Semiglobal High-Gain Hybrid Observer for a Class of Hybrid Dynamical Systems with Unknown Jump Times

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***Abstract*—We study the problem of observer design for hybrid dynamical systems in the challenging setting where the times at which jumps occur are unknown or, equivalently, cannot be detected immediately. We show that when the solutions of interest admit a uniform dwell-time and when the flows are strongly differentially observable, a sufficiently fast high-gain observer can be designed to estimate the state during flow, but using the output of the system near its jump times is counterproductive. To solve this problem, we propose to "disconnect" the high-gain observer when its estimate gets close to the jump set. More precisely, the proposed hybrid observer generates an estimate that, during flow, is obtained via the high-gain observer and, around jump times, is obtained by integrating forward the flow map of the system, until reaching the jump set. Under appropriate assumptions around the jump set of the system, we show that the proposed observer guarantees local uniform asymptotic stability of an appropriately defined zero-error set. Then, we develop a method to turn any such local hybrid observer into a semiglobal hybrid observer. This observer operates sequentially by first employing a continuous-time high-gain observer, and then, after a finite amount of time, solely determined by the current estimate, employing the available local hybrid observer. The capabilities and performance of the proposed hybrid observer are illustrated on a hybrid dynamical system modeling a spiking neuron model.**

I. Introduction

A. Background

THE problem of designing observers for general hybrid systems presenting both a continuous-time behavior and a discrete-time behavior is still largely unsolved, mainly due to the fact that the system jump times, that is, the times at which discrete events occur in the system solution, generally depend on its initial condition, which is unknown in the context of observer design. When the system jump times are known or can be detected immediately, it is natural to design an observer that is synchronized with the system, namely, a hybrid observer whose jumps are triggered at the same time as those of the system. This has been done under assumptions on the time elapsed between successive jumps (reverse/average dwell-time for instance) in a large variety of contexts, including impulsive (possibly switched) systems [1], [20], [27], sampled continuous-time systems [11], [23], [26], and general hybrid systems [7], [8], [24], among others. Because the observer jumps at the same time as the system, both observer and system solutions are defined on the same (hybrid) time domain, which facilitates the analysis of the estimation error and the design of an observer. Unfortunately, exact synchronization between the system and the observer is usually difficult to achieve in practice, due to noisy/delayed jump detection. Robustness with respect to delays in the jump triggering has been studied in [7], but only practical stability outside the delay intervals may be expected. In other contexts, it may even be impossible to detect the jumps of the system from the measurements available.

When the observer jumps are not triggered at the same time as those of the system, the mismatch of time domains between the system and observer solutions makes the formulation of observability and, in turn, observer design very challenging [4]. In the particular context of switched systems, a lot of work appeared to design observers able to estimate the switching signal [2], [22], [28], among many others. On the other hand, very few observer results exist for general hybrid systems [14] when the system jump times are unknown. Exceptions are [18], [21], where the existence of a change of coordinates transforming the jump map into the identity map is studied, thus allowing the use of a continuous-time observer in those new coordinates. Also in [12], an observer with non-synchronized jumps is designed for billiard-type systems, but the knowledge of the system jump times is still needed to trigger the jumps.

B. Contributions

Motivated by the shortcomings mentioned above, we propose hybrid observers that can be designed without relying on the knowledge or perfect detection of the jumps of the hybrid dynamical system. For starters, in the preliminary version of this work [5], we proposed a local hybrid observer for a class of hybrid dynamical systems modeled as in [14], with linear flow, jump, and output maps, assuming the flow dynamics are

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P. Bernard (pauline.bernard@minesparis.psl.eu) is with Centre Automatique et Systèmes (CAS), Mines Paris, Université PSL, France; R. G. Sanfelice (ricardo@ucsc.edu) is with Department of Electrical and Computer Engineering, University of California, Santa Cruz, CA 95064, USA.

observable and solutions of interest admit a uniform dwell-time and evolve within a compact set. We showed that, while it is tempting to use a sufficiently fast observer during flow and trigger the jumps when its estimate reaches the jump set, using the system output around the jump times to define the innovation term is actually counterproductive. Indeed, an arbitrarily small mismatch between the jump times of the system and of the observer typically leads to large estimation errors, even in nominal (noise-free) conditions. Under appropriate assumptions on the behavior of solutions around the jump set, we proposed a hybrid observer that – via an innovation term – properly injects the measurements during continuous evolution of the hybrid system and, around the jump times, produces an (open-loop) estimate until it reaches the jump set.

In this paper, we make the following contributions:

1) We show that the local design of [5] can be extended to handle the larger class of nonlinear hybrid systems with strongly differentially observable flow dynamics, for which a sufficiently fast continuous-time high-gain observer can be used during flow. Because the latter is generally not in the system coordinates an additional state is added to the observer and the hybrid logic appropriately adapted.
2) Unlike in [5] where only local asymptotic convergence to an appropriate zero-error set is presented, we prove here local uniform asymptotic stability.
3) We then propose a novel general method enabling to adapt any local asymptotic observer designed for this class of systems into a semiglobal one. The idea is to run a preliminary high-gain observer and "switch" to the local hybrid observer at an appropriate time. But this "switching" time has to be chosen with care in order to ensure that the estimation error is sufficiently small at that time, no matter when the system may have jumped in the meantime. We thus provide an algorithm to choose this "switching time" based on the value of the high-gain estimate and show semiglobal convergence of the obtained hybrid observer.

Finally, the performance of such a design are illustrated in simulations for different scenarios of initial conditions and compared to a more standard synchronous observer design with delayed jump detection, on an example featuring a spiking neuron.

C. Notation and Preliminaries

We denote $\mathbb{R}$ (resp. $\mathbb{N}$) the set of real numbers (resp. integers), $\mathbb{R}_{\geq 0} := [0,+\infty)$, $\mathbb{R}_{>0} := (0,+\infty)$, and $\mathbb{N}_{>0} =: \mathbb{N} \setminus \{0\}$. For x in $\mathbb{R}^n$ and $\mathcal{A}$ subset of $\mathbb{R}^n$, $|x|_{\mathcal{A}}$ denotes the distance from x to $\mathcal{A}$ and $\partial\mathcal{A}$ the boundary of $\mathcal{A}$. For a symmetric real matrix P, $\underline{\lambda}(P)$ (resp. $\overline{\lambda}(P)$) stands for its smallest (resp. largest) eigenvalue. For a C^1 map $V : \mathbb{R}^n \to \mathbb{R}$, $L_{\mathcal{F}}V(x) := \frac{dV}{dx}(x)\mathcal{F}(x)$ denotes the Lie derivative along the vector field $\mathcal{F}$. For a differential equation $\dot{x} = f(x)$ with f locally Lipschitz, $\Psi_f(x,\tau)$ denotes the value at time τ of the solution initialized at x. We recall that a C^1 map $T : \mathcal{O} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq n$ is an immersion on $\mathcal{O}$ if $\frac{\partial T}{\partial x}(x)$ is full-rank for all $x \in \mathcal{O}$. We consider hybrid dynamical systems of the form (1) introduced in Section II-A, as defined in [14], whose solutions are defined on so-called *hybrid time-domains*. A subset E of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time-domain* if $E = \bigcup_{j=0}^{j_m-1}([t_j,t_{j+1}],j)$ for some finite sequence of times $t_0 \leq t_1 \leq \ldots \leq t_{j_m}$, and it is a hybrid time domain if for any $(t_m,j_m) \in E$, $E \cap [0,t_m] \times \{0,1,\ldots,j_m\}$ is a compact hybrid time domain. For a solution $(t,j) \mapsto x(t,j)$ (see [14, Definition 2.6]), we denote $\operatorname{dom} x$ its domain, $\operatorname{dom}_t x$ (resp. $\operatorname{dom}_j x$) its projection on the time (resp. jump) component, and $I^j := \{t \in \operatorname{dom}_t x : (t,j) \in \operatorname{dom} x\}$ the jth interval of flow. We say that x is *maximal* if there does not exist any other solution x' defined on $\operatorname{dom} x' \supset \operatorname{dom} x$ and agreeing with x on $\operatorname{dom} x$. We say that x is *t-complete* if $\operatorname{dom}_t x$ is unbounded and that it has a dwell-time $\tau_m > 0$ if it flows at least τ_m units of time in between consecutive jumps. Finally, as defined in [4], [6], x^{r} is a *j-reparametrization* of x if there exists a function $\rho : \mathbb{N} \to \mathbb{N}$ verifying $\rho(0) = 0$, $\rho(j+1) - \rho(j) \in \{0,1\}$, and such that $x^{\mathrm{r}}(t,j) = x(t,\rho(j))$ for all $(t,j) \in \operatorname{dom} x^{\mathrm{r}}$. If in addition $\operatorname{dom} x = \bigcup_{(t,j)\in\operatorname{dom} x^{\mathrm{r}}}(t,\rho(j))$, then it is a *full j*-reparametrization.

II. Problem Statement

A. Framework

We consider a hybrid system of the form [14]

$$\mathcal{H}\begin{cases} \dot{x} = f(x) & x \in C \\ x^+ = g(x) & x \in D \end{cases}, \qquad y = h(x) \tag{1}$$

with state $x \in \mathbb{R}^{n_x}$ and output $y \in \mathbb{R}^{n_y}$, flow and jump maps f and g locally Lipschitz, output map h, flow and jump sets C and D. For this broad class of hybrid dynamical systems, denoted $\mathcal{H} = (C,f,D,g,h)$, we are interested in estimating the state of $\mathcal{H}$ when its solutions are initialized in a bounded subset $\mathcal{X}_0 \subset C \cup D$. We denote $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ the set of maximal solutions of $\mathcal{H}$ with initial condition in $\mathcal{X}_0$.

If the system jump times were known or immediately detected as in [7], [8], [24], one could design an observer for (1) of the form

$$\hat{\mathcal{H}}\begin{cases} \dot{\xi} \in \mathcal{F}(\xi,y) & \text{when } \mathcal{H} \text{ flows} \\ \xi^+ \in \mathcal{G}(\xi,y) & \text{when } \mathcal{H} \text{ jumps} \end{cases}, \qquad \hat{x} = \mathcal{T}(\xi) \tag{2}$$

that is synchronized with the system, for some maps $\mathcal{F},\mathcal{G} : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_\xi}$ and $\mathcal{T} : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_x}$ to be chosen such that $\hat{x}$ asymptotically reconstructs the system state x. The advantage of such a setting is that the observer and system solutions are defined on the same domain, which facilitates the analysis of the estimation error.

Unfortunately, exact synchronization between the system and the observer is usually difficult to achieve in practice, due to noisy/delayed jump detection. Motivated by this issue, in this paper, we design an observer whose jumps are triggered based on its own estimate of the system state, rather than an exogenous signal. More precisely, we aim for a hybrid observer of the form

$$\hat{\mathcal{H}}\begin{cases} \dot{\xi} \in \mathcal{F}(\xi,y) & (\xi,y) \in \mathcal{C} \\ \xi^+ \in \mathcal{G}(\xi,y) & (\xi,y) \in \mathcal{D} \end{cases}, \qquad \hat{x} = \mathcal{T}(\xi) \tag{3}$$

for some maps $\mathcal{F},\mathcal{G},\mathcal{T}$ and sets $\mathcal{C},\mathcal{D}$ to be designed. In this setting, $y = h(x)$ is a hybrid signal defined on the domain of the system solution $\operatorname{dom} x$, which typically differs from the domain of the observer solutions given by (3).

To cope with this mismatch, following [4], [6], solutions to (3) are defined as pairs (ξ, y^{r}), where $\operatorname{dom}\xi = \operatorname{dom} y^{\mathrm{r}}$ and y^{r} is a j-reparametrization of y. More precisely, jumps in the domain of the input y (namely, system jumps) trigger a jump in the domain of the solution to (3), with the observer state ξ either reset identically (trivial jump) or in $\mathcal{G}(\xi, y)$ depending on the logic defined in [4], [6]. Similarly, ξ may jump using $\mathcal{G}$ at times that are not jump times of the system solution x, so such trivial jumps are also added in the domain of y leading to its reparametrization y^{r}. This purely analytical process enables us to build a j-reparametrization of the system solution x and its output y on the same domain as the observer solution. However, this artificial addition of trivial jumps for analysis purposes does not change the fact that the system and observer solutions "truly" jump using g and $\mathcal{G}$, respectively, at different times. Consequently, even when $\hat{x}$ "tends" to x asymptotically, $\hat{x}$ may always be slightly ahead/behind x around the jump times, which typically prevents the estimation error $\hat{x} - x$ to converge to zero asymptotically around the jump times (see this *peaking* phenomenon also in the context of tracking [10]).

As in [4], we consider more general notions of convergence of (x, ξ) to appropriate *observation sets* $\mathcal{A}$ that are as close as possible to the ideal zero-error set

$$\mathcal{A}_{\text{ideal}} = \{(x, \xi) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \ : \ \mathcal{T}(\xi) = x\}$$

where $\hat{x} = x$. Our goal is formulated as follows.

Problem statement: Given $\mathcal{H} = (C, f, D, g, h)$ and $\mathcal{X}_0 \subset C \cup D$, design an observation set $\mathcal{A} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$, hybrid data $(\mathcal{C}, \mathcal{F}, \mathcal{D}, \mathcal{G}, \mathcal{T})$ and an initializations set $\Xi_0 \subset \mathbb{R}^{n_\xi}$ such that for any maximal solution x to $\mathcal{H}$ with $x(0,0) \in \mathcal{X}_0$ and any maximal solution (ξ, y^{r}) to $\hat{\mathcal{H}}$ defined by (3) with input $y = h(x)$ and with $\xi(0,0) \in \Xi_0$, there exists a full j-reparametrization $(x^{\mathrm{r}}, \xi^{\mathrm{r}})$ of x and ξ such that $\operatorname{dom} x^{\mathrm{r}} = \operatorname{dom}\xi^{\mathrm{r}}$ and

$$\lim_{t+j\to\infty} |(x^{\mathrm{r}}, \xi^{\mathrm{r}})(t,j)|_{\mathcal{A}} = 0 \ .$$

B. Assumptions

The following assumption describes the class of hybrid systems considered in this study.

Assumption 2.1: Given $\mathcal{H} = (C, f, D, g, h)$ and $\mathcal{X}_0 \subset C \cup D$, there exist $\tau_m > 0$ and a compact subset $\mathcal{X}$ of $C \cup D$ such that any solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$

- is t-complete with dwell-time τ_m; and
- remains in $\mathcal{X}$ at all times.

The uniform dwell-time condition enables our design to rely on an arbitrarily fast continuous-time observer. Under well-posedness, the existence of such a dwell-time is guaranteed if $g(D) \cap D = \emptyset$ using [25, Lemma 2.7] and the fact that the solutions evolve in the compact set $\mathcal{X}$.

We assume that a "high-gain" continuous-time observer $\dot{z} = \mathcal{F}_\ell(z, y)$ is available, allowing to estimate (arbitrarily fast) a certain function T of the state during flow. More precisely, we make the following assumption.

Assumption 2.2: Given $\mathcal{H} = (C, f, D, g, h)$, there exist $\lambda > 0$, $\ell_0 > 0$, rational functions $\underline{c}$ and $\overline{c}$, an open set $\mathcal{O}$ containing $\mathrm{cl}(C \cup D)$, an injective immersion $T : \mathcal{O} \to \mathbb{R}^{n_z}$, and for all $\ell > \ell_0$, maps $\mathcal{F}_\ell : \mathbb{R}^{n_z} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_z}$ and $V_\ell : \mathcal{O} \times \mathbb{R}^{n_z} \to \mathbb{R}$ such that

$$\underline{c}(\ell)|z - T(x)|^2 \leq V_\ell(x, z) \leq \overline{c}(\ell)|z - T(x)|^2 \qquad \forall (x, z) \in \mathcal{X} \times \mathbb{R}^{n_z} \tag{4a}$$

$$L_{F_\ell} V_\ell(x, z) \leq -\ell\lambda V_\ell(x, z) \quad \forall (x, z) \in (C \cap \mathcal{X}) \times \mathbb{R}^{n_z} \tag{4b}$$

with $F_\ell(x, z) = (f(x), \mathcal{F}_\ell(z, h(x)))$.

