathcal{O}$ be given by

$$g(\lambda, x_a, x_b) = S(\lambda, x_a) - S(\lambda, x_b) \ . \tag{16}$$

Let $\lambda \in \Theta$ and $\mathcal{C} \subset \mathcal{O}$. Let $\mathcal{D}_{\lambda,\ell}$ be the sequence of open sets defined as

$$\mathcal{D}_{\lambda,\ell} = \left\{ (x_a, x_b) \in \mathcal{O}^2, x_a \neq x_b, \frac{\partial^k g}{\partial \lambda^k}(\lambda, x_a, x_b) = 0, k = 0, \dots, \ell \right\} \ .$$

We will show that there exists m such that $\mathcal{D}_{\lambda,m} \cap (\mathcal{C} \times \mathcal{C}) = \emptyset$ which implies that $T_{\lambda,m}$ is injective in $\mathcal{C}$. Note that we have

$$\mathcal{D}_{\lambda,\ell+1} \subset \mathcal{D}_{\lambda,\ell} \ .$$

The map $g(\lambda, \cdot, \cdot)$ being analytic since $S = T_0$ on $\Theta \times \mathcal{O}$, $(\mathcal{D}_{\lambda,\ell})_{\ell \in \mathbb{N}}$ is a decreasing sequence of analytic subsets of $\mathcal{O}^2 \subset \mathbb{R}^{2m}$. The ring of analytic functions being Notherian [24, Corollary 1, p.99], $(\mathcal{D}_{\lambda,\ell})_{\ell \in \mathbb{N}}$ is a stationary sequence in all compact subsets, i.e. there exists $m^\star$ in $\mathbb{N}$ such that, for all $m \geqslant m^\star$,

$$\mathcal{D}_{\lambda,m+\ell} \cap (\mathcal{C} \times \mathcal{C}) = \mathcal{D}_{\lambda,m} \cap (\mathcal{C} \times \mathcal{C}) \ , \ \forall \ell \in \mathbb{N} \ .$$

Assume $\mathcal{D}_{\lambda,m} \cap (\mathcal{C} \times \mathcal{C})$ non-empty and take $(x_a, x_b) \in \mathcal{D}_{\lambda,m} \cap (\mathcal{C} \times \mathcal{C})$. We have $\frac{\partial^k g}{\partial \lambda^k}(\lambda, x_a, x_b) = 0$ for all k. Since, moreover $g(\cdot, x_a, x_b)$ is analytic, this implies that $g(\lambda, x_a, x_b) = 0$ for all $\lambda < 0$. On another hand, with (14) and by injectivity of the Laplace transform, this implies that $s \mapsto h(X(x_a, s)) - h(X(x_b, s)) = 0$ for s in $(-\infty, 0]$ and $x_a \neq x_b$. This is a contradiction with the observability assumption. This implies that $\mathcal{D}_{\lambda,m} \cap (\mathcal{C} \times \mathcal{C}) = \emptyset$. $\square$

Remark 3.6 *In Theorem 3.5, the existence of an analytic solution* $x \mapsto T_0(\lambda, x)$ *of* (13) *is assumed for* $\lambda \in \Theta$. *In the proof, it is shown that* T_0 *actually coincides with the map* S, *defined by* (14). *By the Lebesgue dominated* *convergence theorem, if* $x \mapsto h \circ X(x,s) \in C^\infty(\mathcal{O},\mathbb{R})$ *for all* $s \leqslant 0$ *and if for each multi-index* α *there exist a continuous map* $M_\alpha : \mathcal{O} \to \mathbb{R}_+$ *and an integrable map* $\varphi_\alpha : \mathbb{R}_- \to \mathbb{R}_+$ *such that*

$$\exp(-\lambda s)\left|\frac{\partial^\alpha (h\circ X)}{\partial x^\alpha}(x,s)\right| \leqslant \varphi_\alpha(s) M_\alpha(x), \quad \forall s \in \mathbb{R}_-,\ \forall x \in \mathcal{O}, \tag{17}$$

then $S(\lambda,\cdot) \in C^\infty(\mathcal{O},\mathbb{R})$ *and its partial derivatives are given by*

$$\frac{\partial^\alpha S}{\partial x^\alpha}(\lambda,x) = \int_{-\infty}^{0} \exp(-\lambda s)\frac{\partial^\alpha (h\circ X)}{\partial x^\alpha}(x,s)ds. \tag{18}$$

Moreover,

$$\left|\frac{\partial^\alpha S}{\partial x^\alpha}(\lambda,x)\right| \leqslant M_\alpha(x)\int_{-\infty}^{0} \varphi_\alpha(s)ds. \tag{19}$$

Hence, if for all compact sets $\mathcal{C} \subset \mathcal{O}$ *there exists a positive constant* γ *such that* $M_\alpha(x)\int_{-\infty}^0 \varphi_\alpha(s)ds \leqslant \gamma^{|\alpha|+1}\alpha!$ *for all* $x \in \mathcal{C}$ *and and all multi-indices* α*, then* $S(\lambda,\cdot)$ *is analytic.*

Example 3.7 *Consider an harmonic oscillator with unknown frequency investigated in [26] and modelled as*

$$\dot x_1 = x_2\ ,\dot x_2 = -x_3 x_1\ ,\dot x_3 = 0\ ,\ y = x_1\ . \tag{20}$$

In that case, for any $\rho > 0$ *and* $\varpi > 0$*, the bounded set*

$$\mathcal{O} = \left\{ x \in \mathbb{R}^3,\ \frac{1}{\rho} < x_3 x_1^2 + x_2^2 < \rho,\ \frac{1}{\varpi} < x_3 < \varpi \right\} \tag{21}$$

is backward invariant along the dynamics. Besides, the map defined on $\mathbb{R}_{<0}\times\mathcal{O}$ *by*

$$T_0(\lambda,x) = \frac{-\lambda x_1 - x_2}{\lambda^2 + x_3} \tag{22}$$

solves the PDE (13)*. According to Theorem 3.5, we know that for any* $\lambda < 0$ *and for any compact subset* $\mathcal{C}$ *of* $\mathcal{O}$*, there exists an integer* m *such that the (unique) solution to* (6) *with* $(A,B) = (A_{\lambda,m}, B_m) \in \mathbb{R}^{m\times m}\times\mathbb{R}^m$ *given in* (12) *is injective on* $\mathcal{C}$*.*

As shown in the proof of Theorem 3.5, $T_{\lambda,m}$ *is built by successively differentiating* T_0 *with respect to* λ *as defined in* (15) *until obtaining an injective map. We show in this example that we can pick* $m = 4$ *and that the associated map* $T_{\lambda,4}$ *defined by*

$$T_{\lambda,4}(x) = \left(T_0(\lambda,x), \frac{\partial T_0}{\partial\lambda}(\lambda,x), \frac{\partial^2 T_0}{\partial\lambda^2}(\lambda,x), \frac{\partial^3 T_0}{\partial\lambda^3}(\lambda,x)\right)$$

is actually injective on $(\mathbb{R}^2\setminus\{(0,0\})\times\mathbb{R}_{\geqslant 0}$*.*

Indeed, consider x_a and x_b in $(\mathbb{R}^2 \setminus \{(0,0\}) \times \mathbb{R}_{\geqslant 0}$. Denote $w_a = \lambda^2 + x_{a,3}$ and $w_b = \lambda^2 + x_{b,3}$. We have

$$\begin{aligned} T_0(\lambda, x_a) = T_0(\lambda, x_b) =: z_1 &\iff \lambda(x_{b,1}w_a - x_{a,1}w_b) = x_{a,2}w_b - x_{b,2}w_a \\ \tfrac{\partial T_0}{\partial \lambda}(\lambda, x_a) = \tfrac{\partial T_0}{\partial \lambda}(\lambda, x_b) =: z_2 &\iff \quad x_{b,1}w_a - x_{a,1}w_b = 2\lambda z_1(w_b - w_a) \end{aligned}$$

which thus gives

$$\begin{cases} T_0(\lambda, x_a) = T_0(\lambda, x_b) \\ \frac{\partial T_0}{\partial \lambda}(\lambda, x_a) = \frac{\partial T_0}{\partial \lambda}(\lambda, x_b) \end{cases} \iff \begin{cases} x_{b,1}w_a - x_{a,1}w_b = 2\lambda z_1(w_b - w_a) \\ x_{a,2}w_b - x_{b,2}w_a = 2\lambda^2 z_1(w_b - w_a) \end{cases}$$

Then, continuing differentiating,

$$\begin{aligned} \frac{\partial^2 T_0}{\partial \lambda^2}(\lambda, x_a) = \frac{\partial^2 T_0}{\partial \lambda^2}(\lambda, x_b) \quad \iff \quad & \frac{2}{w_a}\left[-2\lambda \frac{\partial T_0}{\partial \lambda}(\lambda, x_a) - T(x_a)\right] \\ & = \frac{2}{w_b}\left[-2\lambda \frac{\partial T_0}{\partial \lambda}(\lambda, x_b) - T(x_b)\right] \end{aligned}$$

and

$$\begin{aligned} & \frac{\partial^3 T_0}{\partial \lambda^3}(\lambda, x_a) = \frac{\partial^3 T_0}{\partial \lambda^3}(\lambda, x_b) \\ \iff \quad & \frac{-4\lambda}{w_a}\left[-2\lambda \frac{\partial T_0}{\partial \lambda}(\lambda, x_a) - T(x_a)\right] + \frac{2}{w_a}\left[-3\frac{\partial T_0}{\partial \lambda}(\lambda, x_a) - 2\lambda \frac{\partial^2 T_0}{\partial \lambda^2}(\lambda, x_a)\right] \\ & = \frac{-4\lambda}{w_b}\left[-2\lambda \frac{\partial T_0}{\partial \lambda}(\lambda, x_b) - T(x_b)\right] + \frac{2}{w_b}\left[-3\frac{\partial T_0}{\partial \lambda}(\lambda, x_b) - 2\lambda \frac{\partial^2 T_0}{\partial \lambda^2}(\lambda, x_b)\right] \end{aligned}$$

Assume therefore that $T_{\lambda,4}(x_a) = T_{\lambda,4}(x_b)$, namely

$$\begin{aligned} & \left(T_0(\lambda, x_a), \frac{\partial T_0}{\partial \lambda}(\lambda, x_a), \frac{\partial^2 T_0}{\partial \lambda^2}(\lambda, x_a), \frac{\partial^3 T_0}{\partial \lambda^3}(x_a)\right) \\ & \qquad = \left(T_0(\lambda, x_b), \frac{\partial T_0}{\partial \lambda}(\lambda, x_b), \frac{\partial^2 T_0}{\partial \lambda^2}(\lambda, x_b), \frac{\partial^3 T_0}{\partial \lambda^3}(\lambda, x_b)\right) \end{aligned}$$

which we denote (z_1, z_2, z_3, z_4). Then, we get two cases :

- *either $-2\lambda z_2 - z_1 \neq 0$ and we get $w_a = w_b$ from the third equality, and then $x_a = x_b$ from the first two;*
- *or $-2\lambda z_2 - z_1 = 0$ and thus $z_3 = 0$, so that the fourth equality provides $z_2\left(\frac{1}{w_a} - \frac{1}{w_b}\right) = 0$. So either $z_2 \neq 0$ and we recover $w_a = w_b$ and conclude as above; or $z_2 = 0$, but then also $z_1 = 0$, and necessarily $x_{a,1} = x_{a,2} = x_{b,1} = x_{b,2} = 0$, which is impossible.*

We conclude that $T_{\lambda,4}$ is injective on $(\mathbb{R}^2 \setminus \{(0,0\}) \times \mathbb{R}_+$ and a KKL observer *can be designed with* $(A, B) = (A_{\lambda,m}, B_m) \in \mathbb{R}^{m\times m} \times \mathbb{R}^m$ *given in* (12) *for* $m = 4$.

4 A remark on a cascaded design procedure for T

Consider now a dynamical system in the cascade form

$$\dot{x} = f(x) \ , \quad \dot{\xi} = F\xi + Gh(x) \tag{23}$$

with $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^{n_\xi}$ and output

$$y_\xi = H\xi$$

and with (F, G, H) in the normal controllability form

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n_\xi-1} \end{pmatrix} , \qquad G = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \gamma \end{pmatrix} \tag{24}$$

$$H = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \tag{25}$$

with $\gamma \neq 0$ and $(a_0, \ldots, a_{n_\xi-1})$ in $\mathbb{R}^{n_\xi}$.

Assuming we know a KKL observer for $\dot{x} = f(x)$ from the output $y = h(x)$, we would like to deduce an observer for (x, ξ) from the output y_ξ. This covers two cases of practical interest:

- $y = h(x)$ is not available but is used as an intermediary step in the design of an observer for (x, ξ) from the real output y_ξ;
- $y = h(x)$ is available, but noisy, and we want to use a filtered version y_ξ of y in the KKL observer.

Due to the controllability form of (F, G, H), if the system $\dot{x} = f(x)$ with output $y = h(x)$ is backward-distinguishable, then, the extended system (23) with state (x, ξ) is also backward-distinguishable. Indeed, intuitively speaking, the past values of y_ξ determine ξ and y uniquely and therefore also x. We could thus use Theorem 3.3 to show the existence of a KKL observer for this extended system. However, the goal of this section is rather to provide an explicit design method when a solution T_0 to the PDE (6) is available for the initial system (f, h) with $A = \lambda$ and $B = 1$ for each λ. More precisely, we exhibit a solution if the following assumption holds for some open set $\mathcal{O}$ containing $\mathrm{cl}(\mathcal{X})$.

Assumption 4.1 *There exist a mapping T_0 in $C^\infty(\mathbb{R}; C^1(\mathbb{R}^n;\mathbb{R}))$ and an open subset $\Theta_0 \subset \mathbb{R}_{<0}$ such that for all (λ, x) in $\Theta_0 \times \mathcal{O}$,*

$$\frac{\partial T_0}{\partial x}(\lambda, x) f(x) = \lambda T_0(\lambda, x) + h(x) \ , \tag{26}$$

and for all (λ, x_a, x_b) in $\Theta_0 \times \mathcal{O}^2$ verifying $x_a \neq x_b$, there exists $k \geqslant n_\xi$ in $\mathbb{N}$ such that

$$\frac{\partial^k T_0}{\partial \lambda^k}(\lambda, x_a) - \frac{\partial^k T_0}{\partial \lambda^k}(\lambda, x_b) \ \neq \ 0 \ . \tag{27}$$

Note that if a solution T_0 to (26) is analytic with respect to λ, then Assumption 4.1 holds as long as for any $(x_a, x_b) \in \mathcal{O}^2$ verifying $x_a \neq x_b$, $T_0(\cdot, x_a) - T_0(\cdot, x_b)$ is not a polynomial of degree strictly less than n_ξ. Under the assumption of backward-distinguishability of Theorem 3.3, it is shown in its proof (see Section 5.2.2) that such an analytic map T_0 always exists in the form (50). Indeed, $T_0(\cdot, x_a) - T_0(\cdot, x_b)$ takes the form of a Laplace transform of some non zero causal signal, hence cannot be a polynomial.

The following theorem shows that this assumption is sufficient to give an explicit expression of an injective mapping T, allowing to obtain an observer for the entire system with state (x, ξ).

Theorem 4.2 *Suppose that Assumption 4.1 holds. Let $\Theta_0^{\mathrm{ext}} = \Theta_0 \backslash \sigma(F)$. Then there exists a zero Lebesgue measure subset $\mathcal{J} \subset (\Theta_0^{\mathrm{ext}})^{2(n+n_\xi)+1}$ such that, for each $\lambda_1, \ldots, \lambda_{2(n+n_\xi)+1}$ in $(\Theta_0^{\mathrm{ext}})^{2(n+n_\xi)+1} \backslash \mathcal{J}$, the map $T : \mathcal{O} \times \mathbb{R}^{n_\xi} \to \mathbb{R}^{2(n+n_\xi)+1}$ defined by*

$$\begin{aligned} &T(x,\xi) = (T_0^{\mathrm{ext}}(\lambda_1, x, \xi), \ldots, T_0^{\mathrm{ext}}(\lambda_{2(n+n_\xi)+1}, x, \xi)), \\ &T_0^{\mathrm{ext}}(\lambda, x, \xi) = H(\lambda I - F)^{-1}(G T_0(\lambda, x) - \xi) \end{aligned}$$

is injective and verifies

$$\frac{\partial T}{\partial (x,\xi)}(x,\xi) \begin{bmatrix} f(x) \\ F\xi + G h(x) \end{bmatrix} = A T(x,\xi) + B H \xi \ , \tag{28}$$

with

$$A = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_{2(n+n_\xi)+1}\right) \ , \ B = \begin{bmatrix} 1 \ \ldots \ 1 \end{bmatrix}^\top \ . \tag{29}$$