Note that the subscript ℓ highlights the dependency of V_ℓ and $\mathcal{F}_\ell$ with respect to the gain ℓ. This gain controls the decay rate in (4b), which can be chosen as large as necessary. The following two examples show that Assumption 2.2 holds when (f, h) is a linear observable pair and, more generally, when (f, h) is *strongly differentially observable*.

Example 2.3 (Linear observable pair): Following [5], assume the flow and output maps of $\mathcal{H}$ are defined by $f(x) = Ax$ and $h(x) = Hx$ with the pair (A, H) *observable*. Define $\mathcal{V} \in \mathbb{R}^{n_x \times n_x}$ a change of coordinates transforming (A, H) into a block-diagonal observable form, namely such that

$$\mathcal{V} A \mathcal{V}^{-1} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{H} \quad , \quad H\mathcal{V}^{-1} = \boldsymbol{H}$$

with $\boldsymbol{A} := \mathrm{blkdiag}(A_1, \ldots, A_{n_y})$, $\boldsymbol{B} := \mathrm{blkdiag}(B_1, \ldots, B_{n_y})$, $\boldsymbol{H} := \mathrm{blkdiag}(H_1, \ldots, H_{n_y})$,

$$A_i = \begin{pmatrix} 0 & 0 & & \ldots & 0 \\ 1 & 0 & & & \\ \vdots & \ddots & \ddots & & \\ 0 & & 1 & 0 & 0 \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n_i \times n_i}$$

$$H_i = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{1 \times n_i} \ ,$$

$D_i \in \mathbb{R}^{n_i \times 1}$, and n_i integers such that $\sum_{i=1}^{n_y} n_i = n_y$. Consider vectors K_i such that $A_i - K_i H_i$ is Hurwitz, and for a positive scalar ℓ, define $\mathcal{L}_i(\ell) := \mathrm{diag}(\ell^{d_i - 1}, \ldots, \ell, 1)$. Then, let us take $\mathcal{F}_\ell$ defined by

$$\mathcal{F}_\ell(z, y) = Az - \mathcal{V}^{-1}(\boldsymbol{B} + \ell\boldsymbol{\mathcal{L}}(\ell)\boldsymbol{K})(Hz - y) \tag{5}$$

where $\boldsymbol{K} := \mathrm{blkdiag}(K_1, \ldots, K_{n_y})$, $\boldsymbol{\mathcal{L}} := \mathrm{blkdiag}(\mathcal{L}_1, \ldots, \mathcal{L}_{n_y})$. Consider a positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$ such that

$$(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{H})^\top P + P(\boldsymbol{A} - \boldsymbol{K}\boldsymbol{H}) \leq -\lambda P$$

for some $\lambda > 0$. Then, for any C, D, g, (4) holds with $T = \mathrm{Id}$,

$$V_\ell(x, z) = (x - z)^\top \mathcal{V}^\top \boldsymbol{\mathcal{L}}(\ell)^{-1} P \boldsymbol{\mathcal{L}}(\ell)^{-1} \mathcal{V}\,(x - z) \ ,$$

$$\ell_0 = 0 \quad , \quad \underline{c}(\ell) = \frac{\underline{\lambda}(\mathcal{V}^\top P \mathcal{V})}{\ell^{2(d-1)}} \quad , \quad \overline{c}(\ell) = \overline{\lambda}(\mathcal{V}^\top P \mathcal{V}) \ ,$$

where $n = \max n_i$. □

Example 2.4 (Strongly differentially observable pair): Assume that $\mathcal{H}$ has $n_y = 1$, and its flow/output pair (f, h) is strongly differentially observable of order n_z on an open set $\mathcal{O} \subset \mathbb{R}^{n_x}$ containing $\mathrm{cl}(C \cup D)$, i.e., the map $T : \mathbb{R}^{n_x} \to \mathbb{R}^{n_z}$ defined by

$$T(x) = (h(x), L_f h(x), ..., L_f^{n_z - 1} h(x)) \tag{6}$$

is an injective immersion on $\mathcal{O}$. Consider a map $\Phi : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$ of the form

$$\Phi(z) = \mathrm{sat} \circ L_f^{n_z} h \circ T_{\mathrm{inv}}(z) \ ,$$

where T_{inv} is a locally Lipschitz left-inverse of T verifying $T_{\mathrm{inv}} \circ T = \mathrm{Id}$ on $\mathcal{X}$, which is guaranteed to exist due to the properties of T, and sat is a bounded C^1 map verifying $\mathrm{sat} = \mathrm{Id}$ on $L_f^{n_z} h(\mathcal{X})$. Then, a high-gain observer [17] can be built for the flow dynamics, with

$$\mathcal{F}_\ell(z, y) = A\, z + B\Phi(z) + \ell \mathcal{L}(\ell) K (y - z_1) \ , \tag{7}$$

$$A = \begin{pmatrix} 0 & 1 & & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{n_z \times n_z} \ , \ B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{n_z}$$

$\mathcal{L}(\ell) = \mathrm{diag}(1, \ell, \ell^2, \dots, \ell^{n_z - 1})$, and K such that $A - KH$ is Hurwitz with $H = (1, 0, \dots, 0)$. Standard high gain computations [17] show that conditions (4) then hold for the Lyapunov function

$$V_\ell(x, z) = (T(x) - z)^\top \mathcal{L}(\ell)^{-1} P \mathcal{L}(\ell)^{-1} (T(x) - z) \ ,$$

with P a positive definite matrix such that

$$(A - KH)^\top P + P(A - KH) \leq -\lambda_0 P$$

for some $\lambda_0 > 0$, $\underline{c}(\ell) = \frac{\underline{\lambda}(P)}{\ell^{2(n_z - 1)}}$, $\overline{c}(\ell) = \overline{\lambda}(P)$, $\lambda > 0$ depending on the Lipschitz constant of Φ, and ℓ larger than a threshold $\ell_0 > 0$ also depending on that Lipschitz constant. Note that the same tools can be used for multi-output triangular normal forms [15]. □

In what follows, we consider a strict compact superset $\mathcal{X}_m$ of $\mathcal{X}$ in $\mathcal{O}$, with $\mathcal{X}$ defined in Assumption 2.1 and $\mathcal{O}$ defined in Assumption 2.2. Because T is an injective immersion on $\mathcal{O}$, it admits a Lipschitz left-inverse $T_{\mathrm{inv}} : \mathbb{R}^{n_z} \to \mathbb{R}^{n_x}$ such that

$$T_{\mathrm{inv}} \circ T(x) = x \qquad \forall x \in \mathcal{X}_m \tag{8}$$

(see for instance [3, Lemma A.12]).

It is shown in [7] that when the jump times of the system are known or detected instantaneously, a possible observer consists of $\hat{\mathcal{H}}$ defined in (2), with the high-gain observer $\mathcal{F} := \mathcal{F}_\ell$ given by Assumption 2.2 during flow, a jump map G of the form

$$\mathcal{G}(z) := \mathrm{sat} \circ T \circ g \circ T_{\mathrm{inv}}(z) \ , \tag{9}$$

with sat a C^1 bounded map verifying $\mathrm{sat} = \mathrm{Id}$ on $T \circ g(\mathcal{X})$, and an estimate given by $\hat{x} = T_{\mathrm{inv}}(z)$. Indeed, for ℓ sufficiently large compared to the Lipschitz constant of $\mathcal{G}$ and the dwell-time τ_m, one can show that the (exponential) decrease of the Lyapunov function V_ℓ during flow wins over its (polynomial) increase at jumps and the estimation error; thus, it asymptotically converges to zero.

Still relying on the dwell-time and the available high-gain continuous-time observer, the goal of this paper is to design a hybrid observer that, unlike (2), does not rely on the detection of the system jumps. More precisely, rather than detecting when $x \in D$, which is not always possible, we propose to use the information that $\hat{x} := T_{\mathrm{inv}}(z) \in D$ to trigger the observer jumps, at least when the initial estimation error is sufficiently small. The construction of a local hybrid observer solving this problem is presented in Section III and the uniform asymptotic stability of its estimation error is proved in Section IV. Then, in Section V, we propose a general procedure to build a semiglobal hybrid observer from such a local design. Their performance is illustrated in simulations in Section VI.

III. Local High-Gain Hybrid Observer

An initial approach for the design of a local observer could be to use the flow map $\mathcal{F}_\ell$ given by Assumption 2.2 during flow, and simply trigger the jumps of the observer when $\hat{x} \in D$, with the jump map $\mathcal{G}$ defined in (9). Indeed, if the estimation error sufficiently decreases during flow, one can expect that the observer jumps will occur close in time to those of the system and somehow the observer will synchronize and converge. However, around the observer jump times, because the system typically jumps slightly sooner or later than the observer, the input y feeding the observer flow map might actually constitute a disturbance and hinder the convergence of the observer. More precisely, assume that $\hat{x}$ and x are both close to D and x jumps first. Then, the observer input y after the jump could steer $\hat{x}$ away from D so as to allow $\hat{x}$ to catch up with x via flow, which could cause $\hat{x}$ to miss its jump. The same reasoning holds in the reverse case where $\hat{x}$ jumps slightly ahead of x, and where the use of y would force $\hat{x}$ to track the value of x before it jumps, instead of simply waiting for x to catch up via a jump.

This issue is dealt with in [12] by making $\hat{x}$ follow a mirrored image of x with respect to D during the jump time mismatches. But this approach solves the problem in a very particular setting of billiard systems, where $g \circ g$ is the identity, and more importantly, the knowledge of the system jump times is necessary to decide whether $\hat{x}$ should follow x or its mirrored image. In the problem tackled in this paper, the system jump times are unknown (and $g \circ g$ is not the identity), so the paradigm of [12] cannot be applied.

A. Open-Loop Estimation around Jump Times

Following the approach presented in our preliminary work in [5], we propose to "disconnect" the high-gain observer when $\hat{x}$ is nearby the jump set D. More precisely, we propose a hybrid mechanism that lets $\hat{x}$ flow in "open-loop" according to f until it naturally reaches D, and only reconnects the correction term in the observer a small amount of time Δ later, in a way that ensures the system has also jumped in the meantime. This process is illustrated in Figure 1. For this strategy to work, we assume that (i) the system eventually reaches D when entering a certain neighborhood of D (see (P1) below), (ii) the system necessarily jumps from D (see (P2)), and with τ_m given by Assumption 2.1, iii) the system takes at least $0 < \tau_m^0 < \tau_m$ units of time to reach that neighborhood again (see (P3)). Similar conditions are used in [13] in the context of trajectory tracking.

More precisely, consider a projection map $\Pi : \mathbb{R}^{n_x} \to \mathrm{cl}(C \cup D)$ verifying $\Pi \circ \Pi = \Pi$, $\Pi(\mathcal{X}_m) \subset \mathcal{X}_m$ and for which

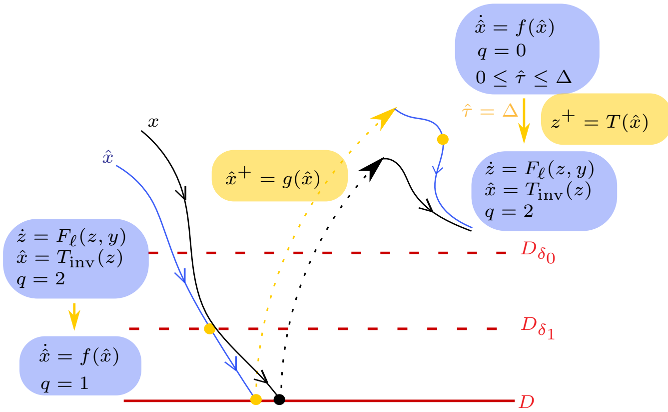


Fig. 1: Sketch of hybrid mechanism in observer (13), with system trajectory in black, observer trajectory in blue/yellow, with blue (resp. yellow) representing observer flow (resp. jumps), in the case where $\hat{x}$ remains in $\text{cl}(C \cup D)$ so that $\Pi(\hat{x}) = \hat{x}$.

there exists $a_p \geq 1$ such that

$$|x - \Pi(\hat{x})| \leq a_p |x - \hat{x}| \qquad \forall (x, \hat{x}) \in \mathcal{X} \times \mathbb{R}^{n_x} . \tag{10}$$

In particular, (10) implies that $\Pi = \text{Id}$ on $\mathcal{X}$. The main role of this projection is to reset the estimate in $\text{cl}(C \cup D)$ at the observer jumps, and to ensure that, given $\delta > 0$, we have, for all $x \in \mathbb{R}^{n_x}$,

$$\Pi(x) \notin D_\delta \quad \Longrightarrow \quad |x|_D > \delta$$

where

$$D_\delta = \{x \in \text{cl}(C \cup D) \,:\, |x|_D \leq \delta\} .$$

We make the following assumption.

Assumption 3.1: Given $\mathcal{H} = (C, f, D, g)$, $\mathcal{X}$ defined in Assumption 2.1 and $\mathcal{X}_m$ verifying (8), there exist $\delta_0 > 0$ and compact sets $\mathcal{X}'_m$, $\mathcal{X}''_m$ such that $\mathcal{X} \subset \mathcal{X}''_m \subset \mathcal{X}'_m \subset \mathcal{X}_m$ with

$$\inf_{x \in \mathcal{X}} |x|_{\partial \mathcal{X}''_m} > 0 \tag{11}$$

and the following hold:

(P1) for any $x \in D_{\delta_0} \cap \mathcal{X}''_m$, there exists $\tau_D \geq 0$ such that
- $\Psi_f(x, \tau_D) \in D$,
- $\Psi_f(x, t) \in (D_{\delta_0} \cap \mathcal{X}'_m) \setminus D$ for all $t \in [0, \tau_D)$,

where Ψ_f is the flow operator along the vector field f. In addition, the map $\mathfrak{T} : D_{\delta_0} \cap \mathcal{X}''_m \to \mathbb{R}_{\geq 0}$ which associates τ_D to each $x \in D_{\delta_0} \cap \mathcal{X}''_m$, is Lipschitz.

(P2) No flow in D_{δ_0} is possible for $\dot{x} = f(x)$ starting from $D \cap \mathcal{X}'_m$, namely there does not exist any solution $t \mapsto x(t)$ such that $x(0) \in D \cap \mathcal{X}'_m$ and $x(t) \in D_{\delta_0}$ on $[0, \epsilon]$ for some $\epsilon > 0$.

(P3) $g(D \cap \mathcal{X}'_m) \cap D_{\delta_0} = \emptyset$ and there exists $\tau_m^0 > 0$ such that every solution $t \mapsto x(t)$ of $\dot{x} = f(x)$ with $x(0) \in g(D \cap \mathcal{X}'_m)$ is defined over the interval $[0, \tau_m^0]$, with $x(t) \in \mathcal{X}_m$ and $\Pi(x(t)) \notin D_{\delta_0}$ for all $t \in [0, \tau_m^0]$.