Proof : First, that for all (λ, x, ξ) in $\Theta_0^{\text{ext}} \times \mathcal{O} \times \mathbb{R}^{n_\xi}$

$$\begin{aligned}
\frac{\partial T_0^{\text{ext}}}{\partial x}(\lambda, x, \xi) f(x) + \frac{\partial T_0^{\text{ext}}}{\partial \xi}(\lambda, x, \xi)(F\xi + Gh(x)) \\
&= H(\lambda I - F)^{-1} G \frac{\partial T_0}{\partial x}(\lambda, x) f(x) - H(\lambda I - F)^{-1}(F\xi + Gh(x)) \\
&= \lambda H(\lambda I - F)^{-1} G T_0(\lambda, x) - H(\lambda I - F)^{-1} F\xi \\
&= \lambda \left(T_0^{\text{ext}}(\lambda, x, \xi) + H(\lambda I - F)^{-1}\xi\right) - H(\lambda I - F)^{-1} F\xi \\
&= \lambda T_0^{\text{ext}}(\lambda, x, \xi) + H\xi.
\end{aligned} \tag{30}$$

We next show injectivity of the mapping T built from T_0^{ext} by picking $2(n + n_\xi) + 1$ distinct λ. For that, our aim is to apply Lemma 5.3. To do so, let $(\lambda, x_a, x_b, \xi_a, \xi_b)$ in $\Theta_0 \times \mathcal{O}^2 \times \mathbb{R}^{2n_\xi}$ verifying $(x_a, \xi_a) \neq (x_b, \xi_b)$. We have for each $\lambda \notin \sigma(F)$

$$H(\lambda I - F)^{-1} G = \frac{\gamma}{d_{n_\xi}(\lambda)} \ , \ d_{n_\xi}(\lambda) = \lambda^{n_\xi} + \sum_{j=0}^{n_\xi - 1} a_j \lambda^j \ .$$

Moreover,

$$H(\lambda I - F)^{-1} = \begin{pmatrix} \frac{p_{n_\xi - 1}(\lambda)}{d_{n_\xi}(\lambda)} & \cdots & \cdots & \frac{p_1(\lambda)}{d_{n_\xi}(\lambda)} & \frac{1}{d_{n_\xi}(\lambda)} \end{pmatrix}$$

where p_j are polynomials of degree j and d_{n_ξ} a polynomial of degree n_ξ. Let us denote (forgetting the dependency in the variables (x_a, ξ_a, x_b, ξ_b))

$$\mathfrak{g}(\lambda) = T_0^{\text{ext}}(\lambda, x_a, \xi_a) - T_0^{\text{ext}}(\lambda, x_b, \xi_b) \ , \ g_0(\lambda) = T_0(\lambda, x_a) - T_0(\lambda, x_b) \ ,$$

and $\tilde{\xi} = (\tilde{\xi}_a, \ldots, \tilde{\xi}_{n_\xi}) = \xi_a - \xi_b$. Note that

$$\mathfrak{g}(\lambda) = \frac{\sum_{j=0}^{n_\xi - 1} \tilde{\xi}_j p_j(\lambda) + \gamma g_0(\lambda)}{d_{n_\xi}(\lambda)} \ . \tag{31}$$

This gives[4]

$$\mathfrak{g}^{(1)}(\lambda) = \frac{d_{n_\xi}^{(1)}(\lambda)}{d_{n_\xi}(\lambda)} \mathfrak{g}(\lambda) + \frac{\sum_{j=1}^{n_\xi - 1} \tilde{\xi}_j p_j^{(1)}(\lambda) + \gamma g_0^{(1)}(\lambda)}{d_{n_\xi}(\lambda)} \ .$$

which more generally gives for all $\ell \in \mathbb{N}$ and some integers $(c_{ir\ell})$

$$\mathfrak{g}^{(\ell)}(\lambda) = \sum_{r=0}^{\ell - 1} \sum_{i=1}^{\ell} c_{ir\ell} \frac{d_{n_\xi}^{(i)}(\lambda)}{d_{n_\xi}(\lambda)} \mathfrak{g}^{(r)}(\lambda) + \frac{\sum_{j=\ell}^{n_\xi - 1} \tilde{\xi}_j p_j^{(\ell)}(\lambda) + \gamma g_0^{(\ell)}(\lambda)}{d_{n_\xi}(\lambda)} \ . \tag{32}$$

The former expression gives for $\ell \geqslant n_\xi$:

$$\mathfrak{g}^{(\ell)}(\lambda) = \sum_{r=0}^{\ell - 1} \sum_{i=1}^{\ell} c_{ir\ell} \frac{d_{n_\xi}^{(i)}(\lambda)}{d_{n_\xi}(\lambda)} \mathfrak{g}^{(r)}(\lambda) + \frac{\gamma}{d_{n_\xi}(\lambda)} g_0^{(\ell)}(\lambda) \tag{33}$$

[4] With the notation $\mathfrak{g}^{(\ell)}(\lambda) = \frac{\partial^\ell \mathfrak{g}}{\partial \lambda^\ell}(\lambda)$.

If $x_a \neq x_b$, with Assumption 4.1, there exists k in $\mathbb{N}$ such that

$$\forall i \in \{0, \dots, k-1\}, \ \frac{\partial^{n_\xi+i} g_0}{\partial \lambda^{n_\xi+i}}(\lambda) = 0 \quad \text{and} \quad \frac{\partial^{n_\xi+k} g_0}{\partial \lambda^{n_\xi+k}}(\lambda) \neq 0 \ . \tag{34}$$

Combining (33) and (34), there exists k in $\mathbb{N}$ such that

$$\frac{\partial^k T_0^{\text{ext}}}{\partial \lambda^k}(\lambda, x_a, \xi_a) - \frac{\partial^k T_0^{\text{ext}}}{\partial \lambda^k}(\lambda, x_b, \xi_b) \ \neq \ 0 \ . \tag{35}$$

Indeed, otherwise, (33) implies that $g_0^{(\ell)}(\lambda) = 0$ for all $\ell \geqslant n_\xi$ which contradicts (34).

Otherwise, $x_a = x_b$ and $\xi_a \neq \xi_b$. We thus have $g_0^{(\ell)}(\lambda) = 0$ for all ℓ, $\tilde{\xi} \neq 0$ and (by (32))

$$\mathfrak{g}^{(\ell)}(\lambda) = \sum_{r=0}^{\ell-1} \sum_{i=1}^{\ell} c_{ir\ell} \frac{d_{n_\xi}^{(i)}(\lambda)}{d_{n_\xi}(\lambda)} \mathfrak{g}^{(r)}(\lambda) + \frac{\sum_{j=\ell}^{n_\xi-1} \tilde{\xi}_j p_j^{(\ell)}(\lambda)}{d_{n_\xi}(\lambda)} \ .$$

Again, this implies (35) for some $k \in \mathbb{N}$. Indeed, otherwise, $\sum_{j=\ell}^{n_\xi-1} \tilde{\xi}_j p_j^{(\ell)}(\lambda) = 0$ for all ℓ which implies that $\tilde{\xi} = 0$ (since each p_j is of degree j), which contradicts $\xi_a \neq \xi_b$.

To conclude, this implies that for all $(\lambda, x_a, x_b, \xi_a, \xi_b)$ in $\Theta_1 \times \mathcal{O}^2 \times \mathbb{R}^{2n_\xi}$ verifying $(x_a, \xi_a) \neq (x_b, \xi_b)$, there exists k in $\mathbb{N}$ such that (35) holds. Applying Lemma 5.3 with $\Upsilon = (\mathcal{O} \times \mathbb{R}^{n_\xi})^2$, $\Theta_i = \Theta_0^{\text{ext}}$ and $g_i(\lambda, x_a, \xi_a, x_b, \xi_b) = T_0^{\text{ext}}(\lambda, x_a, \xi_a) - T_0^{\text{ext}}(\lambda, x_b, \xi_b)$ for $i \in \{1, \dots, 2(n + n_\xi) + 1\}$, we obtain the result. □