Note that only the largest compact set $\mathcal{X}_m$ and the parameters δ_0 and τ_m^0 are design parameters. The existence of the intermediary compact sets $\mathcal{X}'_m$ and $\mathcal{X}''_m$ is only required for analysis in order to use the Lipschitz constants of the nonlinear maps along the estimate trajectory. In particular, the strict inclusion of $\mathcal{X}$ into $\mathcal{X}''_m$ given by (11) allows to say that if the estimation error $\hat{x} - x$ is sufficiently small, $x \in \mathcal{X}$ implies $\hat{x} \in \mathcal{X}''_m$. Also, the reason why we need to differentiate the compact sets $\mathcal{X}'_m$ and $\mathcal{X}''_m$ is that there may not exist an invariant compact set by the flow $\dot{x} = f(x)$ in (P1). Indeed, we only know by Assumption 2.1 that solutions initialized in $\mathcal{X}_0$ remain in $\mathcal{X}$ but there is no reason why other solutions, in particular $\hat{x}$ flowing with f, should remain in a given compact set. So we assume solutions starting in $D_{\delta_0} \cap \mathcal{X}''_m$ remain in a (possibly larger) compact set $\mathcal{X}'_m$ until they reach D, and then remain in a possibly even larger $\mathcal{X}_m$ after their jump while they flow on the interval $[0, \tau_m^0]$. Note that Assumption 3.1 was shown in [5] to hold for the model of a bouncing ball, modulo a regularization of its jump set D encoding the absence of Zeno. It is also shown to hold for a spiking neuron model in Section VI.

Remark 3.2: Sufficient conditions on the data f, C, and D ensuring the Lipschitzness of the *time-to-impact* function $\mathfrak{T}$ in (P1) are given in a more general context in [19] and references therein. Actually, when there exists a continuously differentiable function $\varpi : \mathbb{R}^{n_x} \to \mathbb{R}$ such that the map $\mathfrak{T}$ is characterized at each $x \in D_{\delta_0} \cap \mathcal{X}''_m$ by $\varpi(\Psi_f(x, \mathfrak{T}(x))) = 0$ with $\Psi_f(x, \mathfrak{T}(x)) \in D \cap \mathcal{X}'_m$, the continuous differentiability of $\mathfrak{T}$ is guaranteed by the implicit function theorem under the *transversality condition*

$$\frac{\partial \varpi}{\partial x}(x) f(x) \neq 0 \qquad \forall x \in D \cap \mathcal{X}'_m . \tag{12}$$

Note that (12) also ensures that no flow is possible in $D \cap \mathcal{X}'_m$, namely (P2) holds. Alternatively, conditions involving the tangent cone of the flow set at points in $C \cap D \cap \mathcal{X}'_m$ and the flow map can be formulated to assure that flow in $D \cap \mathcal{X}'_m$ is not possible.

B. Hybrid Observer Construction

Suppose Assumption 3.1 holds with δ_0 and τ_m^0, and pick $0 < \delta_1 < \delta_0$ and $0 < \Delta < \frac{\tau_m^0}{2}$. To implement the observation strategy explained in the previous section, we define our hybrid observer with state (z, χ, τ, q), where z and χ are used to construct the estimate $\hat{x}$ of the system state x, τ is a resettable timer, and $q \in \{0, 1, 2\}$ is a logic variable that describes the "operating mode" of the observer. The hybrid observer operates as follows (see also Figure 1 in the case where $\Pi(\hat{x}) = \hat{x}$):

- When $q = 2$, z flows governed by the high-gain map $\mathcal{F}_\ell$ and $\hat{x}$ is obtained by inverting T, namely $\hat{x} = T_{\text{inv}}(z)$. In this mode, the estimate $\hat{x}$ approaches x over the flow interval for sufficiently large values of ℓ. On the other hand, the states χ and τ are unused and, hence, remain constant. For simplicity, τ is forced to be always 0 when $q = 2$.
- When $\Pi(\hat{x})$ reaches D_{δ_1}, the observer jumps to mode $q = 1$ with χ initialized at $\Pi(\hat{x})$, so as to store the current estimate of x. During this mode $q = 1$, χ flows governed by f and the estimate $\hat{x}$ is given by χ. On the other hand, z and τ are unused and remain constant, with still $\tau = 0$.
- When χ reaches D, which we know happens in finite flow time thanks to (P1), χ is reset to $g(\chi)$ and the mode changes

to $q=0$. Indeed, no flow is possible from D in C. During this mode $q=0$, the timer τ counts flow time and χ flows governed again by f for Δ units of time. The estimate $\hat{x}$ is still given by χ and z is still unused, so it is kept constant.

- When the timer expires, namely when $\tau=\Delta$, the observer jumps back to mode $q=2$, with z updated to $T(\Pi(\chi))$.

Naturally, since q is a logic variable, it is constant during flow in each mode. In the following, we refer to the period of time where z flows governed by $\mathcal{F}_\ell$ and $q=2$ as the *high-gain phase*, and to the period of time where χ flows governed by f with $q\in\{0,1\}$ as the *open-loop phase*.

This observation strategy is modeled by the following hybrid system, denoted $\hat{\mathcal{H}}$:

$$\begin{pmatrix}\dot{z}\\ \dot{\chi}\\ \dot{\tau}\\ \dot{q}\end{pmatrix} = \begin{cases}\begin{pmatrix}\mathcal{F}_\ell(z,y)\\ 0\\ 0\\ 0\end{pmatrix} & \text{if } (z,\chi,\tau,q)\in\mathcal{C}_2\\ \begin{pmatrix}0\\ f(\chi)\\ 0\\ 0\end{pmatrix} & \text{if } (z,\chi,\tau,q)\in\mathcal{C}_1\\ \begin{pmatrix}0\\ f(\chi)\\ 1\\ 0\end{pmatrix} & \text{if } (z,\chi,\tau,q)\in\mathcal{C}_0\end{cases} \tag{13a}$$

$$\begin{pmatrix}z^+\\ \chi^+\\ \tau^+\\ q^+\end{pmatrix} = \begin{cases}\begin{pmatrix}z\\ \Pi(T_{\rm inv}(z))\\ 0\\ 1\end{pmatrix} & \text{if } (z,\chi,\tau,q)\in\mathcal{D}_2\\ \begin{pmatrix}z\\ g(\chi)\\ 0\\ 0\end{pmatrix} & \text{if } (z,\chi,\tau,q)\in\mathcal{D}_1\\ \begin{pmatrix}T(\Pi(\chi))\\ \chi\\ 0\\ 2\end{pmatrix} & \text{if } (z,\chi,\tau,q)\in\mathcal{D}_0\end{cases} \tag{13b}$$

with estimate given by

$$\hat{x} = \mathcal{T}(z,\chi,q) := \begin{cases} T_{\rm inv}(z) & \text{if } q=2\\ \chi & \text{if } q\in\{0,1\}\end{cases} \tag{13c}$$

the (disjoint) flow sets defined by

$$\begin{aligned}\mathcal{C}_2 &= C^z_{\delta_1}\times\mathbb{R}^{n_x}\times\{0\}\times\{2\}\\ \mathcal{C}_1 &= \mathbb{R}^{n_z}\times D_{\delta_0}\times\{0\}\times\{1\}\\ \mathcal{C}_0 &= \mathbb{R}^{n_z}\times\mathbb{R}^{n_x}\times[0,\Delta]\times\{0\}\end{aligned}$$

and the (disjoint) jump sets by

$$\begin{aligned}\mathcal{D}_2 &= D^z_{\delta_1}\times\mathbb{R}^{n_x}\times\{0\}\times\{2\}\\ \mathcal{D}_1 &= \mathbb{R}^{n_z}\times D\times\{0\}\times\{1\}\\ \mathcal{D}_0 &= \mathbb{R}^{n_z}\times C_{\delta_0}\times[\Delta,+\infty)\times\{0\}\end{aligned}$$

where

$$C_{\delta_0} = \{x\in\mathbb{R}^{n_x} \,:\, \Pi(x)\in \mathrm{cl}(\mathbb{R}^{n_x}\setminus D_{\delta_0})\} \tag{14a}$$
$$C^z_{\delta_1} = \{z\in\mathbb{R}^{n_z} \,:\, \Pi(T_{\rm inv}(z))\in \mathrm{cl}(\mathbb{R}^{n_x}\setminus D_{\delta_1})\} \tag{14b}$$
$$D^z_{\delta_1} := \{z\in\mathbb{R}^{n_z} \,:\, \Pi(T_{\rm inv}(z))\in D_{\delta_1}\}\ . \tag{14c}$$

Of course, the system $\mathcal{H}$ evolves simultaneously with the observer $\hat{\mathcal{H}}$, with jumps that are not necessarily synchronized with those of the observer. However, as long as the estimation error $\hat{x}-x$ is sufficiently small, the hybrid observer $\hat{\mathcal{H}}$ in (13) guarantees the following properties:

i) When the observer flows in mode $q=2$, $|\Pi(T_{\rm inv}(z))|_D = |\Pi(\hat{x})|_D \geq \delta_1$ so $x\notin D$ and system $\mathcal{H}$ is also flowing, with y evolving continuously;

ii) When the observer enters mode $q=1$, $|\Pi(T_{\rm inv}(z))|_D = |\Pi(\hat{x})|_D = \delta_1$, so $x\in D_{\delta_0}$ and from (P1)-(P2), x jumps in a near future, some time during the open-loop phase where $q\in\{1,0\}$;

iii) Once x has jumped, the observer has time to finish the open-loop phase with $q\in\{1,0\}$ and to start again the high-gain phase with $q=2$, before x reenters D_{δ_0} (according to (P3) and the fact that $\Delta<\tau^0_m/2$).

Item iii) ensures that the estimation error has time to decrease via the use of the high-gain observer $\mathcal{F}_\ell$ before another open-loop phase starts.

It is interesting to note that the definition of the flow and jump sets ensure a certain robustness of implementation because $0<\delta_1<\delta_0$. Indeed, at the end of a high-gain phase in $\mathcal{C}_2$, a jump occurs in $\mathcal{D}_2$ when $\Pi(T_{\rm inv}(z))$ reaches D_{δ_1}. At this point, χ is reset to $\Pi(T_{\rm inv}(z))$, thus it belongs to D_{δ_1}, and flow is allowed for χ in the strictly larger set D_{δ_0} by definition of $\mathcal{C}_1$. Similarly, a jump cannot happen before χ has reached D by definition of $\mathcal{D}_1$. In other words, at the beginning of the open-loop phase, χ is initialized "$\delta_0-\delta_1$"-away from the boundary of the flow set and "δ_1"-away from the jump set, which leads to a robustness margin equal to $\min\{\delta_1,\delta_0-\delta_1\}$.

Remark 3.3: Note that, compared to [5], an additional state $\chi\in\mathbb{R}^{n_x}$ is used in the open-loop phases with $q\in\{1,0\}$, because the state z of the high-gain observer is generally in other coordinates with possibly $n_z>n_x$ and cannot be used during that time. If z is directly in the x-coordinates, with $T_{\rm inv}=\mathrm{Id}$, then χ can be removed as in [5].

C. Locally Asymptotically Stable Observer

With $\mathcal{O}$ and T given by Assumption 2.2, let us consider

$$\mathcal{A} = \mathcal{A}_2\cup\mathcal{A}_{10}\cup\mathcal{A}_1\cup\mathcal{A}_0 \tag{15}$$

where

$$\begin{aligned}\mathcal{A}_2 &= \{(x,z,\chi,q)\in\mathcal{O}\times\mathbb{R}^{n_z}\times\mathbb{R}^{n_x}\times\{2\} \,:\, z=T(x)\}\\ \mathcal{A}_{10} &= \{(x,z,\chi,q)\in\mathbb{R}^{n_x}\times\mathbb{R}^{n_z}\times\mathbb{R}^{n_x}\times\{1,0\} \,:\, \chi=x\}\\ \mathcal{A}_1 &= \{(x,z,\chi,q)\in g(D)\times\mathbb{R}^{n_z}\times D\times\{1\} \,:\, x=g(\chi)\}\\ \mathcal{A}_0 &= \{(x,z,\chi,q)\in D\times\mathbb{R}^{n_z}\times g(D)\times\{0\} \,:\, \chi=g(x)\}\end{aligned}$$

which, according to (8), are such that for each $x\in\mathcal{X}$, with $\mathcal{X}$ defined in Assumption 2.1, and $\hat{x}$ defined in (13c),

$$\begin{aligned}(x,z,\chi,q)\in\mathcal{A}_2\cup\mathcal{A}_{10} &\implies \hat{x}=x\\ (x,z,\chi,q)\in\mathcal{A}_1 &\implies \hat{x}\in D \text{ and } x=g(\hat{x})\\ (x,z,\chi,q)\in\mathcal{A}_0 &\implies x\in D \text{ and } \hat{x}=g(x)\end{aligned}$$

The sets $\mathcal{A}_2$ and $\mathcal{A}_{10}$ correspond to a zero estimation error, while the sets $\mathcal{A}_1$ and $\mathcal{A}_0$ correspond to $\hat{x}$ being one jump right ahead or behind of x. Unless exact synchronization of the system and observer jump times is achieved, including $\mathcal{A}_1$ and $\mathcal{A}_0$ cannot be avoided in an asymptotic analysis of a hybrid observer, since such errors are inevitable arbitrarily close to the jump times. This is known as the *peaking* phenomenon.

The following theorem shows that for ℓ sufficiently large, $\mathcal{A}$ is locally uniformly asymptotically stable for the interconnection of $\mathcal{H}$ and $\hat{\mathcal{H}}$.

Theorem 3.4: Suppose Assumptions 2.1, 2.2, and 3.1 hold with $\mathcal{X}_0$, ℓ_0, δ_0, τ_m^0 and $\mathcal{X}_m''$. Pick $0 < \delta_1 < \delta_0$ and $0 < \Delta < \frac{\tau_m^0}{2}$. Then, there exists $\ell^* \geq \ell_0$ such that for all $\ell > \ell^*$, there exist $\beta_\ell \in \mathcal{KL}$ and $\epsilon_\ell > 0$ such that for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$, any maximal solution $\phi := (z, \chi, \tau, q)$ to $\hat{\mathcal{H}}$ defined by (13) with input $y = h(x)$ and

$$\phi(0,0) \in \mathcal{C}_2 \cup \left(\mathbb{R}^{n_z} \times (D_{\delta_1} \cap \mathcal{X}_m'') \times \{0\} \times \{1\}\right)$$

such that

$$|(x, z, \chi, q)(0,0)|_{\mathcal{A}} < \epsilon_\ell \tag{16}$$

is t-complete and there exists a full j-reparametrization x^{r} of x such that for all $(t,j) \in \operatorname{dom}\phi$,

$$\begin{aligned}|(x^{\mathrm{r}}(t,j), z(t,j), \chi(t,j), q(t,j))|_{\mathcal{A}}& \\ \leq \beta_\ell(|(x, z, \chi, q)(0,0)|_{\mathcal{A}}, t+j)& \ .\end{aligned} \tag{17}$$

In other words, (17) says that during the high-gain phases where $q = 2$, z asymptotically converges to $T(x)$ (captured by $\mathcal{A}_2$), and during the open-loop phases where $q \in \{0,1\}$, χ either asymptotically converges to x (captured by $\mathcal{A}_{10}$), or is a jump ahead/behind x during the jump time mismatches (captured by $\mathcal{A}_1$ and $\mathcal{A}_0$). However, thanks to $\mathfrak{T}$ being Lipschitz, the length of those time mismatches asymptotically goes to zero.