Example 4.3 *Consider again an harmonic oscillator with unknown frequency given in* (20)*, but this time with a simple filter in the form*

$$\dot{\xi} = -a\xi + y \tag{36}$$

with $a > 0$*. Note that the function* T_0 *given in* (22) *is solution of* (26)*, is analytic and sastisfies Assumption 4.1. With the former theorem, we know that for almost all* 9 *negative real numbers* λ_i *(different from* $-a$*) the system*

$$(\hat{x}, \hat{\xi}) = T^{\text{inv}}(z_1, \dots, z_9) \ , \ \dot{z}_i = \lambda_i z_i + \xi \ ,$$

where T^{inv} *is any continuous function which satisfies*

$$T^{\text{inv}}(T_0^{\text{ext}}(\lambda_1, x, \xi), \dots, T_0^{\text{ext}}(\lambda_9, x, \xi)) = (x, \xi)$$

where

$$T_0^{\text{ext}}(\lambda, x, \xi) = \frac{1}{\lambda + a}\left[\frac{-\lambda x_1 - x_b}{\lambda^2 + x_3} - \xi\right] ,$$

is an observer. In fact, on this example, 9 different eigenvalues are not required to get injectivity of the mapping $(x, \xi) \mapsto (T_0^{\text{ext}}(\lambda_1, x, \xi), \ldots, T_0^{\text{ext}}(\lambda_9, x, \xi))$*. Indeed, we have for all* (x_a, x_b) *in* $\mathcal{O}$ *defined in* (21) *and all* λ

$$T_0(\lambda, x_a) - T_0(\lambda, x_b) = \frac{1}{(\lambda^2 + x_{a,3})(\lambda^2 + x_{b,3})}\begin{bmatrix}1 \ \lambda \ \lambda^2 \ \lambda^3\end{bmatrix} v(x_a, x_b)$$

with

$$v(x_a, x_b) = \begin{bmatrix} x_{a,3}x_{b,2} - x_{b,3}x_{a,2} \\ x_{a,3}x_{b,1} - x_{b,3}x_{a,1} \\ x_{b,2} - x_{a,2} \\ x_{b,1} - x_{a,1} \end{bmatrix}$$

It yields for all $\lambda_i \neq -a$, $i = 1, \ldots, 5$, *for all* (x_a, ξ_a, x_b, ξ_b) *in* $(\mathcal{O} \times \mathbb{R})^2$,

$$\begin{aligned} T(x_a, \xi_a) - T(x_b, \xi_b) &= \begin{bmatrix} T_0^{\text{ext}}(\lambda_1, x_a, \xi_a) - T_0^{\text{ext}}(\lambda_1, x_b, \xi_b) \\ \vdots \\ T_0^{\text{ext}}(\lambda_5, x_a, \xi_a) - T_0^{\text{ext}}(\lambda_5, x_b, \xi_b) \end{bmatrix} \\ &= \mathfrak{D}(\lambda_1, \ldots, \lambda_5)\mathfrak{V}(\lambda_1, \ldots, \lambda_5)\begin{bmatrix} v(x_a, x_b) + w(x_a, x_b)(\xi_b - \xi_a) \\ \xi_b - \xi_a \end{bmatrix} . \end{aligned}$$

where $\mathfrak{V}$ *is the Vandermonde matrix*

$$\mathfrak{V}(\lambda_1, \ldots, \lambda_5) = \begin{bmatrix} 1 \ \lambda_1 \ \lambda_1^2 \ \lambda_1^3 \ \lambda_1^4 \\ \vdots \\ 1 \ \lambda_5 \ \lambda_5^2 \ \lambda_5^3 \ \lambda_5^4 \end{bmatrix} ,$$

which is invertible as soon as the λ_i*'s are all different and*

$$\mathfrak{D}(\lambda_1, \ldots, \lambda_5) = \text{diag}\left\{\frac{1}{(\lambda_i + a)(\lambda_i^2 + x_{a,3})(\lambda_i^2 + x_{b,3})}\right\} .$$

is also invertible and well defined for (x_a, ξ_a, x_b, ξ_b) *in* $(\mathcal{O} \times \mathbb{R})^2$*. Note that injectivity of the mapping* T *is obtained since from the former expression* $T(x_a, \xi_a) - T(x_b, \xi_b) = 0$ *implies that* $\xi_b = \xi_a$ *and* $v(x_a, x_b) = 0$*. Moreover, for* (x_a, x_b) *in* $\mathcal{O}^2$, $v(x_a, x_b) = 0$ *implies* $x_a = x_b$.

5 Proof of Theorem 3.3

5.1 A proof based on diagonalization

The proof relies on two main ideas:

- almost any matrix A of dimension $2n+1$ is diagonalizable, with a spectrum decomposed into 2ℓ complex conjugate eigenvalues and $2(n-\ell)+1$ real eigenvalues for some $\ell \in \{0, \dots, n\}$;
- a generic choice of ℓ complex eigenvalues and $2(n-\ell)+1$ real eigenvalues for ℓ describing $\{0, \dots, n\}$ yields a generic choice of matrix A of dimension $2n+1$.

The construction of the zero-measure set $\mathcal{J}$ allowing to prove Theorem 3.3 is thus based on the following preliminary result which investigates the existence of an injective solution to (6) in the case where A is a diagonal Hurwitz matrix with ℓ complex eigenvalues with real part smaller than $-\rho$ and $2(n-\ell)+1$ real eigenvalues smaller than $-\rho$. In that case, to define the observer state space, for ℓ in $\{0, \dots, n\}$, we introduce

$$\Omega_\ell = \mathbb{C}^\ell \times \mathbb{R}^{2(n-\ell)+1} \ . \tag{37}$$

Also, given a positive real number ρ and ℓ in $\{0, \dots, n\}$, we consider the set $\Omega_{\ell,\rho}$ defined as (see (9))

$$\Omega_{\ell,\rho} = \mathbb{C}_\rho^\ell \times \mathbb{R}_\rho^{2(n-\ell)+1} \ . \tag{38}$$

The following result can be stated.

Proposition 5.1 *Assume System (3) satisfies Assumption 3.1. Then there exist a positive real number ρ such that for each ℓ in $\{0, \dots, n\}$, there exists a zero Lebesgue measure subset $\mathcal{I}_\ell$ of $\Omega_{\ell,\rho}$ such that for each $(\lambda_1, \dots, \lambda_{2n-\ell+1})$ in $\Omega_{\ell,\rho} \setminus \mathcal{I}_\ell$, there exists an injective C^1 function $T_{\text{diag}} : \mathcal{O} \mapsto \Omega_\ell$ verifying* (6) *with*

$$A_{\text{diag}} = \text{diag}\left(\lambda_1 \ \dots, \lambda_{2n-\ell+1}\right) \ , \ B_{\text{diag}} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \ . \tag{39}$$

Note that the first ℓ components of the map T_{diag} provided by Proposition 5.1 are complex valued. Considering the real and imaginary parts of these components, we obtain a map $T_{\text{real}} : \mathcal{O} \to \mathbb{R}^{2n+1}$ which is an injective C^1

solution to (6) with

$$A_{\text{real}} = \operatorname{diag}(\Lambda_1, \dots, \Lambda_\ell, \lambda_{2\ell+1}, \lambda_{2n-\ell+1}) \quad , \quad B_{\text{real}} = \begin{bmatrix} B_1 \\ \vdots \\ B_{2n-\ell+1} \end{bmatrix} \tag{40}$$

where Λ_i and B_i take the form

$$\Lambda_i = \begin{bmatrix} \operatorname{Re}(\lambda_i) & -\operatorname{Im}(\lambda_i) \\ \operatorname{Im}(\lambda_i) & \operatorname{Re}(\lambda_i) \end{bmatrix} \quad , \quad B_i = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{for } i \in \{1, \dots, \ell\} \\ 1 & \text{for } i \in \{\ell+1, \dots, 2n-\ell+1\} \ . \end{cases}$$

The proof of Proposition 5.1 is postponed to Section 5.2. In the meantime, we prove Theorem 3.2 by translating the generic choice of eigenvalues in Proposition 5.1 into a generic choice of the pair (A, B) of dimension $2n+1$.

Lemma 5.2 *For ℓ in $\{0, \dots, n\}$, let $\mathcal{I}_\ell$ be a zero measure subset of Ω_ℓ. The set of matrices in $\mathbb{R}^{(2n+1)\times(2n+1)}$ with characteristic polynomial*

$$\prod_{j=1}^{\ell} (s^2 - 2\operatorname{Re}(\lambda_j)s + \operatorname{Im}(\lambda_j)^2) \times \prod_{j=2\ell+1}^{2n-\ell+1} (s - \lambda_j)$$

for some $(\lambda_1, \dots, \lambda_{2n-\ell+1}) \in \mathcal{I}_\ell$ is of zero-measure in $\mathbb{R}^{(2n+1)\times(2n+1)}$.