Remark 3.5: The analysis of the estimation error heavily relies on items i), ii), and iii) described above and, thus, necessitates a sufficiently small initial error, guaranteeing that $\hat{x}$ is only one jump ahead or behind x. One may proceed with initialization of (z, χ, τ, q) as follows. If we know that at the initial time, x is not about to jump or has not just jumped (namely $x(0,0)$ is not close to either D or $g(D)$), one may initialize (z, χ, τ, q) to $q(0,0) = 2$, $\tau(0,0) = 0$ and $z(0,0) = T(\hat{x}_0)$ with $\hat{x}_0 \in \mathcal{X} \setminus D_{\delta_1}$ such that the estimation error $\hat{x}_0 - x(0,0)$ is sufficiently small to have (16) hold. On the other hand, if we know that $x(0,0)$ is in D_{δ_0} or close to $g(D)$, one should initialize (z, χ, τ, q) to $q(0,0) = 1$, $\tau(0,0) = 0$ and $\chi(0,0) \in D_{\delta_1} \cap \mathcal{X}$ such that either $\chi(0,0) - x(0,0)$ or $g(\chi(0,0)) - x(0,0)$ is sufficiently small according to (16).

IV. Proof of Theorem 3.4

In this entire section, we impose Assumptions 2.1, 2.2, and 3.1. We will start by showing that as long as the high-gain Lyapunov function V_ℓ is sufficiently small during high-gain phases, solutions to $\hat{\mathcal{H}}$ are t-complete and alternating between high-gain phases and open-loop phases. This is done by following a solution and considering all possible cases (see Lemma 4.1). Then, we study the evolution of the estimation error through one cycle of high-gain and open-loop phases, and finally, by iterating over those cycles, prove that a sufficient large gain and small initial estimation error ensure the Lyapunov function remains small enough and $\mathcal{A}$ is asymptotically stable.

Consider $0 < \varepsilon < \min\{\delta_1, \delta_0 - \delta_1\}$ such that

$$\mathcal{X}_\varepsilon := \{x \in \mathbb{R}^{n_x} \ : \ |x|_{\mathcal{X}} \leq \varepsilon\} \subset \mathcal{X}_m'' \ ,$$

with $\mathcal{X}$ defined in Assumption 2.1 (possible according to (11)). Then, for all $(x, \chi) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$,

(a) If $x \in D$ and $|x - \chi| \leq \varepsilon$, then $|\chi|_D < \delta_1$
(b) If $|\chi|_D \leq \delta_1$ and $|x - \chi| \leq \varepsilon$, then $|x|_D \leq \delta_0$.
(c) If $x \in \mathcal{X}$ and $|x - \chi| \leq \varepsilon$, then $\chi \in \mathcal{X}_\varepsilon \subset \mathcal{X}_m''$.

Denoting a_{inv} the Lipschitz constant of T_{inv}, let

$$v_\ell := \underline{c}(\ell)\left(\frac{\varepsilon}{a_p a_{\mathrm{inv}}}\right)^2 \tag{18}$$

where a_p and $\underline{c}(\ell)$ are defined in (10) and Assumption 2.2 respectively. Then, from (4a), (8), and (10), $V_\ell(x,z) \leq v_\ell$ with $(x,z) \in \mathcal{X} \times \mathbb{R}^{n_z}$ implies

$$\begin{aligned}|x - \Pi(T_{\mathrm{inv}}(z))| &\leq a_p|x - T_{\mathrm{inv}}(z)| \\ &\leq a_p|T_{\mathrm{inv}}(T(x)) - T_{\mathrm{inv}}(z)| \\ &\leq a_p a_{\mathrm{inv}}|z - T(x)| \leq \varepsilon \ .\end{aligned}$$

Therefore, items (a), (b), and (c) hold when $V_\ell(x,z) \leq v_\ell$ for $\chi = \Pi(T_{\mathrm{inv}}(z))$ and, since $a_p \geq 1$, also for $\chi = T_{\mathrm{inv}}(z)$.

A. t-Completeness of Observer Solutions

Let us start by showing that, given a solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$, any observer solution $\phi := (z, \chi, \tau, q)$ verifying $V_\ell(x,z) \leq v_\ell$ during the high-gain phases is t-complete, with the mode q sequentially taking values $2 \to 1 \to 0$ or $1 \to 0 \to 2$ depending on the initial condition. Note that by definition of solutions to hybrid systems with hybrid input y given in [6], jumps of the input (and thus here of the system $\mathcal{H}$) can trigger artificial trivial jumps in the observer solution, which leads us to consider in the next lemma a sub-parametrization ϕ^{sub} which describes the behavior of ϕ without those trivial jumps.

Lemma 4.1: Consider $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ and a maximal solution $\phi = (z, \chi, \tau, q)$ of $\hat{\mathcal{H}}$ with input $y = h(x)$ such that for all $(t,j) \in \operatorname{dom}\phi$ such that $q(t,j) = 2$ and all $j' \in \mathbb{N}$ such that $(t,j') \in \operatorname{dom} x$, $V_\ell(x(t,j'), z(t,j)) \leq v_\ell$ with v_ℓ defined in (18)*. Then, ϕ is t-complete, and is a full j-reparametrization of a hybrid arc ϕ^{sub} verifying either*

$$q^{\mathrm{sub}}(t,j) = 2 - j \pmod 3 \qquad \forall (t,j) \in \operatorname{dom}\phi^{\mathrm{sub}} \tag{19a}$$

if $\phi(0,0) \in \mathcal{C}_2$, or

$$q^{\mathrm{sub}}(t,j) = 1 - j \pmod 3 \qquad \forall (t,j) \in \operatorname{dom}\phi^{\mathrm{sub}} \tag{19b}$$

otherwise.

B. Evolution of Estimation Error through a Cycle $q = 2 \to 1 \to 0 \to 2$

Consider the positive maps $W : \mathcal{O} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \{0,1,2\} \to \mathbb{R}_{\geq 0}$ and $U : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \{0,1\} \to \mathbb{R}_{\geq 0}$ defined by

$$W(x,z,\chi,q) = \begin{cases} V_\ell(x,z) & \text{if } q = 2 \\ |\chi - x|^2 & \text{if } q \in \{0,1\} \end{cases}$$

$$U(x,\chi,q) = \begin{cases} |g(\chi)-x| + |x|_{g(D)} + |\chi|_D & \text{if } q=1 \\ |\chi - g(x)| + |x|_D + |\chi|_{g(D)} & \text{if } q=0 \end{cases}$$

which verify

$$W(x,z,\chi,q) = 0 \iff (x,z,\chi,q) \in \mathcal{A}_2 \cup \mathcal{A}_{10} \ , \quad \text{(20a)}$$
$$U(x,\chi,q) = 0 \iff (x,z,\chi,q) \in \mathcal{A}_1 \cup \mathcal{A}_0 \ . \quad \text{(20b)}$$

In high-gain phases, when $q=2$, we know from Assumption 2.2 that V_ℓ and thus W decrease exponentially at rate $\ell\lambda$. The following lemma describes the evolution of the estimation error through an open-loop phase. This error is measured by W or U depending on the period: U is used when x is one jump ahead or behind χ, i.e., during the interval of time with jump mismatch, and W is used the rest of the time. The next lemma roughly says that if the observer starts an open-loop phase with V_ℓ sufficiently small, it will go through modes 1, 0 and then back to 2, and V_ℓ will have grown at most by $a\frac{\overline{c}(\ell)}{\underline{c}(\ell)}$ through this process. Besides, x will have jumped only once in the meantime. Then, either ϕ flows in mode 2 for all time, or it reaches another open-loop phase with a dwell-time of at least $\tau_m^0 - 2\Delta$, so that V_ℓ grows at most by $a\frac{\overline{c}(\ell)}{\underline{c}(\ell)}e^{-\ell\lambda(\tau_m^0-2\Delta)}$ over the full cycle "open-loop + high-gain" phases.

Lemma 4.2: Consider $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ and a maximal solution $\phi = (z,\chi,\tau,q)$ of $\hat{\mathcal{H}}$ with input $y = h(x)$. Consider a transition from mode 2 to 1, namely a jump $j \in \operatorname{dom}_j \phi$ such that $q(t_j,j-1) = 2$ and $q(t_j,j) = 1$. Assume

$$V_\ell(x(t_j,j'),z(t_j,j-1)) < \min\left\{v_\ell, \frac{\underline{c}(\ell)}{a_p^2 a_{\rm inv}^2}\frac{\Delta^2}{a_\tau^2}\right\}$$

with $j' \in \mathbb{N}$ such that $(t_j,j') \in \operatorname{dom} x$, v_ℓ defined in (18), $a_{\rm inv}$ the Lipschitz constant of $T_{\rm inv}$, a_p defined in (10), and a_τ the Lipschitz constant of $\mathfrak{T}$ on $\mathcal{X}_m'' \cap D_{\delta_0}$ given by (P1). Then, $j+3 \in \operatorname{dom}_j \phi$ and x jumps exactly once in the time interval (t_j, t_{j+3}), i.e., j' is unique such that

$$(t_j,j') \in \operatorname{dom} x \qquad , \qquad (t_{j+3},j'+1) \in \operatorname{dom} x \ .$$

More precisely, $q(t,j) = 1$ for all $t \in [t_j,t_{j+1}]$, $q(t,j+2) = 0$ for all $t \in [t_{j+2},t_{j+3}]$, $q(t_{j+3},j+3) = 2$, and there exist $a_1, a_0, a_{01} > 0$ (independent from ℓ, x and ϕ) such that

$$|\chi(t,j) - x(t,j')| \leq a_1|\chi(t_j,j) - x(t_j,j')| \quad \forall t \in [t_j,t_{j+1}]$$
$$|\chi(t,j+2) - x(t,j'+1)| \leq a_0|\chi(t_j,j) - x(t_j,j')| \quad \forall t \in [t_{j+2},t_{j+3}]$$

and[1]

- *either x jumps before ϕ reaches $\mathcal{D}_1$, so $j+1$ is a trivial jump in ϕ, $q(t,j+1) = 1$ for all $t \in [t_{j+1},t_{j+2}]$, and during the interval with jump mismatch*

$$U(x(t,j'+1),\chi(t,j+1),q(t,j+1)) \leq a_{01}|\chi(t_j,j) - x(t_j,j')| \quad \forall t \in [t_{j+1},t_{j+2}]$$

- *or x jumps after ϕ has reached $\mathcal{D}_1$ so $j+2$ is a trivial jump in ϕ, $q(t,j+1) = 0$ for all $t \in [t_{j+1},t_{j+2}]$, and during the interval with jump mismatch*

$$U(x(t,j'),\chi(t,j+1),q(t,j+1)) \leq a_{01}|\chi(t_j,j) - x(t_j,j')| \quad \forall t \in [t_{j+1},t_{j+2}]$$

Moreover, there exists $a > 0$ (independent from ℓ, x and ϕ) such that

$$V_\ell(x(t_{j+3},j'+1),z(t_{j+3},j+3)) \leq a\frac{\overline{c}(\ell)}{\underline{c}(\ell)}V_\ell(x(t_j,j'),z(t_j,j-1)) \ , \quad (21)$$

with $\overline{c}(\ell), \underline{c}(\ell)$ given by Assumption 2.2. Finally, if in addition,

$$V_\ell(x(t_j,j'),z(t_j,j-1)) < \frac{1}{a}\frac{\underline{c}(\ell)}{\overline{c}(\ell)}v_\ell \ , \quad (22)$$

then, either $q(t,j+3) = 2$ for all $t \geq t_{j+3}$, or $j+4 \in \operatorname{dom}_j \phi$ with $q(t,j+3) = 2$ for all $t \in [t_{j+3},t_{j+4}]$, $q(t_{j+4},j+4) = 1$, and

$$t_{j+4} - t_{j+3} \geq \tau_m^0 - 2\Delta > 0$$

so that

$$V_\ell(x(t_{j+4},j'+1),z(t_{j+4},j+3)) \leq a\frac{\overline{c}(\ell)}{\underline{c}(\ell)}e^{-\ell\lambda(\tau_m^0-2\Delta)}V_\ell(x(t_j,j'),z(t_j,j-1)) \ . \quad (23)$$

C. Iterating Cycles

Exploiting exponential growth over polynomial growth, let us pick ℓ sufficiently large such that

$$\mu_\ell := ae^{-\ell\lambda(\tau_m^0-2\Delta)}\frac{\overline{c}(\ell)}{\underline{c}(\ell)} < 1 \quad \text{(24a)}$$

and v_1 sufficiently small such that

$$v_1 < \frac{\underline{c}(\ell)}{a_p^2 a_{\rm inv}^2}\min\left\{\frac{\Delta^2}{a_\tau^2} \ , \ \frac{1}{a}\frac{\underline{c}(\ell)}{\overline{c}(\ell)}\varepsilon^2\right\} \quad \text{(24b)}$$

Then, if we ensure that the observer verifies $V_\ell(x,z) < v_1$ before each transition from high-gain to open-loop phase, we ensure that i) $V_\ell(x,z) < v_\ell$ after the open-loop phase (once it has switched back to $q=2$) thanks to (24b) and (18), and that ii) $V_\ell(x,z) < v_1$ again at the next transition from high-gain to open-loop phase, namely when $q = 2 \to 1$, thanks to (24a). Then, these transitions can be iterated by repetitively using Lemma 4.2. Therefore, denoting v_k the value of V_ℓ before the kth transition $q = 2 \to 1$, we have $v_k \leq \mu_\ell^{k-1}v_1$.

If $q(0,0) = 2$, choosing $|(x,z,\chi,q)(0,0)|_{\mathcal{A}}$ sufficiently small ensures that $|(T(x)-z)(0,0)|$, and thus V_ℓ, are initially smaller than v_1. Because V_ℓ decreases in high-gain phase when $q=2$, the observer necessarily verifies $V_\ell(x,z) < v_1$ at its first transition to open-loop $q = 2 \to 1$. On the other hand, if $q(0,0) = 1$, by assumption $\chi(0,0) \in D_{\delta_1} \cap \mathcal{X}_m''$ and choosing $|(x,z,\chi,q)(0,0)|_{\mathcal{A}}$ sufficiently small ensures that $|(x-\chi)(0,0)|$ and thus W is initially sufficiently small, so that all the previously described steps are still valid and V_ℓ is also smaller than v_1 at its first transition to open-loop $q = 2 \to 1$.

[1] x could also jump at the exact time where ϕ reaches $\mathcal{D}_1$, but in that case it is equivalent to counting two jumps, one after the other, for easiness of presentation, since the jump maps are independent.

D. Asymptotic stability of $\mathcal{A}$

For each system jump, the observer solution jumps four times: three times for its own transitions between modes $2 \to 1 \to 0 \to 2$, with its state reset using the observer jump map (13b), and once trivially at the system jump with its state reset via the identity map. For each $j \in \operatorname{dom}_j \phi$, let us thus denote

$$k(j) := \begin{cases} \lfloor \frac{1}{4}(j+3) \rfloor & , \text{ if } q(0,0) = 2 \\ \lfloor \frac{1}{4} j \rfloor & , \text{ if } q(0,0) = 1. \end{cases}$$

At each hybrid time $(t,j) \in \operatorname{dom} \phi$, the integer $k(j)$ represents the number of transitions from high-gain to open-loop phase, i.e., $q = 2 \to 1$, that have occurred in the observer solution since the start. In the following, we denote x^{r} the j-reparametrization of x such that $\operatorname{dom} x^{\mathrm{r}} = \operatorname{dom} \phi$, containing trivial jumps whenever ϕ jumps and x does not.