Proof : Let $\mathbb{R}_{2n+1}[s]^{\text{unit}}$ be the set of unitary real polynomials of degree $2n+1$. Consider $\phi : \Omega_{\ell,\rho} \to \mathbb{R}_{2n+1}[s]^{\text{unit}}$ such that

$$\phi(\lambda_1, \dots, \lambda_{2n-\ell+1}) : s \mapsto \prod_{j=1}^{\ell} (s^2 - 2\operatorname{Re}(\lambda_j)s + \operatorname{Im}(\lambda_j)^2) \times \prod_{j=2\ell+1}^{2n-\ell+1} (s - \lambda_j)$$

The map ϕ associates to a list of ℓ complex roots and $2(n-\ell)+1$ real roots, the unitary polynomial of degree $2n+1$ with real coefficients having those roots. By identifying $\mathbb{R}_{2n+1}[s]^{\text{unit}}$ with a list of $2n+1$ coefficients in $\mathbb{R}^{2n+1}$, ϕ is C^1 from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+1}$. From which, we concludes that $\phi(\mathcal{I}_\ell)$ is a zero measure subset of $\mathbb{R}_{2n+1}[s]^{\text{unit}}$ assimilated to $\mathbb{R}^{2n+1}$ (see for instance [13, Theorem 3 in §3]). Consider now $\Phi : \mathbb{R}^{(2n+1)\times(2n+1)} \to \mathbb{R}_{2n+1}[s]^{\text{unit}}$ defined as

$$\Phi(A) = \det(A - sI_{2n+1})$$

This map is C^∞ (still identifying $\mathbb{R}_{2n+1}[s]^{\text{unit}}$ with $\mathbb{R}^{2n+1}$) Let us show that it is a submersion almost everywhere. All the coefficients of $\det(A - sI_{2n+1})$ are polynomials of the coefficients of A. It follows that $\frac{\partial\Phi}{\partial A}(A)$ is a rectangular matrix of dimension $(2n+1)\times(2n+1)^2$ whose coefficients are polynomials of the coefficients of A. The set of matrices A such that $\text{rank}\frac{\partial\Phi}{\partial A}(A) < 2n+1$ is characterized by the determinant of each minor being zero, which is thus an algebraic set of zero-measure. Hence Φ is a submersion. With[5] [25, Theorem 1], we therefore conclude that the set $\mathcal{S}_\ell$ defined as

$$\mathcal{S}_\ell = \{A, \Phi(A) \in \phi(\mathcal{I}_\ell)\} \tag{41}$$

is a zero Lebesgue measure subset of $\mathbb{R}^{(2n+1)\times(2n+1)}$. □

Let ρ be a positive real number, and for each ℓ in $\{0, \dots, n\}$, let $\mathcal{I}_\ell$ be a zero measure subset of $\Omega_{\ell,\rho}$ as given by Proposition 5.1 and consider the sets

$$\mathcal{J}_{NC} = \{(A, B) \in \mathbb{R}^{(2n+1)\times(2n+1)} \times \mathbb{R}^{2n+1}, (A, B) \text{ is not controllable}\}\ , \tag{42}$$

$$\mathcal{J}_{ND} = \{(A, B) \in \mathbb{R}^{(2n+1)\times(2n+1)} \times \mathbb{R}^{2n+1}, A \text{ is not diagonalizable in } \mathbb{C}\}\ , \tag{43}$$

$$\mathcal{J}_\ell = \{(A, B) \in \mathbb{R}^{(2n+1)\times(2n+1)} \times \mathbb{R}^{2n+1}, A \in \mathcal{S}_\ell\}\ . \tag{44}$$

It is well-known that $\mathcal{J}_{NC}$ and $\mathcal{J}_{ND}$ are of zero-measure. Applying Lemma 5.2, we conclude that the set $\mathcal{J} = \mathcal{J}_{NC} \cup \mathcal{J}_{ND} \cup (\bigcup_{\ell=0}^{n} \mathcal{J}_\ell)$ is of zero Lebesgue measure in $\mathbb{R}^{(2n+1)\times(2n+1)} \times \mathbb{R}^{2n+1}$.

Consider now (A, B) in $\mathbb{R}^{(2n+1)\times(2n+1)} \times \mathbb{R}^{2n+1} \setminus \mathcal{J}$, such that $A + \rho I_{2n+1}$ is Hurwitz. We wish to transform (A, B) into $(A_{\text{real}}, B_{\text{real}})$ defined in (40) in order to apply Proposition 5.1. The spectrum of A can be decomposed into 2ℓ complex conjugate eigenvalues and $2(n-\ell)+1$ real eigenvalues for some ℓ in $\{0, \dots, n\}$. By definition of $\mathcal{J}$, A is diagonalizable in $\mathbb{C}$, so there exist $(\lambda_1, \dots, \lambda_{2n-\ell+1})$ in $\Omega_{\ell,\rho}$ and an invertible matrix P in $\mathbb{R}^{(2n+1)\times(2n+1)}$ such that

$$A_{\text{real}} = P^{-1} A P$$

with A_{real} defined in (40). Let

$$\tilde{B} = P^{-1} B = \begin{bmatrix} \tilde{B}_1 \\ \vdots \\ \tilde{B}_{2n-\ell+1} \end{bmatrix}$$

[5] Let $\Phi : U \subset \mathbb{R}^k \to \mathbb{R}^{k'}$ of class $C^{k-k'+1}$, where $k' \leqslant k$. Then, the pre-image of any zero-measure set is of zero-measure if and only if Φ is a submersion almost everywhere, i.e.,

$$\text{rank}\frac{\partial\Phi}{\partial x}(x) = k' \quad \text{for almost all} \quad x \in U$$

with

$$\tilde{B}_i = \begin{cases} \begin{bmatrix} \tilde{b}_{i,1} \\ \tilde{b}_{i,2} \end{bmatrix} \in \mathbb{R}^2 & \text{for } i \in \{1, \dots, \ell\} \\ \tilde{b}_i \in \mathbb{R} & \text{for } i \in \{\ell+1, \dots, 2n-\ell+1\} \ . \end{cases}$$

and

$$M = \operatorname{diag}(M_1, \dots, M_{2n-\ell+1})$$

with

$$M_i = \begin{cases} \begin{bmatrix} \tilde{b}_{i,1} & -\tilde{b}_{i,2} \\ \tilde{b}_{i,2} & \tilde{b}_{i,1} \end{bmatrix} & \text{for } i \in \{1, \dots, \ell\} \\ \tilde{b}_i & \text{for } i \in \{\ell+1, \dots, 2n-\ell+1\} \end{cases}$$

so that $MA_{\text{real}} = A_{\text{real}}M$ and $MB_{\text{real}} = \tilde{B}$.

Since $A \notin \mathcal{J}_\ell$, the vector $(\lambda_1, \dots, \lambda_{2n-\ell+1})$ with (41) is not in $\mathcal{I}_\ell$. Hence, according to Proposition 5.1, there exists an injective C^1 function $T_{\text{real}} : \mathcal{O} \mapsto \mathbb{R}^{2n+1}$ such that, for all x in $\mathcal{X}$,

$$\frac{\partial T_{\text{real}}}{\partial x} f(x) = A_{\text{real}} T_{\text{real}}(x) + B_{\text{real}} h(x) \ . \tag{45}$$

with B_{real} as in (40).

Finally, let $T : \mathcal{O} \mapsto \mathbb{R}^{2n+1}$ be the mapping

$$T(x) = PMT_{\text{real}}(x) \ .$$

Since the pair (A, B) is controllable, and P invertible, the pair $(A_{\text{real}}, \tilde{B})$ is also controllable. Hence, this yields that for all i, $\tilde{B}_i \neq 0$. Consequently, the matrix M is invertible. Thus T is injective on $\mathcal{O}$. Besides, for all x in $\mathcal{X}$,

$$\begin{aligned} \frac{\partial T}{\partial x}(x) f(x) &= PMA_{\text{real}} T_{\text{real}}(x) + PMB_{\text{real}} h(x) \ , \\ &= PA_{\text{real}} P^{-1} PMT_{\text{real}}(x) + P\tilde{B} h(x) \ , \\ &= AT(x) + Bh(x) \ . \end{aligned}$$

5.2 Proof of Proposition 5.1

5.2.1 Some variations on Coron's lemma

The proof of Proposition 5.1 is based on the following lemma.

Lemma 5.3 *Let Υ be an open subset of $\mathbb{R}^{2n}$, and Θ_i, g_i, p_i be such that for all $i \in \{1, \dots, m\}$,*

- *either* Θ_i *is an open subset of* $\mathbb{R}$, $g_i : \Theta_i \times \Upsilon \to \mathbb{R}$ *is in* $C^\infty(\mathbb{R}; C^1(\mathbb{R}^{2n}; \mathbb{R}))$ *and* $p_i = 1$ *;*
- *or* Θ_i *is an open subset of* $\mathbb{C}$, $g_i : \Theta_i \times \Upsilon \to \mathbb{C}$ *is holomorphic with respect to* λ *and* C^1 *with respect to* x*, and* $p_i = 2$.

Then, if $\sum_i p_i \geqslant 2n + 1$*, and if for all* $i \in \{1, \dots, m\}$*, for all* $(\lambda, x) \in \Theta_i \times \Upsilon$*, there exists* $k_i \in \mathbb{N}$ *such that*

$$\frac{\partial^{k_i} g_i}{\partial \lambda^{k_i}}(\lambda, x) \neq 0 \tag{46}$$

then the following set has zero Lebesgue measure in $\prod_{i=1}^m \Theta_i$*:*

$$\mathcal{I} = \bigcup_{x \in \Upsilon} \left\{ (\lambda_i)_{i \in \{1,\dots,m\}} \in \prod_{i=1}^m \Theta_i : \; g_i(\lambda_i, x) = 0 \quad \forall i \in \{1, \dots, m\} \right\} . \tag{47}$$

This lemma is an extension of [12, Lemma 3.2] as well as the version given in [4, Lemma 3.2]:

- In those previous versions, the functions g_i were the same for each i but this does not make any significant difference in the proof.
- In [12, Lemma 3.2], the functions g_i are in $C^\infty(\mathbb{R} \times \mathbb{R}^{2n}; \mathbb{R})$ instead of $C^\infty(\mathbb{R}; C^1(\mathbb{R}^{2n}; \mathbb{R}))$ here. This loss of regularity is not a problem. Instead of the Malgrange theorem of [15], we employ the one obtained in [23].
- In [4, Lemma 3.2], the functions g_i are holomorphic with respect to λ and C^1 with respect to x.

Apart from these modifications, the proof follows readily and is based on the fact that $\mathcal{I}$ is contained in the countable union of image sets through C^1 functions taking values in a real submanifold of dimension $2n$ of $\prod_i \Theta_i$ (which is a real manifold of dimension $\sum_i p_i \geqslant 2n + 1$). Hence the result is obtained from a variation of Sard theorem. The proof is provided in Appendix B.

Now following [4], the idea to prove Proposition 5.1 is first to exhibit a C^1 solution to (6) with $(A_{\text{diag}}, B_{\text{diag}})$ as in (39). This solution is parameterized by the $(\lambda_i)'s$. With the distinguishability assumption and the use of Lemma 5.3, it is then shown that generically this function is injective on $\mathcal{O}$.

5.2.2 *Construction of* T_{diag}

Let $\delta_b > \delta_d$ be any positive real number where δ_d is given by Assumption 3.1. Let $\rho = \max_{x \in \mathcal{O} + \delta_b} \left| \frac{\partial \breve{f}}{\partial x}(x) \right|$ where $\breve{f} = \chi f$ and where $\chi : \mathbb{R}^n \to \mathbb{R}$ is a C^∞

function such that

$$\chi(x) = \begin{cases} 0\ , & x \notin \mathcal{O} + \delta_b \\ 1\ , & x \in \mathcal{O} + \delta_d\ . \end{cases} \tag{48}$$

Fix ℓ in $\{1, \dots, n\}$. For each $(\lambda_1, \dots, \lambda_{2n-\ell+1})$ in $\Omega_{\ell,\rho}$, we can define the mapping $T_{\rm diag} : \mathcal{O} \mapsto \Omega_{\ell,\rho}$ defined as

$$T_{\rm diag}(x) = (T_0(\lambda_1, x), \dots, T_0(\lambda_{2n-\ell+1}, x)) \tag{49}$$

with $T_0 : \mathbb{C}_\rho \times \mathcal{O} \to \mathbb{R}$ defined as

$$T_0(\lambda, x) \ = \ \int_{-\infty}^{0} \exp(-\lambda s) h(\breve{X}(x, s)) ds \tag{50}$$

where $\breve{X} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the flow of the modified system

$$\dot{x} = \breve{f}(x) = \chi(x) f(x)\ . \tag{51}$$

To prove Proposition 5.1, we need to show that $T_{\rm diag}$ is solution to the PDE (6) and also that it has enough regularity to apply Lemma 5.3 to obtain injectivity. First recall the following fact.

Proposition 5.4 ([3, Proposition 3.3]) *The function $T_{\rm diag}$ is C^1 and satisfies* (6) *with $(A_{\rm diag}, B_{\rm diag})$ given in* (39).

Moreover,

- The map $T_0(\cdot, x)$ is holomorphic on $\mathbb{C}_\rho$ for each $x \in \mathcal{O}$.
- The restriction of $T_0(\cdot, x)$ to $\mathbb{R}_\rho$ is C^∞ (actually, analytic) for each $x \in \mathcal{O}$. Moreover, for all $k \in \mathbb{N}$, $\frac{\partial^k T_0}{\partial \lambda^k}(\lambda, \cdot)$ can be shown to be C^1 for any $\lambda \in \mathbb{R}_\rho$ by following readily the proof of [3, Proposition 3.3].

To prove Proposition 5.1, it now remains to show injectivity by applying Lemma 5.3.

5.2.3 Injectivity of $T_{\rm diag}$

Let $\Upsilon = \{(x_a, x_b) \in \mathcal{O}^2, x_a \neq x_b\}$. Let also $\Theta_i = \mathbb{C}_\rho$ for $i \in \{1, \dots, \ell\}$ and $\Theta_i = \mathbb{R}_\rho$ for $i \in \{\ell+1, \dots, 2n-\ell+1\}$. Let $g_i : \Theta_i \times \Upsilon \mapsto \mathbb{C}$ for $i \in \{1, \dots, \ell\}$ and $g_i : \Theta_i \times \Upsilon \mapsto \mathbb{R}$ for $i \in \{\ell+1, \dots, 2n-\ell+1\}$ be defined by

$$g_i(\lambda, x_a, x_b) = T_0(\lambda, x_a) - T_0(\lambda, x_b)\ , \tag{52}$$

$$= \int_{-\infty}^{0} \exp(-(\lambda + \rho)s) \Delta(x_a, x_b, s) ds\ , \tag{53}$$

for all $(x_a, x_b) \in \Upsilon$ and all $\lambda \in \Theta_i$, where

$$\Delta(x_a, x_b, s) = \exp(\rho s)\left[h(\breve{X}(x_a, s)) - h(\breve{X}(x_b, s))\right] .$$

By Assumption 3.1, for all (x_a, x_b) in Υ there exists a negative time t in $\left(\max\left\{\sigma^-_{\mathcal{O}+\delta_d}(x_a), \sigma^-_{\mathcal{O}+\delta_d}(x_b)\right\}, 0\right]$ such that :

$$h(X(x_a, t)) \neq h(X(x_b, t)) .$$

Moreover, by definition of χ in (51), $X(x, s) = \breve{X}(x, s)$ for all $x \in \mathcal{O}$ and all $s \in (\sigma^-_{\mathcal{O}+\delta_d}(x), 0]$. It yields that for all (x_a, x_b) in Υ, there exists $s < 0$ such that $\Delta(x_a, x_b, s) \neq 0$.

From there, two cases may be distinguished.

- For $i \in \{1, \ldots, \ell\}$, for each $(x_a, x_b) \in \Upsilon$, $g_i(\cdot, x_a, x_b)$ is holomorphic (since $T_0(\cdot, x)$ is holomorphic for each $x \in \mathcal{O}$) and consequently, there exists k_i such that (46) is satisfied.
- For $i \in \{\ell + 1, \ldots, 2n - \ell + 1\}$, similarly to the proof of the injectivity of the Laplace transform, for all $\lambda \in \mathbb{R}_\rho$, with $u = \exp(s)$, yields

  $$g_i(\lambda, x_a, x_b) = \int_0^1 u^{-(\lambda+\rho)-1}\bar{\Delta}(u)du$$

  where $\bar{\Delta}$ is a continuous function defined by $\bar{\Delta}(u) = \Delta(x_a, x_b, \ln(u))$ for $u > 0$ and $\bar{\Delta}(0) = 0$. We deduce that $g_i(\cdot, x_a, x_b)$ is not identically zero on $\mathbb{R}_\rho$. Indeed, otherwise, picking $\lambda = -(j + \rho + 1)$ for each $j \in \mathbb{N}$, we get

  $$\int_0^1 u^j \bar{\Delta}(u)du = 0 .$$

  By Stone-Weierstrass theorem, for each $\epsilon > 0$, there exists a polynomial P_ϵ such that

  $$|\bar{\Delta}(u) - P_\epsilon(u)| \leqslant \epsilon \ , \ \forall u \in [0, 1].$$

  Hence,

  $$\int_0^1 \bar{\Delta}(u)^2 du = \int_0^1 \bar{\Delta}(u)(\bar{\Delta}(u) - P_\epsilon(u))du \leqslant \max_{u\in[0,1]} |\bar{\Delta}(u)|\epsilon .$$

  The former inequality being true for all ϵ, it yields that $\bar{\Delta}$ is identically zero on $[0, 1]$, which is a contradiction since Δ is not identically zero. Therefore, $g_i(\cdot, x_a, x_b)$ is not identically zero on $\mathbb{R}_\rho$. Since moreover g_i is analytic, it yields that there exists k_i such that (46) is satisfied.