According to (20) and by continuity, there exists $\mathcal{K}$-maps ρ, ρ' such that on the compact set containing the solutions,

$$\rho'(W(x,z,\chi,q)) \leq |(x,z,\chi,q)|_{\mathcal{A}_2 \cup \mathcal{A}_{10}} \leq \rho(W(x,z,\chi,q))$$
$$\rho'(U(x,\chi,q)) \leq |(x,z,\chi,q)|_{\mathcal{A}_0 \cup \mathcal{A}_1} \leq \rho(U(x,\chi,q))$$

and

$$\rho'(\min\{W(x,z,\chi,q), U(x,\chi,q)\}) \leq |(x,z,\chi,q)|_{\mathcal{A}} \leq \rho\big(\min\{W(x,z,\chi,q), U(x,\chi,q)\}\big)$$

If $k(j) = 0$ for all $j \in \operatorname{dom}_j \phi$, it means no transition from mode 2 to 1 occurs, namely the solution is eventually continuous in mode 2. Exponential decrease of W during this high-gain phase allows to conclude. Now, assume instead there exists $j_1 \in \mathbb{N}$ such that $k(j_1) = 1$, namely the jump index marking the first transition between $q = 2$ and $q = 1$, and denote $w_1 := W(x^{\mathrm{r}}(t_{j_1}, j_1 - 1), z(t_{j_1}, j_1 - 1), \chi(t_{j_1}, j_1 - 1), 2) = V_\ell(x^{\mathrm{r}}(t_{j_1}, j_1 - 1), z(t_{j_1}, j_1 - 1))$, the value of the Lyapunov function right before this transition. By applying Lemma 4.2 iteratively, on each cycle of "open-loop + high-gain" phases happening after hybrid time (t_{j_1}, j_1), we deduce that there exists $\pi_\ell \geq 1$ (depending on $\ell, a, a_0, a_1, a_{01}$) such that for all $(t,j) \in \operatorname{dom} \phi$ with $k(j) \geq 1$, one of the following holds:

- $W(x^{\mathrm{r}}(t,j), z(t,j), \chi(t,j), 1) \leq \pi_\ell \mu_\ell^{k(j)-1} w_1$ during the first period of the open-loop phase where $q = 1$ and neither x nor χ has jumped,
- $U(x^{\mathrm{r}}(t,j), z(t,j), \chi(t,j), q(t,j)) \leq \pi_\ell \mu_\ell^{k(j)-1} v_1$ during the jump time mismatch,
- $W(x^{\mathrm{r}}(t,j), z(t,j), \chi(t,j), 0) \leq \pi_\ell \mu_\ell^{k(j)-1} w_1$ in the last period of the open-loop phase where $q = 0$ and both x and χ have jumped,
- $W(x^{\mathrm{r}}(t,j), z(t,j), \chi(t,j), 2) \leq \pi_\ell e^{-\lambda \ell (t - t_{j-1})} \mu_\ell^{k(j)-1} w_1$ if $q(t,j) = 2$.

Those upper bounds ensure asymptotic convergence to $\mathcal{A}$ since the mode $q = 2$ is persistently visited, $\mu_\ell < 1$ and t is unbounded by t-completeness of solutions. However, in order to show asymptotic stability as in (17), we need a $\mathcal{KL}$-bound with respect to both $t + j$ and the initial condition. First, concerning the initial condition, note that if $q(0,0) = 2$, we have $j_1 = 1$ and

$$w_1 \leq e^{-\lambda \ell t_1} W(x(0,0), z(0,0), \chi(0,0), 2) \ .$$

Otherwise, if $q(0,0) = 1$, we have $j_1 = 3$, and

$$w_1 \leq \pi_\ell e^{-\lambda \ell (t_3 - t_2)} W(x(0,0), z(0,0), \chi(0,0), 1)$$

On the other hand, concerning $t + j$, we need to be more precise about the exponential decrease during each high-gain phase. In particular, the bounds by $\mu_\ell^{k(j)-1}$ only account for a decrease of W by $e^{-\ell \lambda \tau'_m}$ during each high-gain phase with $q = 2$ (see the definition of μ_ℓ in (24a)), while a high-gain phase could last for more than τ'_m amount of time. In order to obtain a $\mathcal{KL}$-bound, we need to reestablish this unaccounted decrease. Because each open-loop phase where $q \in \{0,1\}$ lasts at most 2Δ, and because τ'_m amount of flow is taken into account in μ_ℓ at each high-gain phase where $q = 2$, extra exponential decrease should be accounted for whenever

$$t \geq (k(j) + 1)(2\Delta + \tau'_m)$$

so that we actually have

$$\begin{aligned} &|(x^{\mathrm{r}}(t,j), z(t,j), \chi(t,j), q(t,j))|_{\mathcal{A}} \\ &\quad \leq \rho\big(\pi_\ell^2 \mu_\ell^{k(j)-1} e^{-\lambda \ell \max\{t - (k(j)+1)(2\Delta + \tau'_m), 0\}} \\ &\qquad\qquad \rho'^{-1}(|(x,z,\chi,q)(0,0)|_{\mathcal{A}})\big) \end{aligned}$$

which gives the result.

V. Semiglobal Hybrid Asymptotic Observer

With the high-gain hybrid observer formulated in Section III assuring local uniform asymptotic stability of the set $\mathcal{A}$ for the interconnection between $\mathcal{H}$ and $\hat{\mathcal{H}}$, in this section we provide a hybrid observer that, through the use of a continuous-time high-gain observer, which we call *preliminary high-gain observer*, enlarges the region of attraction and leads to a semiglobal result. The resulting observer, which we refer to as *semiglobal hybrid observer*, sequentially switches from the preliminary high-gain observer to the local hybrid observer when (16) holds. A key challenge in enlarging the basin of attraction is the presence of jumps of $\mathcal{H}$ when the preliminary high-gain observer is used, which may prevent the state estimate from converging to the state of $\mathcal{H}$. To overcome this issue, we exploit the properties in Assumption 2.1, as we describe below. Also, since the system is no longer in its initialization set $\mathcal{X}_0$ at the switching time where the local observer is launched, and since this switching time will be chosen smaller than τ_m, we introduce an additional initialization set $\mathcal{X}'_0$.

Assumption 5.1: There exists a compact subset $\mathcal{X}'_0$ of $\mathcal{X}$ such that any solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ remains in $\mathcal{X}'_0$ for $(t,j) \in \operatorname{dom} x$ such that $t \in [0, \tau_m]$.

Since the construction of the semiglobal hybrid observer we provide works for any observer that assures local convergence to the set $\mathcal{A}$, we design it for a general local hybrid observer. To this end, we assume the existence of a local hybrid observer

$$\hat{\mathcal{H}} \begin{cases} \dot{\xi} \in \mathcal{F}(\xi, y) & (\xi, y) \in \mathcal{C} \\ \xi^+ \in \mathcal{G}(\xi, y) & (\xi, y) \in \mathcal{D} \end{cases} \quad , \quad \hat{x} = \mathcal{T}(\xi) \tag{25}$$

for $\mathcal{H}$, for instance, through the design proposed in Section III. Similar to the properties in Theorem 3.4, we assume that $\hat{\mathcal{H}}$ induces the following local convergence property; see

Section II-A for the definition of solutions to (25) with hybrid input y.

Assumption 5.2: Consider $\mathcal{A} \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$. Let $\mathcal{X}_0'$ come from Assumption 5.1. There exists a set $\Xi_0 \subset \mathbb{R}^{n_\xi}$ and $\epsilon > 0$ such that for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0')$, any maximal solution (ξ, y^{r}) to $\hat{\mathcal{H}}$ defined by (25) with input $y = h(x)$ and with $\xi(0,0) \in \Xi_0$ such that $\hat{x}(0,0) := \mathcal{T}(\xi(0,0))$ verifies

$$|\hat{x}(0,0) - x(0,0)| < \epsilon \ , \tag{26}$$

is t-complete and there exists a full j-reparametrization $(x^{\mathrm{r}}, \xi^{\mathrm{r}})$ of x and ξ such that $\operatorname{dom} x^{\mathrm{r}} = \operatorname{dom} \xi^{\mathrm{r}}$ and

$$\lim_{t+j\to\infty} |(x^{\mathrm{r}}, \xi^{\mathrm{r}})(t,j)|_{\mathcal{A}} = 0 \ .$$

In other words, $\hat{\mathcal{H}}$ is a local observer relative to $\mathcal{A}$ in the sense of [4].

Example 5.3: Assume any solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0')$, with $\mathcal{X}_0'$ from Assumption 5.1, still verifies Assumption 2.1, namely is t-complete with dwell-time τ_m and remains in $\mathcal{X}$ at all times. If Assumptions 2.2 and 3.1 hold, we can apply Theorem 3.4 from $\mathcal{X}_0'$. Therefore, $\hat{\mathcal{H}}$ defined by (13) with state $\xi = (z, \chi, \tau, q)$ is a local observer for $\mathcal{H}$. Indeed, for ℓ sufficiently large, Assumption 5.2 is verified for $\mathcal{A}$ defined in (15), with

$$\Xi_0 = \mathcal{C}_2 \cup \left(\mathbb{R}^{n_z} \times (D_{\delta_1} \cap \mathcal{X}_m'') \times \{0\} \times \{1\}\right)$$

for $\mathcal{X}_m''$ from Assumption 3.1, and ϵ sufficiently small in (26) for (16) to hold.

A. Preliminary High-Gain observer and Switching Logic

Building upon the local observer in (25) satisfying Assumption 5.2, we design an observer that ensures semiglobal asymptotic convergence of the estimation error. Given an arbitrary compact set $\mathcal{Z}_0 \subset \mathbb{R}^{n_z}$, we propose the following time-driven logic:

1) Run a *preliminary* high-gain observer

$$\dot{z}_p = \mathcal{F}_{\ell_p}(z_p, y) \quad , \quad \hat{x} = T_{\mathrm{inv}}(z_p) \tag{27}$$

on the time interval $[0, t_p]$, initialized in $\mathcal{Z}_0$, with $\mathcal{F}_{\ell_p}$ and T_{inv} defined in Assumption 2.2 and (8);

2) After time t_p, launch the local observer $\hat{\mathcal{H}}$ defined in (25), initialized at $\xi_0 \in \Xi_0$ such that $\mathcal{T}(\xi_0) = \hat{x}(t_p)$,

with t_p chosen in a way that ensures the estimation error $\hat{x} - x$ provided by (27) at time t_p, is sufficiently small to verify (26), so that $\hat{\mathcal{H}}$ can be launched from an appropriate initialization guaranteeing asymptotic convergence to $\mathcal{A}$. In the following, we consider a map $\Xi : \mathbb{R}^{n_x} \to \Xi_0$ such that for any $\hat{x} \in \mathbb{R}^{n_x}$, $\xi_0 := \Xi(\hat{x})$ verifies $\mathcal{T}(\xi_0) = \hat{x}$.

The observer (27) guarantees that the Lyapunov function V_{ℓ_p} defined in Assumption 2.2 decreases arbitrarily fast for a sufficiently large gain ℓ_p, while the system $\mathcal{H}$ flows. However, jumps of $\mathcal{H}$ could make V_{ℓ_p} increase as (27) is not designed to account for the effects of those jumps. And because the jump times of $\mathcal{H}$ are unknown, we cannot adapt t_p to the first jump of $\mathcal{H}$ as it may happen any time. Fortunately, under Assumption 2.1, we know that $\mathcal{H}$ can jump only once in the interval $[0, \tau_m]$. Therefore, choosing $t_p \leq \tau_m$ ensures that only one jump of $\mathcal{H}$ should be handled in the error analysis on the interval $[0, t_p)$.

A first idea could be to choose ℓ_p sufficiently large for V_{ℓ_p} to decrease by a sufficient amount over a time window smaller than τ_m and fix $t_p = \tau_m$. The problem is when the system $\mathcal{H}$ jumps right before τ_m: V_{ℓ_p} may increase at the jump and does not have enough time to decrease again before $\hat{\mathcal{H}}$ is launched at time τ_m.

A second idea could be to choose ℓ_p sufficiently large for V_{ℓ_p} to decrease by a sufficient amount over a time window smaller than $\frac{\tau_m}{2}$ and then authorize an immediate switch to $\hat{\mathcal{H}}$ as soon as $\Pi(\hat{x}) \in D_{\delta_1}$ in the time interval $\left[\frac{\tau_m}{2}, \tau_m\right]$, or at time τ_m at the latest if this does not happen. Indeed, the fact that $\Pi(\hat{x}) \in D_{\delta_1}$ announces a system jump if the estimation error $\hat{x} - x$ is sufficiently small. But this strategy raises a similar issue as above if $\mathcal{H}$ jumps right before $\frac{\tau_m}{2}$, since $\Pi(\hat{x})$ could happen to be near D right after $\frac{\tau_m}{2}$ without having $\hat{x} - x$ sufficiently small to launch $\hat{\mathcal{H}}$. Actually, in this latter case, it would have been better to stick to the first strategy and wait till time τ_m in order for the estimation error to decrease again after the system jump.

Due to those reasons, we propose to combine both strategies and, considering $\tau_1 < \tau_2$ in $(0, \tau_m)$, switch to $\hat{\mathcal{H}}$ either

- at time τ_m if the event "$\Pi(\hat{x}) \in D_{\delta_1}$" happens in the time window $[\tau_1, \tau_2]$ or does not happen at all over the window $[0, \tau_m]$; or
- as soon as $\Pi(\hat{x}) \in D_{\delta_1}$ if such an event happens after τ_2.

To make this strategy more precise, and following the definition of $\mathfrak{T}$ in (P1), let $\Delta_{\max}$ be an upper bound of $\mathfrak{T}$ on $D_{\delta_0} \cap \mathcal{X}$, namely an upper bound of the time elapsed between the time where a solution x of $\mathcal{H}$ enters $D_{\delta_0} \cap \mathcal{X}$ and the time it reaches D. The existence of $\Delta_{\max}$ is guaranteed by (P1) and it verifies $\Delta_{\max} \leq a_\tau \delta_0$ where a_τ is the Lipschitz constant of $\mathfrak{T}$. Consider $0 < \delta_1 < \delta_0$ sufficiently small such that $\Delta_{\max} < \tau_m$, and times $\tau_1, \tau_2 \in [0, \tau_m]$, such that

$$0 < \tau_1 < \tau_2 < \tau_2 + \Delta_{\max} < \tau_m \ .$$

We show that for ℓ_p sufficiently large, the switching time t_p to $\hat{\mathcal{H}}$ can be chosen in $[\tau_2, \tau_m]$ based on when $\Pi(\hat{x}) \in D_{\delta_1}$, with $\hat{x}$ provided by (27). The strategy is described in Algorithm 1 (for simplicity, we index the solutions to (27) on continuous time only, even though its input y is an hybrid signal).