We can finally apply Lemma 5.3, to obtain the set $\mathcal{I}_\ell$ given in (47). By definition of g_i and of $\mathcal{I}_\ell$, we conclude that the map T_{diag} defined in (49) is injective on $\mathcal{O}$ for any $(\lambda_1, \ldots, \lambda_{2n-\ell+1})$ in $\Omega_{\ell,\rho} \setminus \mathcal{I}_\ell$. This concludes the proof.

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A Proof of Theorem 1.1

Since T is injective on the compact set $\mathrm{cl}(\mathcal{X})$, there exists a continuous map $T^{\mathrm{inv}} : \mathbb{R}^m \to \mathbb{R}^n$ such that (5) holds (see [22] for instance). According to (6) holding on $\mathcal{X}$, and because A is Hurwitz,

$$\lim_{t\to+\infty} |Z(z,x,t) - T(X(x,t))| = 0.$$

Consider $\delta > 0$. Since $X(x,t) \in \mathcal{X}$ for all $t \geqslant 0$, there exists $\bar{t} \geqslant 0$ such that for all $t > \bar{t}$, $Z(z,x,t) \in T(\mathcal{X}) + \delta$. Besides, T^{inv} is continuous on the compact set $\mathrm{cl}(T(\mathcal{X}) + \delta)$, so there exists a class-$\mathcal{K}$ map ρ such that

$$|T^{\mathrm{inv}}(z_a) - T^{\mathrm{inv}}(z_b)| \leqslant \rho(|z_a - z_b|) \quad \forall (z_a, z_b) \in \mathrm{cl}(T(\mathcal{X}) + \delta) \times \mathrm{cl}(T(\mathcal{X}) + \delta)\ .$$

Applying this inequality with $z_a = T(X(x,t))$ and $z_b = Z(z,x,t)$ for $t > \bar{t}$ then gives the result using (5).

B Proof of Lemma 5.3

This part should be removed from the final version of the paper. It is a reproduction of the proof given in [4] with the small update related to the use of real or complex valued functions. The differences are in blue.

Let $\bar{\Theta} = \prod_i \Theta_i$. Assume that $\sum_i p_i \geqslant 2n + 1$. The idea of the proof is to show that the set $\mathcal{I}$ is contained in a countable union of sets which have zero Lebesgue measure.

Given $(\epsilon, \underline{\Lambda}, \underline{x})$ in $\mathbb{R}_{>0} \times \bar{\Theta} \times \Upsilon$, we denote by $S_{\epsilon,\underline{\Lambda},\underline{x}}$ the set :

$$S_{\epsilon,\underline{\Lambda},\underline{x}} = \bigcup_{x\in\mathcal{B}_\epsilon(\underline{x})} \{\Lambda \in \mathcal{B}_\epsilon(\underline{\Lambda}) : \quad g_\ell(\lambda_\ell, x) = 0 \quad \forall \ell \in \{1,\dots,m\}\}\ . \tag{B.1}$$

Assume for the time being that, for each pair $(\underline{\Lambda}, \underline{x})$ in $\Upsilon \times \bar{\Theta}$, we can find a positive real number ϵ and a countable family of C^1 functions $\sigma_i : \mathcal{B}_\epsilon(\underline{x}) \to \bar{\Theta}$, such that we have :

$$S_{\epsilon,\underline{\Lambda},\underline{x}} \subset \bigcup_{i\in\mathbb{N}} \sigma_i(\mathcal{B}_\epsilon(\underline{x}))\ . \tag{B.2}$$

The family $(\mathcal{B}_\epsilon(\underline{\Lambda}) \times \mathcal{B}_\epsilon(\underline{x}))_{(\underline{\Lambda},\underline{x})\in\bar{\Theta}\times\Upsilon}$ is a covering of $\bar{\Theta} \times \Upsilon$ by open subsets. From Lindelöf Theorem (see [9, Lemma 4.1] for instance), there exists a countable family $\left\{(\underline{\Lambda}_j, \underline{x}_j)\right\}_{j\in\mathbb{N}}$ such that we have :

$$\bar{\Theta} \times \Upsilon \ \subset \ \bigcup_{j\in\mathbb{N}} \mathcal{B}_{\epsilon_j}(\underline{\Lambda}_j) \times \mathcal{B}_{\epsilon_j}(\underline{x}_j) \ ,$$

where ϵ_j denotes the ϵ associated to the pair $(\underline{\Lambda}_j \times \underline{x}_j)$. With (B.2), it follows that we have :

$$\mathcal{I} \ \subset \ \bigcup_{j\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \sigma_{i,j}(\mathcal{B}_{\epsilon_j}(\underline{x}_j)) \ ,$$

where $\sigma_{i,j}$ denotes the ith function σ associated with the pair $(\underline{\Lambda}_j, \underline{x}_j)$. The set $\sigma_{i,j}(\mathcal{B}_{\epsilon_j}(\underline{x}_j))$ is the image, contained in $\bar{\Theta}$, a real manifold of dimension $\sum_i p_i \geqslant 2n+1$, by a C^1 function of $\mathcal{B}_{\epsilon_j}(\underline{x}_j)$, a real manifold of dimension $2n$. From a variation on Sard's Theorem (see [13, Theorem 3 in §3] for instance), this image $\sigma_{i,j}(\mathcal{B}_{\epsilon_j}(\underline{x}_j))$ has zero Lebesgue measure in $\bar{\Theta}$. So S, being a countable union of such zero Lebesgue measure subsets, has zero Lebesgue measure.

So all we have to do to establish Lemma 5.3 is to prove the existence of ε and the functions σ_i satisfying (B.2) for each pair $(\underline{x}, \underline{\Lambda})$in $\Upsilon \times \bar{\Theta}$. For ε, we consider two cases :

(1) Consider a pair $(\underline{\Lambda}, \underline{x})$ such that $g_j(\underline{\lambda}_\ell, \underline{x})$ is non zero. By continuity of g_j, we can find a positive real number ϵ such that $g(\lambda_\ell, x)$ is also non zero for all Λ in $\mathcal{B}_\epsilon(\underline{\lambda}) \subset \Theta_i$ and all x in $\mathcal{B}_\epsilon(\underline{x})$. In this case, the set $S_{\epsilon,\underline{\Lambda},\underline{x}}$ is empty.

(2) Consider a pair $(\underline{\Lambda}, \underline{x})$ such that $g(\underline{\lambda}_\ell, \underline{x})$ is zero. From the assumption (46), for each ℓ, there exists an integer k_ℓ satisfying :

$$\frac{\partial^i g_{j_\ell}}{\partial \lambda^i}(\underline{\lambda}_\ell, \underline{x}) \ = \ 0 \qquad \forall i \in \{0, \dots, k_\ell - 1\} \qquad , \qquad \frac{\partial^{k_\ell} g_{j_\ell}}{\partial \lambda^{k_\ell}}(\underline{\lambda}_\ell, \underline{x}) \ \neq \ 0 \ .$$

For each ℓ in $\{1, \dots, m\}$, two cases may be distinguished.

(a) If $\Theta_\ell = \mathbb{C}$, following the Weierstrass Preparation Theorem (see [15, Theorem IV.1.1] [6] for instance), we know the existence of a positive real number ϵ_ℓ, a function $q_\ell : \mathcal{B}_{\epsilon_\ell}(\underline{\lambda}_\ell) \times \mathcal{B}_{\epsilon_\ell}(\underline{x}) \to \mathbb{C}$, and k_ℓ C^1 functions $a_j^\ell : \mathbb{R}^{2n} \to \mathbb{C}$ satisfying, for all (λ, x) in $B_{\epsilon_\ell}(\underline{\lambda}_\ell) \times \mathcal{B}_{\epsilon_\ell}(\underline{x})$,

$$q_\ell(\lambda, x)\, g_\ell(\lambda, x) \ = \ (\lambda - \underline{\lambda}_\ell)^{k_\ell} \ + \ \sum_{j=0}^{k_\ell - 1} a_j^\ell(x)(\lambda - \underline{\lambda}_\ell)^j \ . \qquad \text{(B.3)}$$

[6] In [15, Theorem IV.1.1], this theorem is stated with the assumption that g_ℓ is holomorphic in both λ and x. However, as far as x is concerned, it can be seen in the proof of this Theorem that we need only the implicit function theorem to apply. So continuous differentiability in x for each λ is enough.

(b) If $\Theta_\ell = \mathbb{R}$, following the Malgrange Preparation Theorem [23], we know the existence of a positive real number ϵ_ℓ, a function $q_\ell : \mathcal{B}_{\epsilon_\ell}(\underline{\lambda}_\ell) \times \mathcal{B}_{\epsilon_\ell}(\underline{x}) \to \mathbb{R}$, and k_ℓ C^1 functions $a_j^\ell : \mathbb{R}^{2n} \to \mathbb{R}$ satisfying, for all (λ, x) in $B_{\epsilon_\ell}(\underline{\lambda}_\ell) \times \mathcal{B}_{\epsilon_\ell}(\underline{x})$,

$$q_\ell(\lambda, x)\, g_\ell(\lambda, x) \;=\; (\lambda - \underline{\lambda}_\ell)^{k_\ell} \;+\; \sum_{j=0}^{k_\ell - 1} a_j^\ell(x)(\lambda - \underline{\lambda}_\ell)^j\,. \tag{B.4}$$

We choose the real number ϵ, to be associated to $(\underline{\Lambda}, \underline{x})$ in the definition of $S_{\epsilon, \underline{\Lambda}, \underline{x}}$, as :

$$\epsilon \;=\; \inf_{\ell \in \{1,\dots,m\}} \epsilon_\ell\,.$$

In the following $P_\ell : \Theta_\ell \times \mathcal{B}_\epsilon(\underline{x}) \to \Theta_\ell$ and $a^\ell : \mathcal{B}_\epsilon(\underline{x}) \to \Theta^{k_\ell}$ denote :

$$P_\ell(\lambda, x) \;=\; (\lambda - \underline{\lambda}_\ell)^{k_\ell} + \sum_{j=0}^{k_\ell - 1} a_j^\ell(x)(\lambda - \underline{\lambda}_\ell)^j \quad , \quad a^\ell(x) \;=\; (a_0^\ell(x), \dots, a_{k_\ell - 1}^\ell(x))\,.$$

With this definition of ε, we have the following implication, for Λ in $\mathcal{B}_\epsilon(\underline{\Lambda})$ and x in $\mathcal{B}_\epsilon(\underline{x})$,

$$g(\lambda_\ell, x) = 0 \quad \forall \ell \in \{1, \dots, m\} \quad \Rightarrow \quad (\lambda_\ell, a^\ell(x)) \in M^\ell \quad \forall \ell \in \{1, \dots, m\} \tag{B.5}$$

where M^ℓ is the set :

$$M^\ell \;=\; \left\{ (\lambda, (b_0, \dots, b_{k_\ell - 1})) \in \Theta_\ell \times \Theta_\ell^{k_\ell} \;:\; (\lambda - \underline{\lambda}_\ell)^{k_\ell} + \sum_{j=0}^{k_\ell - 1} b_j (\lambda - \underline{\lambda}_\ell)^j = 0 \right\} \tag{B.6}$$

Our interest in this set follows from the following Lemma, which we prove later on,

Lemma B.1 *Let M be the set defined as :*

$$M \;=\; \left\{ (\lambda, b_0, \dots, b_{k-1}) \in \Theta \times \Theta^k \;:\; \lambda^k + \sum_{j=0}^{k-1} b_j \lambda^j = 0 \right\}\,.$$

where $\Theta = \mathbb{C}$ or $\Theta = \mathbb{R}$. There exists a countable family $\{M_m\}_{m \in \mathbb{N}}$ of regular submanifolds of Θ^k and a countable family of C^1 functions $\rho_m : M_m \to \Theta$ such that we have the inclusion :

$$M \;\subset\; \bigcup_{m \in \mathbb{N}} \bigcup_{b \in M_m} \{(\rho_m(b), b)\}\,. \tag{B.7}$$

So, for each ℓ in $\{1, \dots, n+1\}$ we have a countable family $\{M_{m_\ell}^\ell\}_{m_\ell \in \mathbb{N}}$ of regular submanifolds of $\mathbb{C}^{k_\ell}$ and a countable family of C^1 functions $\rho_{m_\ell}^\ell : M_{m_\ell}^\ell \to \mathbb{C}$

such that, for each x in $\mathcal{B}_\epsilon(\underline{x})$, if $P_\ell(\lambda_\ell, x)$ is zero, then there exists an integer m_ℓ such that we have :

$$a^\ell(x) \in M^\ell_{m_\ell} \quad , \quad \lambda_\ell = \rho^\ell_{m_\ell}(a^\ell(x)) \ . \tag{B.8}$$

Hence, with (B.5), if :

$$g(\lambda_\ell, x) = 0 \qquad \forall \ell \in \{1, \dots, m\} \ ,$$

then there exists an m-tuple $\mu = (m_1, \dots, m_m)$ of integers satisfying :

$$a^\ell(x) \in M^\ell_{m_\ell} \ , \quad \lambda_\ell = \rho^\ell_{m_\ell}(a^\ell(x)) \qquad \forall \ell \in \{1, \dots, m\} \ .$$

So, by letting :

$$S^\mu_{\epsilon,\underline{\Lambda},\underline{x}} = \bigcup_{\left\{x \in \mathcal{B}_\epsilon(\underline{x}) : \ a^\ell(x) \in M^\ell_{m_\ell} \ \forall \ell \in \{1,\dots,n+1\}\right\}} \left\{(\rho^1_{m_1}(a^1(x)), \dots, \rho^m_{m_m}(a^m(x))\right\} \tag{B.9}$$

we have established :

$$S_{\epsilon,\underline{\Lambda},\underline{x}} \subset \bigcup_{\mu \in \mathbb{N}^m} S^\mu_{\epsilon,\underline{\Lambda},\underline{x}} \ . \tag{B.10}$$

Comparing (B.2) with (B.10) and using the definition (B.9), we see that a candidate for the function σ_i is :

$$\sigma_i(x) = \left(\rho^\ell_{m_\ell}\left(R_{M^\ell_{m_\ell}}(a^\ell(x))\right)\right)_{\ell \in \{1,\dots,m\}}$$

where i happens to be the m-tuple μ and $R_{M^\ell_{m_\ell}} : \Theta^{k_\ell}_\ell \to M^\ell_{m_\ell}$ is a "restriction" to $M^\ell_{m_\ell}$ since we have to consider only those $a^\ell(x)$ which are in $M^\ell_{m_\ell}$. Finding such functions $R_{M^\ell_{m_\ell}}$ such that σ_i is C^1 may not be possible. But, following [12, Lemma 3.3], we know the existence, for each ℓ, of a countable family of C^1 functions $R^\ell_\nu : \Theta^{k_\ell}_\ell \to M^\ell_{m_\ell}$ such that we get :

$$S^\mu_{\epsilon,\underline{\Lambda},\underline{x}} \subset \bigcup_{\nu \in \mathbb{N}} \left\{\left(\rho^\ell_{m_\ell}\left(R^\ell_\nu(a^\ell(\mathcal{B}_\epsilon(\underline{x})))\right)\right)_{\ell \in \{1,\dots,n+1\}}\right\} \ .$$

In other words the family of functions σ_i is actually given by the family :

$$\sigma_{\mu,\nu} = \left(\rho^\ell_{m_\ell} \circ R^\ell_\nu \circ a_\ell\right)_{\ell \in \{1,\dots,n+1\}}$$

i.e. we have :

$$S_{\epsilon,\underline{\Lambda},\underline{x}} \subset \bigcup_{\mu \in \mathbb{N}^{n+1}} \bigcup_{\nu \in \mathbb{N}} \sigma_{\mu,\nu}(\mathcal{B}_\epsilon(\underline{x})) \ .$$

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