B. Semiglobal Hybrid Observer Construction

The solutions produced by Algorithm 1 are generated by a hybrid observer $\hat{\mathcal{H}}_{\mathrm{sg}}$, defined below in (28), with state $\xi_{\mathrm{sg}} = (z_p, \tau_p, q_p, q_w, \xi)$, where z_p is the state of the preliminary observer (27), ξ is the state of the local observer $\hat{\mathcal{H}}$, $\tau_p \in [0, \tau_m]$ is a timer, $q_w \in \{0,1\}$ is a warning state used to determine the time t_p to switch to $\hat{\mathcal{H}}$, and $q_p \in \{0,1\}$ is used to indicate whether the observer is in preliminary mode ($q_p = 1$) or local mode ($q_p = 0$). Actually, in order to check the robustness of our algorithm, we also include the solutions obtained by allowing either $q_w \leftarrow 1$ or $t_p \leftarrow t$ at the frontier time τ_2 in Algorithm 1. This allows to obtain an outer semicontinuous jump map with a closed jump set. Given the

Algorithm 1 Semiglobal observer strategy

Pick $0 < \tau_1 < \tau_2 < \tau_m$ such that $\tau_2 + \Delta_{\max} < \tau_m$.
Pick $z_p(0) \in \mathcal{Z}_0$.
$q_w \leftarrow 0$, $t_p \leftarrow \tau_m$, $t \leftarrow 0$.
while $t < t_p$ **do**
  Run (27) with input $y = h(x)$ and output $\hat{x}$.
  if $\Pi(\hat{x}(t)) \in D_{\delta_1}$ and $q_w = 0$ **then**
    if $t \in [\tau_1, \tau_2)$ **then**
      $q_w \leftarrow 1$
    else if $t \in [\tau_2, \tau_m]$ **then**
      $t_p \leftarrow t$.
    end if
  end if
end while
Pick $\xi(t_p, 0) \in \Xi_0$ such that $\mathcal{T}(\xi(0,0)) = \Pi(\hat{x}(t_p))$.
while $t \geq t_p$ **do**
  Run $\hat{\mathcal{H}}$ in (25) with input $y = h(x)$ and output $\hat{x}$.
end while

local and preliminary observers (25),(27), the dynamics of the observer $\hat{\mathcal{H}}_{\rm sg}$ are thus defined by

$$\begin{pmatrix} \dot{z}_p \\ \dot{\tau}_p \\ \dot{q}_p \\ \dot{q}_w \\ \dot{\xi} \end{pmatrix} = \begin{cases} \begin{pmatrix} \mathcal{F}_{\ell_p}(z_p, y) \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \text{if } \xi_{\rm sg} \in \mathcal{C}_{{\rm sg},1} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \mathcal{F}(\xi, y) \end{pmatrix} & \text{if } \xi_{\rm sg} \in \mathcal{C}_{{\rm sg},0} \end{cases} \tag{28a}$$

$$\begin{pmatrix} z_p^+ \\ \tau_p^+ \\ q_p^+ \\ q_w^+ \\ \xi^+ \end{pmatrix} \in \begin{cases} \begin{pmatrix} z_p \\ \tau_p \\ 0 \\ 0 \\ \Xi(\Pi(T_{\rm inv}(z_p))) \end{pmatrix} & \text{if } \xi_{\rm sg} \in \mathcal{D}_{{\rm sg},1} \setminus \mathcal{D}^w_{{\rm sg},1} \\ \begin{pmatrix} z_p \\ \tau_p \\ q_p \\ 1 \\ \xi \end{pmatrix} & \text{if } \xi_{\rm sg} \in \mathcal{D}^w_{{\rm sg},1} \setminus \mathcal{D}_{{\rm sg},1} \\ \begin{pmatrix} z_p \\ \tau_p \\ q_p \\ 1 \\ \xi \end{pmatrix} \cup \begin{pmatrix} z_p \\ \tau_p \\ 0 \\ 0 \\ \Xi(\Pi(T_{\rm inv}(z_p))) \end{pmatrix} & \text{if } \xi_{\rm sg} \in \mathcal{D}_{{\rm sg},1} \cap \mathcal{D}^w_{{\rm sg},1} \\ \begin{pmatrix} z_p \\ \tau_p \\ q_p \\ q_w \\ \mathcal{G}(\xi, y) \end{pmatrix} & \text{if } \xi_{\rm sg} \in \mathcal{D}_{{\rm sg},0} \end{cases} \tag{28b}$$

with estimate given by

$$\hat{x} = \mathcal{T}_{\rm sg}(\xi_{\rm sg}) := \begin{cases} T_{\rm inv}(z_p) & \text{if } q_p = 1 \\ \mathcal{T}(\xi) & \text{if } q_p = 0 \end{cases} \tag{28c}$$

the flow and jump sets defined by

$$\begin{aligned} \mathcal{C}_{{\rm sg},1} &= \left(\mathbb{R}^{n_z} \times [0, \tau_m] \times \{1\} \times \{1\} \times \mathbb{R}^{n_\xi}\right) \\ &\qquad \cup \left(\mathbb{R}^{n_z} \times [0, \tau_1] \times \{1\} \times \{0\} \times \mathbb{R}^{n_\xi}\right) \\ &\qquad \cup \left(C^z_{\delta_1} \times [\tau_1, \tau_m] \times \{1\} \times \{0\} \times \mathbb{R}^{n_\xi}\right) \\ \mathcal{C}_{{\rm sg},0} &= \mathbb{R}^{n_z} \times [0, \tau_m] \times \{0\} \times \{0,1\} \times \mathcal{C} \\ \mathcal{D}_{{\rm sg},1} &= \left(\mathbb{R}^{n_z} \times \{\tau_m\} \times \{1\} \times \{0,1\} \times \mathbb{R}^{n_\xi}\right) \\ &\qquad \cup \left(D^z_{\delta_1} \times [\tau_2, \tau_m] \times \{1\} \times \{0\} \times \mathbb{R}^{n_\xi}\right) \\ \mathcal{D}^w_{{\rm sg},1} &= D^z_{\delta_1} \times [\tau_1, \tau_2] \times \{1\} \times \{0\} \times \mathbb{R}^{n_\xi} \\ \mathcal{D}_{{\rm sg},0} &= \mathbb{R}^{n_z} \times [0, \tau_m] \times \{0\} \times \{0,1\} \times \mathcal{D} \end{aligned}$$

where $\mathcal{C}$ and $\mathcal{D}$ are defined in (25) and $C^z_{\delta_1}$ and $D^z_{\delta_1}$ in (14).

Solutions $\hat{\mathcal{H}}_{\rm sg}$ initialized in $\mathcal{Z}_0 \times \{0\} \times \{1\} \times \{0\} \times \mathbb{R}^{n_\xi}$ flow in $\mathcal{C}_{{\rm sg},1}$ with $q_p = 1$ and $q_w = 0$ until either

- the timer τ_p reaches τ_m; or
- $\Pi(\hat{x}) \in D_{\delta_1}$ at some time in $[\tau_1, \tau_m]$.

In the former case, the solution is in $\mathcal{D}_{{\rm sg},1} \setminus \mathcal{D}^w_{{\rm sg},1}$ so that q_p is reset to 0, marking the end of the interval over which the preliminary observer is used, ξ is reset to $\Xi(\Pi(\hat{x}))$, and the solution then evolves according to $\hat{\mathcal{H}}$. In the latter case, either

- $\tau_p \in [\tau_1, \tau_2)$, the solution is in $\mathcal{D}^w_{{\rm sg},1} \setminus \mathcal{D}_{{\rm sg},1}$ and, after the jump, q_p remains at 1 and q_w is equal to 1; or
- $\tau_p \in (\tau_2, \tau_3]$, the solution is in $\mathcal{D}_{{\rm sg},1} \setminus \mathcal{D}^w_{{\rm sg},1}$, q_p is reset to 0, marking the end of the preliminary observer mode; or
- $\tau_p = \tau_2$, the solution is in $\mathcal{D}_{{\rm sg},1} \cup \mathcal{D}^w_{{\rm sg},1}$ and we have a choice between the former two items.

In the first item, after q_w has been reset to 1, the preliminary observer evolves continuously with the solution in $\mathcal{C}_{{\rm sg},1}$ until the timer τ_p reaches τ_m. When this happens, the solution is in $\mathcal{D}_{{\rm sg},1}$, so q_p is reset to 0 and $\hat{\mathcal{H}}$ is launched from $\Xi(\Pi(\hat{x}))$.

C. Semiglobal Result

The hybrid observer $\hat{\mathcal{H}}_{\rm sg}$ guarantees the following semiglobal property for the set $\mathcal{A}$.

Theorem 5.4: Suppose Assumptions 2.1, 2.2, 3.1, 5.1 and 5.2 hold with δ_0 sufficiently small such that $\Delta_{\max} < \tau_m$ (guaranteed since $\Delta_{\max} \leq a_\tau \delta_0$ where a_τ is the Lipschitz constant of $\mathfrak{T}$). Consider a compact subset $\mathcal{Z}_0$ of $\mathbb{R}^{n_z}$ and $0 < \tau_1 < \tau_2 < \tau_m$ such that $\tau_2 + \Delta_{\max} < \tau_m$. Then, there exists $\ell^* > \ell_0$ such that for all $\ell_p > \ell^*$ and for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$, any solution $\phi = (z_p, \tau_p, q_p, q_w, \xi)$ to $\hat{\mathcal{H}}_{\rm sg}$ defined in (28) and initialized in $\mathcal{Z}_0 \times \{0\} \times \{1\} \times \{0\} \times \mathbb{R}^{n_\xi}$ is t-complete and verifies

$$\lim_{t+j \to \infty} |(x^{\rm r}, \xi^{\rm r})(t,j)|_{\mathcal{A}} = 0 \tag{29}$$

for some full j-reparametrizations $x^{\rm r}$ and $\phi^{\rm r} = (z_p^{\rm r}, \tau_p^{\rm r}, q_p^{\rm r}, q_w^{\rm r}, \xi^{\rm r})$ of x and ϕ respectively such that $\operatorname{dom} x^{\rm r} = \operatorname{dom} \phi^{\rm r}$.

Note that this result ensures asymptotic convergence, but not stability. In particular, if $\hat{\mathcal{H}}$ is chosen as in (13), the stability provided by Theorem 3.4 ensures stability of $|(x,\xi)|_{\mathcal{A}}$ with respect to its value at time t_p, but not with respect to its initial condition. Indeed, the preliminary observer (27) running on $[0, t_p]$ may miss up to one jump of x. In other words, $|(x,\xi)(0,0)|_{\mathcal{A}} = 0$ does not imply $|(x,\xi)|_{\mathcal{A}}$ remains zero. In order to ensure semiglobal stability, other observers must be designed able to follow the jumps of $\mathcal{H}$ from the start.

On the other hand, the observer data is chosen verifying the *hybrid basic conditions*, namely with closed sets, and outer semicontinuous locally bounded maps (see [14, Assumption 6.5]). Due to the absence of stability with respect to the initial error, we cannot directly claim robustness through [14, Theorem 7.21]. However, the observer sequentially uses robust elements (robust preliminary high-gain observer, hysteresis in the choice of t_p, robust local observer) which provide robustness as observed in simulations in Section VI, although of course, this design does not escape from the well-known limitations of the performance of high-gain observers in the presence of output noise.

Proof: Consider $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ and ϕ a solution to (28) initialized in $\mathcal{Z}_0 \times \{0\} \times \{1\} \times \{0\} \times \mathbb{R}^{n_\xi}$ with input $y = h(x)$. The solution ϕ evolves as detailed above the statement of Theorem 5.4, with first a preliminary observer mode ($q_p = 1$) and then a local observer mode ($q_p = 0$). Let us denote $t_p \in [\tau_1, \tau_m]$ the time at which the preliminary observer stops and the associated jump $j_p \in \{1, 2\}$ verifying

$$q_p(t_p, j_p - 1) = 1 \quad , \quad q_p(t_p, j_p) = 0 \ .$$

If $j_p = 2$, it means that $\Pi(\hat{x})$ has been in D_{δ_1} in $[\tau_1, \tau_2]$, leading to the warning state q_w being reset to 1 and t_p necessarily equal to τ_m. On the other hand, if $j_p = 1$, it means that no warning state has been used (i.e., $q_w(t, 0) = 0$ for all $t \in [0, t_p]$) and $t_p \in [\tau_2, \tau_m]$. Then, for all $t \geq t_p$ and all $j \geq j_p$ such that $(t, j) \in \operatorname{dom} \phi$, $(t, j) \mapsto \xi(t, j)$ evolves according to the local observer dynamics $\hat{\mathcal{H}}$. By Assumption 2.1, the system solution x jumps at most once on $[0, t_p]$ and $x(t_p, j') \in \mathcal{X}$, for $j' \in \{0, 1\}$ such that $(t_p, j') \in \operatorname{dom} x$. Since ξ is initialized to $\Xi(\Pi(\hat{x}(t_p, j_p)))$ at the beginning of the local observer mode, Assumptions 5.1 and 5.2 imply the result if

$$|\Pi(\hat{x}(t_p, j_p)) - x(t_p, j')| < \epsilon \ . \tag{30}$$

Without loss of generality, we assume $0 < \epsilon < \min\{\delta_1, \delta_0 - \delta_1\}$. Then, similarly to Section IV, for all $(x, \chi) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$,

(a') If $x \in D$ and $|x - \chi| \leq \epsilon$, then $|\chi|_D < \delta_1$.
(b') If $|\chi|_D \leq \delta_1$ and $|x - \chi| \leq \epsilon$, then $|x|_D \leq \delta_0$.

Still denoting a_{inv} the Lipschitz constant of T_{inv}, let

$$v_{\ell_p} := \underline{c}(\ell) \left(\frac{\epsilon}{a_p a_{\text{inv}}} \right)^2 \tag{31}$$

where a_p and $\underline{c}(\ell)$ are defined in (10) and in Assumption 2.2, respectively. Then, for all $(x, z) \in \mathcal{X} \times \mathbb{R}^{n_z}$,

$$V_{\ell_p}(x, z) < v_{\ell_p} \implies |x - \Pi(T_{\text{inv}}(z))| \leq a_p |x - T_{\text{inv}}(z)| < \epsilon \ .$$

In particular, it is enough to show that

$$V_{\ell_p}(x(t_p, j'), z_p(t_p, j_p)) < v_{\ell_p} \tag{32}$$

to ensure (30) holds.

Let us study the evolution of $V_{\ell_p}(x, z_p)$ on $[0, t_p]$. Define

$$r_0 = \max_{(x,z) \in \mathcal{X}_0 \times \mathcal{Z}_0} |T(x) - z| \quad , \quad r_1 = \max_{x \in \mathcal{X}} |T(g(x)) - T(x)| \ .$$

Before the first jump of x, i.e. for all $t \in [0, \min\{t_1(x), \tau_m\}]$, and for $j \in \{0, 1\}$ such that $(t, j) \in \operatorname{dom} \phi$,

$$\begin{aligned} V_{\ell_p}(x(t,0), z_p(t,j)) &\leq e^{-\ell_p \lambda t} V_{\ell_p}(x_0(0,0), z_p(0,0)) \\ &\leq \overline{c}(\ell_p) e^{-\ell_p \lambda t} |z_p(0,0) - T(x(0,0))|^2 \\ &\leq \overline{c}(\ell_p) r_0^2 e^{-\ell_p \lambda t} \end{aligned} \tag{33a}$$

Then, if $t_1(x) < t_p$, i.e., if the first jump of the system happens before t_p, x jumps to $g(x)$ while z_p does not jump. Since for all $(x, z) \in \mathcal{X} \times \mathbb{R}^{n_z}$,

$$\begin{aligned} V_{\ell_p}(g(x), z) &\leq \overline{c}(\ell_p) |z - T(g(x))|^2 \\ &\leq \frac{\overline{c}(\ell_p)}{2} (|T(x) - T(g(x))|^2 + |z - T(x)|^2) \\ &\leq \frac{\overline{c}(\ell_p)}{2} \left(r_1^2 + \frac{1}{\underline{c}(\ell_p)} V_{\ell_p}(x, z) \right) \end{aligned}$$

we have for all $t \in [t_1(x), t_p]$, and for $j \in \{0, 1\}$ such that $(t, j) \in \operatorname{dom} \phi$,

$$\begin{aligned} &V_{\ell_p}(x(t,1), z_p(t,j)) \\ &\quad \leq \frac{\overline{c}(\ell_p)}{2} \left(r_1^2 + \frac{\overline{c}(\ell_p)}{\underline{c}(\ell_p)} r_0^2 e^{-\ell_p \lambda t_1(x)} \right) e^{-\ell_p \lambda (t - t_1(x))} \\ &\quad \leq \frac{\overline{c}(\ell_p)}{2} \left(r_1^2 e^{-\ell_p \lambda (t - t_1(x))} + \frac{\overline{c}(\ell_p)}{\underline{c}(\ell_p)} r_0^2 e^{-\ell_p \lambda t} \right) \ . \end{aligned} \tag{33b}$$

Denoting $\tau_3 := \tau_2 + \Delta_{\max} < \tau_m$ and exploiting exponential decay over polynomial growth, pick ℓ_p sufficiently large such that

$$\overline{c}(\ell_p) r_0^2 e^{-\ell_p \lambda \tau_1} < v_{\ell_p} \tag{34a}$$

$$\frac{\overline{c}(\ell_p)}{2} \left(r_1^2 e^{-\ell_p \lambda \min\{\tau_2 - \tau_1, \tau_m - \tau_3\}} + \frac{\overline{c}(\ell_p)}{\underline{c}(\ell_p)} r_0^2 e^{-\ell_p \lambda \tau_2} \right) < v_{\ell_p} \tag{34b}$$

We have the following cases:

- if $t_1(x) \leq \tau_1$, (32) holds according to (33b) and (34b) since $t_p \in [\tau_2, \tau_m]$ and thus $t_p - t_1(x) \geq \tau_2 - \tau_1$.
- if $\tau_1 < t_1(x) \leq \tau_2$, we have according to (33a) and (34a),

  $$V_{\ell_p}(x(t,0), z_p(t,0)) < v_{\ell_p}$$

  for any $\tau_1 \leq t \leq t_1(x)$, and thus, according to item (a'), there exists $t^* \in [\tau_1, t_1(x)) \subseteq [\tau_1, \tau_2)$ such that $\Pi(T_{\text{inv}}(z_p(t^*, 0))) \in D_{\delta_1}$. According to the behavior of solutions described above (see also Algorithm 1), $t_p = \tau_m$ and (32) holds at time t_p thanks to (33b) and (34b) since $t_p - t_1(x) \geq \tau_m - \tau_2 > \tau_m - \tau_3$.
- if $\tau_2 \leq t_1(x) \leq \tau_m$, still through (33a), (34a),

  $$V_{\ell_p}(x(t,0), z_p(t,0)) < v_{\ell_p}$$

  for any $\tau_1 \leq t \leq t_1(x)$, and with item (a'), there exists a minimal time $t^* \in [\tau_1, t_1(x)) \subseteq [\tau_1, \tau_m]$ such that $\Pi(T_{\text{inv}}(z_p(t^*, 0))) \in D_{\delta_1}$. We distinguish two cases:
  - If $t^* \in [\tau_1, \tau_2]$, then, either $t_p = \tau_m$ or $t_p = t^*$, and since $x(t^*, 0) \in D_{\delta_0}$ by item (b'), x jumps at a time

    $$t_1(x) \leq t^* + \Delta_{\max} \leq \tau_2 + \Delta_{\max} = \tau_3$$

    by definition of $\Delta_{\max}$. Therefore, if $t_p = \tau_m$, (32) holds thanks to (33b) and (34b) since $t_p - t_1(x) \geq \tau_m - \tau_3$.

And if $t_p = t^*$, (32) holds according to (33a) and (34a) since $\tau_1 \le t_p \le t_1(x)$.
  - If on the other hand, $t^* \in (\tau_2, \tau_m]$, then necessarily $t_p = t^*$ and again, (32) holds according to (33a) and (34a) since $\tau_1 \le t_p \le t_1(x)$.
- if $t_1(x) \ge \tau_m$, then $t_p \le t_1(x)$ and (32) holds according to (33a) and (34a).

We conclude that in all cases, (32) holds and the result is proved. ■

The parameters τ_i in Algorithm 1 can be chosen arbitrarily as long as $\tau_m - \tau_2 > \Delta_{\max}$, namely, in sight of Assumption 3.1, as long as any solution entering D_{δ_0} before τ_2 reaches D and jumps before τ_m. But of course, the smaller τ_1, $\tau_2 - \tau_1$ and $\tau_m - \tau_3$, the larger ℓ_p must be taken according to (34).

Remark 5.5: Note that if $\hat{\mathcal{H}}$ defined in (13) with state $\xi = (z, \chi, \tau, q)$ is chosen as local observer, the implementation of its semiglobal version $\hat{\mathcal{H}}_{\rm sg}$ defined in (28) can be simplified by using the states z and q of $\hat{\mathcal{H}}$ in place of z_p and q_p, respectively, with, for instance $q \in \{0, 1, 2, 3\}$, where $q = 3$ denotes the "local observer mode".

VI. Example

A. Bouncing ball

The local observer (13) was illustrated in [5] on a bouncing ball with state $(x_1, x_2) \in \mathbb{R}^2$ and data given by

$$\begin{aligned} &f(x) = (x_2, -\mathfrak{g})\,,\; g(x) = -x\,,\; h(x) = x_1 \\ &C = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 \ge 0\} \\ &D = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 = 0\,,\; x_2 \le 0\} \end{aligned} \tag{35}$$

In order to satisfy Assumption 3.1 and encode the absence of Zeno behavior in the solutions of interest, the set D was replaced in the observer by $D_m := \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 = 0\,,\, x_2 \le -m\}$ for $m > 0$. The semiglobal observer (28) presented in this paper is implemented in `https://github.com/HybridSystemsLab/NonSyncHybridHighGainObserver`.

B. Spiking neuron

We consider the parameterized nonlinear model of a spiking neuron presented in [16] given by a hybrid system $\mathcal{H}$ as in (1) with state $(x_1, x_2) \in \mathbb{R}^2$ and data given by

$$\begin{aligned} &f(x) = \big(0.04x_1^2 + 5x_1 + 140 - x_2 + I_{\rm ext}\,,\; a(bx_1 - x_2)\big) \\ &g(x) = (c, x_2 + d) \qquad , \qquad h(x) = x_1 \\ &C = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 \le v_m\} \\ &D = \{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_1 = v_m\} \end{aligned} \tag{36}$$

where x_1 is the membrane potential, x_2 is the recovery variable, and $I_{\rm ext}$ represents the (constant) synaptic current or injected DC current. The value of the input $I_{\rm ext}$ and the model parameters a, b, c, and d, as well as the threshold voltage v_m characterize the neuron type and its firing pattern. We refer the reader to [16] for units and more physical details. Here, we consider $I_{\rm ext} = 10, a = 0.02, b = 0.2, c = -55, d = 4$, and $v_m = 30$, leading to an intrinsically bursting neuron. The solutions are known to remain in a physical compact set $\mathcal{X}$ and have a uniform dwell-time, thus verifying Assumption 2.1. A solution is plotted on Figure 2.

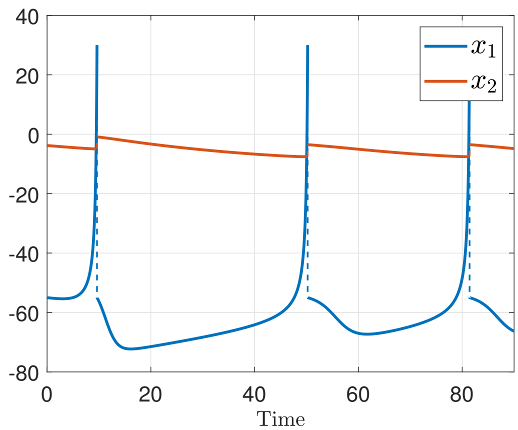


Fig. 2: Solution to hybrid system (36) with initial condition $x(0, 0) = (-55, -6)$.

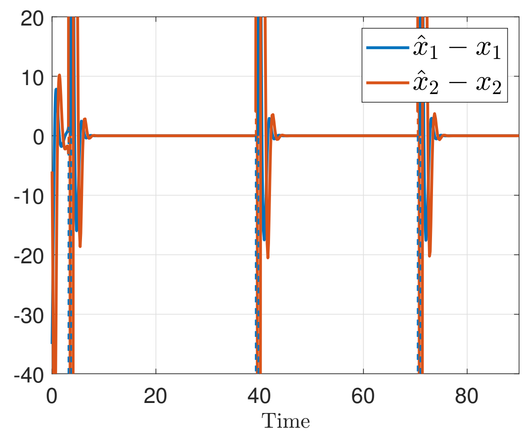


Fig. 3: Estimation error for system (36) with observer (2) where $\mathcal{F} = \mathcal{F}_\ell$ defined in (7) with $\ell = 4$ and $K = (1, 1)^\top$, $x(0, 0) = (-55, -6)$, $\hat{x}_0 = (-20, 0)$, $z(0, 0) = T(\hat{x}_0)$ and delay in the jump detection of 0.5 units of time.

The map

$$T(x) = (h(x), L_f h(x)) = (x_1, 0.04x_1^2 + 5x_1 + 140 - x_2 + I_{\rm ext})$$

is a diffeomorphism on $\mathbb{R}^2$, so that the flow dynamics admit a high-gain observer (7) as detailed in Example 2.4. Since the jump times can be detected from the jumps of the output $y = x_1$, it is proposed in [7] to use an observer of the type (2), with $\mathcal{F} = \mathcal{F}_\ell$ given by the high-gain observer, $\mathcal{G}$ defined in (9), and jumps triggered at the same time as the those of the system. However, because of the unstable quadratic term $0.04x_1^2$ in the flow dynamics, slight delays in the jump detection deteriorate very quickly the estimate around each jump, as illustrated on Figure 3.

Instead, we would like to implement observer (13), which does not rely on instantaneous jump detection and that automatically synchronizes its jumps with those of the system. Because of the quadratic term in f, items (P1) and (P2) of Assumption 3.1 hold for any choice of δ_0 and δ_1. However,

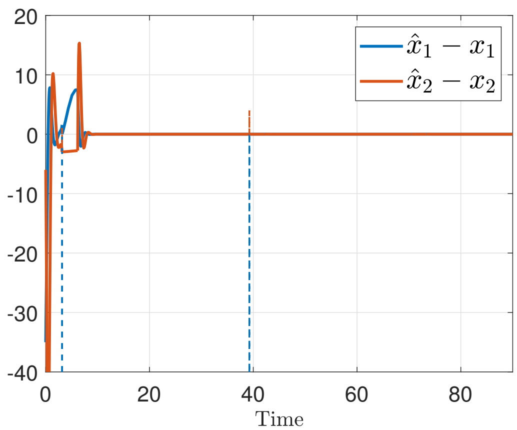


Fig. 4: Estimation error for system (36) with observer (13) with $\ell = 4$ and $K = (1,1)^\top$ in $\mathcal{F}_\ell$ defined in (7), $\delta_0 = 5$, $\delta_1 = 3$, $\Delta = 3$, and initial conditions $x(0,0) = (-55,-6)$, $\chi(0,0) = (-20,0)$, $z(0,0) = T(\chi(0,0))$, $q(0,0) = 2$.

some care should be taken in the choice of $\Delta < \frac{\tau_m^0}{2}$ with τ_m^0 given in (P3). Indeed, the flow dynamics exhibit finite-time escape in open-loop, so Δ should be smaller than the minimal time needed for the flow dynamics to escape in finite-time from (c, x_2) in a "larger" compact set $\mathcal{X}_m$ in which the observer trajectory should evolve. Unfortunately, these choices depend on the magnitude of the initial error. Figure 4 shows the results of a simulation with the same initial conditions and same high-gain map $\mathcal{F}_\ell$ as in Figure 3, but this time with observer (13), which does not rely on the jump detection and which automatically synchronizes its jumps with those of the system. Compared to Figure 3, we observe that the mismatch of jump times marked by vertical dashed lines and the estimation error asymptotically converge to 0.

A major difference between both designs is that the former is global while the latter is only local. In particular, as mentioned above, for too large initial estimation errors, observer trajectories could explode in finite time during the first open-loop phase (pick $\chi(0,0) = (24,0)$ in the previous simulation to witness finite-time escape). It may also happen that the observer "misses" a jump of the system and simply catches up afterwards with the high-gain observer without any guarantee from the analysis. In order to avoid this, or do it in a "safe" way that ensures convergence, we run a preliminary continuous-time high-gain observer and launch observer (13) at a well-chosen time t_p according to Algorithm 1 given in Section V, namely we implement the semiglobal observer (28). Figures 5,6 illustrates[2] the behavior of the observer depending on the first jump time of the system and shows that the algorithm choosing t_p is effective to ensure asymptotic convergence in each case. When the local observer is launched at time τ_m and the system jumps before that time, two transients occur: one at the initial time, and another after the system jump (see Figures 5a-5b). In particular, a possible system jump is successfully detected between τ_1 and τ_2 in Figure 5b), leading to the warning state q_w being turned on and the switching time t_p fixed at τ_m for safety. On the other hand, in Figure 5c), a system jump is successfully anticipated after τ_2 and the local observer directly switched on, thus avoiding another transient.

Finally, the robustness of this design was also tested in presence of output noise in the three previous scenarios. Figure 7 reproduces the errors plotted on Figure 5 but in presence of output noise and on a longer time horizon. As expected, the noise is amplified through the high-gain observers and peaking is observed around the jump times since the jumps of the system and the observer no longer asymptotically synchronize.

[2]Simulations available at `https://github.com/HybridSystemsLab/NonSyncHybridHighGainObserver`

VII. Conclusion

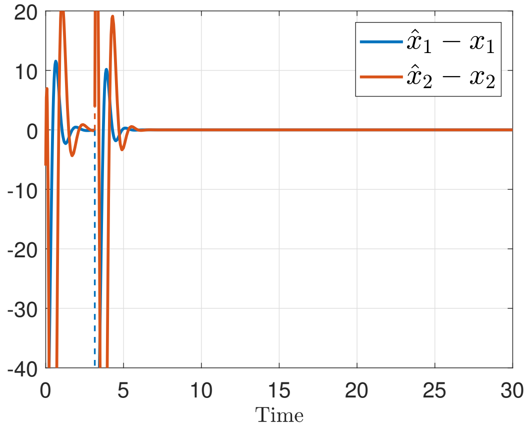
We have proposed a semiglobal hybrid observer for hybrid dynamical systems whose jump times are unknown and whose pair of flow/output maps is differentially observable. The observer combines a preliminary continuous-time high-gain observer and a local hybrid observer which relies on a high-gain observer of the flow and jumps triggered based on the observer state, in a way that "disconnects" the correction term around the jump times. Compared to designs in [7], [8], [24] where the observer jumps are synchronized with those of the system, this novel observer avoids the problems of delayed/noisy detection of the system jump times. Its robustness to noise was also tested in simulations. Of course, this design heavily relies on high-gain flow-based observers, with the full state being differentially observable during flow. The possibility of designing non-synchronized hybrid observers for larger classes of systems, exploiting both flow and jumps, remains to be investigated. Also, because the switching to the local hybrid observer happens only once, it is not robust to large pointwise disturbances. Actually, if such a disturbance is detected, the observer should be reinitialized from the start. More robust back and forth switching strategies could be studied, with the repeated challenge of detecting when to switch and of handling the unknown system jump times.

Appendix

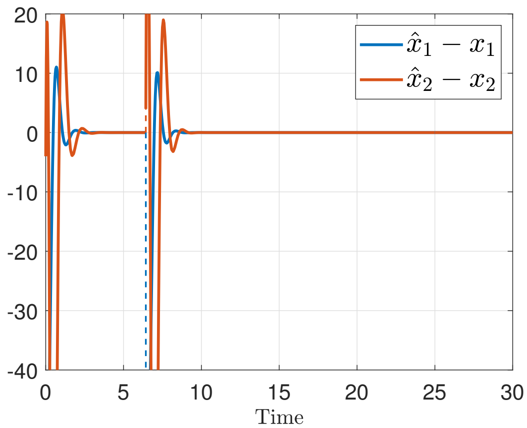
A. Proof of Lemma 4.1

Consider first the case where $\phi(0,0) \in \mathcal{C}_2$, i.e., $q(0,0) = 2$.

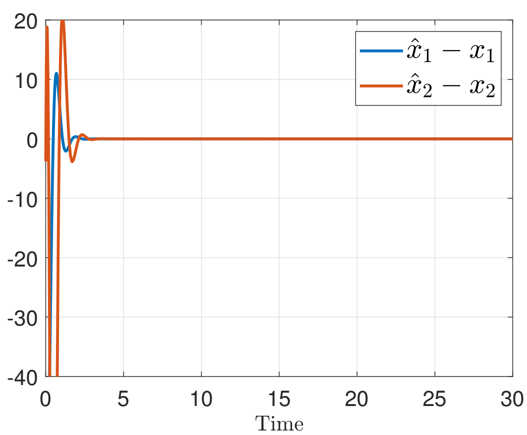
- By definition of $\hat{\mathcal{H}}$, $\phi(t,0) \in \mathcal{C}_2$ for all $t \in I^0$. During that time, since $\Pi(T_{\rm inv}(z(t,0))) \in {\rm cl}(C \cup D) \cap {\rm cl}(\mathbb{R}^{n_x} \setminus D_{\delta_1})$ by definition of Π and $\mathcal{C}_2$, $|\Pi(T_{\rm inv}(z(t,0)))|_D \geq \delta_1$. Therefore, since $V(x(t,0), z(t,0)) \leq v_\ell$ by assumption, $x(t,0) \notin D$ by item (a) and both x and ϕ flow simultaneously. Thus, z evolves according to the continuous-time high-gain observer $\mathcal{F}_\ell$ with input $y = h(x)$, which, due to x flowing in $C \setminus D$, is a continuous signal during this period. On the other hand, the components χ, τ, q of the observer solution ϕ remain constant with mode $q = 2$. This is the high-gain phase, and defining $\phi^{\rm sub}(t,0) = \phi(t,0)$ for $t \in I^0$, we have (19a) for $j = 0$.
- If $I^0 = [0, +\infty)$, then ϕ is t-complete and the proof is concluded. Otherwise, since y is bounded due to the system solution x being bounded, ϕ cannot explode in finite time and according to the previous item, the input y does not



(a) $x(0,0) = (-55,-6)$, $\hat{x}(0,0) = (24,0)$: the system jumps before τ_1 and the local observer is launched at time τ_m.

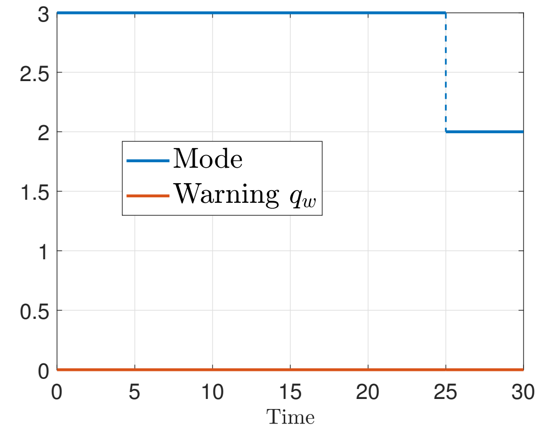


(b) $x(0,0) = (-55,-4)$, $\hat{x}(0,0) = (27,0)$: the system jumps between τ_1 and τ_2, the warning state is turned on around $t = 6.42$ when $\Pi(\hat{x}) \in D_{\delta_1}$ and the local observer is launched at time τ_m.

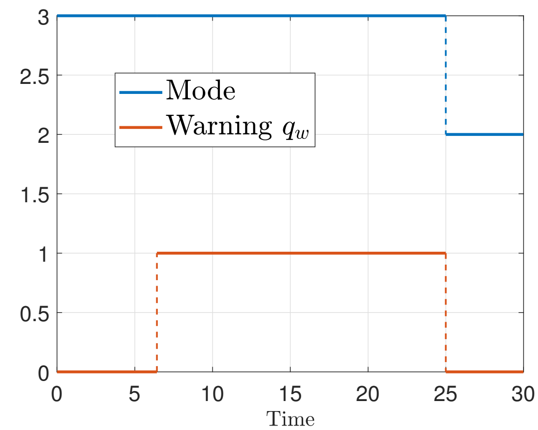


(c) $x(0,0) = (-55,-3.8)$, $\hat{x}(0,0) = (27,0)$: the system jumps after τ_2, and the local observer is launched right before around time $t = 9.61$ as soon as $\Pi(\hat{x}) \in D_{\delta_1}$.

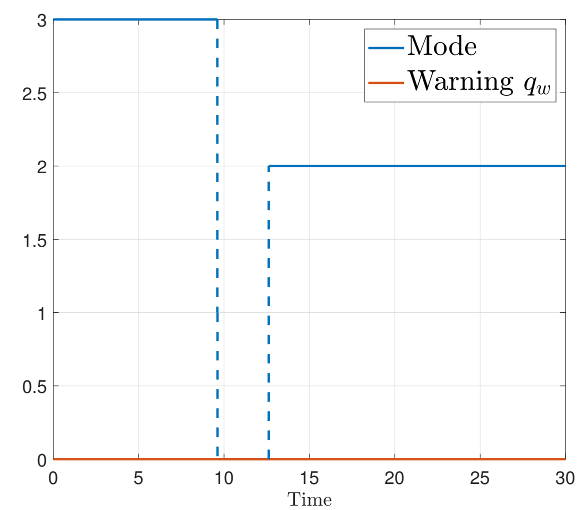
Fig. 5: Estimation errors for system (36) with observer (28) designed for the local observer $\hat{\mathcal{H}}$ given by (13) with $\ell = \ell_p = 6$ and $K = (1,1)^\top$ in $\mathcal{F}_\ell = \mathcal{F}_{\ell_p}$ defined in (7), $\delta_0 = 5$, $\delta_1 = 3$, $\Delta = 3$, $\tau_m = 25$, $\tau_1 = \tau_m/4$, $\tau_2 = \tau_m/3$, and different initial conditions with $z_p(0,0) = T(\hat{x}(0,0))$. The y-axis is chopped on $[-40, 20]$ for better overall view, but the error during the first transient typically goes down to -130.



(a) $x(0,0) = (-55,-6)$, $\hat{x}(0,0) = (24,0)$: the system jumps before τ_1 and the local observer is launched at time τ_m.



(b) $x(0,0) = (-55,-4)$, $\hat{x}(0,0) = (27,0)$: the system jumps between τ_1 and τ_2, the warning state is turned on around $t = 6.42$ when $\Pi(\hat{x}) \in D_{\delta_1}$ and the local observer is launched at time τ_m.



(c) $x(0,0) = (-55,-3.8)$, $\hat{x}(0,0) = (27,0)$: the system jumps after τ_2, and the local observer is launched right before around time $t = 9.61$ as soon as $\Pi(\hat{x}) \in D_{\delta_1}$.

Fig. 6: Logic variables observer (28) designed for the local observer $\hat{\mathcal{H}}$ given by (13) with $\ell = \ell_p = 6$, $K = (1,1)^\top$, $\delta_0 = 5$, $\delta_1 = 3$, $\Delta = 3$, $\tau_m = 25$, $\tau_1 = \tau_m/4$, $\tau_2 = \tau_m/3$, and different initial conditions with $z_p(0,0) = T(\hat{x}(0,0))$.

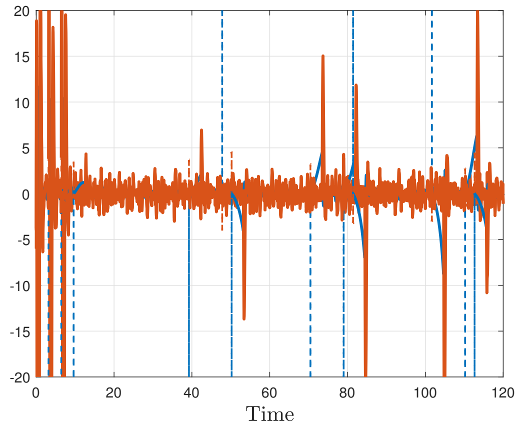


Fig. 7: Estimation errors for system (36) with observer (28) in the three scenarios of Figure 5 with band-limited white noise of power 0.1 and sample time 0.1.

jump on I^0, so necessarily, by definition of $\mathcal{C}_2$, there exists t_1 such that $\phi(t_1,0) \in \mathcal{D}_2$ and ϕ jumps. At this point, $|\,\Pi(T_{\rm inv}(z(t_1,0)))|_D = \delta_1$, so $\chi^+ = \Pi(T_{\rm inv}(z(t_1,0))) \in D_{\delta_1} \setminus D$ and $q^+ = 1$. Therefore, $(z^+,\chi^+,\tau^+,q^+) \in \mathcal{C}_1 \setminus \mathcal{D}_1$ and $\phi(\cdot,1)$ necessarily flows. Besides, from item (c), $\chi^+ \in \mathcal{X}_m''$. The open-loop phase starts. In this phase, the behavior of ϕ is independent from that of the input y, i.e., that of x. Therefore, we concentrate on the behavior of ϕ, putting aside the jumps of x possibly happening during that phase. Indeed, such jumps may only trigger trivial jumps in ϕ according to [6] and do not alter the values taken by ϕ. Thus, ϕ may be analyzed as in an autonomous hybrid system during this phase, which actually corresponds to $\phi^{\rm sub}$ in the statement of the lemma.

- While $\phi(\cdot,1)$ flows in $\mathcal{C}_1$, the states z,τ,q remain constant. Since $\chi(t_1,1) \in D_{\delta_0} \cap \mathcal{X}_m''$ and χ flows with f, we know by (P1) of Assumption 3.1 that χ remains in $D_{\delta_0} \cap \mathcal{X}_m'$ and reaches $D \cap \mathcal{X}_m'$ in finite time. Besides, ϕ cannot jump before χ has reached D according to the definition of $\mathcal{D}_1$.
- When χ reaches $D \cap \mathcal{X}_m'$, we know by (P2) in Assumption 3.1 that ϕ can no longer flow. Since at this point $\phi(t_2,1) \in \mathcal{D}_1$, ϕ jumps with $\chi^+ = g(\chi)$ and $q^+ = 0$.
- From there, $\phi(t_2,2) \in \mathcal{C}_0 \setminus \mathcal{D}_0$, with $\tau = 0$ and $\chi \in g(D \cap \mathcal{X}_m')$, so ϕ can only flow as long as $\tau \leq \Delta$, namely during Δ units of time. Since $\Delta < \tau_m^0$, χ can indeed flow with f during that time and remains in $\mathcal{X}_m$ with $\Pi(\chi) \notin D_{\delta_0}$ according to (P3).
- Thus, when τ reaches Δ, we have $\Pi(\chi) \notin D_{\delta_0}$, i.e., $\phi(t_3,2) \in \mathcal{D}_0$ and since no flow is possible in $\mathcal{C}_0$ when $\tau = \Delta$, ϕ jumps with $z^+ = T(\Pi(\chi))$, $\tau^+ = 0$ and $q^+ = 2$. Since $\chi \in \mathcal{X}_m$ and $\Pi(\mathcal{X}_m) \subset \mathcal{X}_m$, $\Pi(\chi) \in \mathcal{X}_m$, and from (8), $\Pi(T_{\rm inv}(z^+)) = \Pi(\Pi(\chi)) = \Pi(\chi) \notin D_{\delta_0}$ so that $\phi(t_3,3) \in \mathcal{C}_2$. This is the end of the open-loop phase, the high-gain phase starts again, and we are back to where the argument started.

On the other hand, by assumption, if $\phi(0,0) \notin \mathcal{C}_2$, $q(0,0) = 1$ and $\chi(0,0) \in D_{\delta_1} \cap \mathcal{X}_m''$ so that the same reasoning holds, starting from the open-loop phase in the third item.

B. Proof of Lemma 4.2

For convenience, we denote $v_1 := V_\ell(x(t_j,j'-1), z(t_j,j-1))$, the value of V_ℓ before the transition to open-loop phase.

By definition of the observer data, $\phi(t_j,j-1) \in \mathcal{D}_2$ with $|\,\Pi(T_{\rm inv}(z(t_j,j-1)))|_D = \delta_1$ and from Assumption 2.1, $x(t_j,j') \in \mathcal{X}$. Since $v_1 \leq v_\ell$, it follows from items (a), (b) and (c) that $x(t_j,j') \in (D_{\delta_0} \setminus D) \cap \mathcal{X}$ and $\chi(t_j,j) = \Pi(T_{\rm inv}(z(t_j,j-1))) \in D_{\delta_1} \cap \mathcal{X}_m''$. In particular, t_j is not a jump time for x. Let us denote $x_1 := x(t_j,j')$ and $\chi_1 := \chi(t_j,j)$ the values of x and χ at the beginning of the open-loop phase. From (P1), flow according to f will lead both x and χ to D with a time mismatch Δ_τ verifying

$$\begin{aligned}\Delta_\tau = |\mathfrak{T}(\chi_1) - \mathfrak{T}(x_1)| &\leq a_\tau |\chi_1 - x_1| \\ &\leq a_\tau |\,\Pi(T_{\rm inv}(z)) - T_{\rm inv}(T(x_1))| \\ &\leq a_\tau a_p |T_{\rm inv}(z) - T_{\rm inv}(T(x_1))| \\ &\leq a_\tau a_p a_{\rm inv} |z - T(x_1)| \\ &\leq a_\tau a_p a_{\rm inv} \sqrt{\frac{v_1}{\underline{c}(\ell)}} < \Delta\end{aligned}$$

with z denoting $z(t_j,j-1)$ for brevity, $a_{\rm inv}$ the Lipschitz constant of $T_{\rm inv}$, a_p defined in (10), and a_τ the Lipschitz constant of $\mathfrak{T}$ on $\mathcal{X}_m'' \cap D_{\delta_0}$. This mode with $q=1$ lasts until χ reaches D, namely $\mathfrak{T}(\chi_1)$ units of time. For all $x_D \in D \cap \mathcal{X}_m''$,

$$|\mathfrak{T}(\chi_1) - \mathfrak{T}(x_D)| = |\mathfrak{T}(\chi_1)| \leq a_\tau |\chi_1 - x_D| \leq a_\tau \delta_1$$

so that the interval of time where $q = 1$ lasts at most $a_\tau \delta_1$. The following mode with $q = 0$ lasts Δ units of time. But since the time mismatch between observer and system jump $\Delta_\tau < \Delta$, the system state x jumps some time before the observer switches back to $q = 2$.

We are now ready to study the evolution of W, namely the estimation error $x - \chi$ during the open-loop phase where $q \in \{0,1\}$. For that, we decompose the analysis in three consecutive periods:

1) both x and χ flow with f until one of them reaches D;
2) one jumps and then the other, with a jump time mismatch Δ_τ;
3) both x and χ flow with f until the observer switches back to $q = 2$.

As long as neither x nor χ reaches D, both flow on the time interval $[t_j, t_{j+1}]$ with f locally Lipschitz in $\mathcal{X}_m'$ with $q = 1$, so that

$$\begin{aligned}\dot{W} = 2(\chi - x)^\top (f(\chi) - f(x)) &\leq 2|\chi - x|\,|f(\chi) - f(x)| \\ &\leq \lambda_f W\end{aligned}$$

where $\lambda_f/2$ is the Lipschitz constant of f on the compact set $\mathcal{X}_m$. This period lasts at most $a_\tau \delta_1$, so $\chi - x$ grows by at most $a_1 := e^{\frac{1}{2}\lambda_f a_\tau \delta_1}$ and denoting $(\chi_1', x_1') := (\chi, x) := (\chi(t_{j+1},j), x(t_{j+1},j'))$ the value of (χ,x) at the end of this first period,

$$W(x_1', z, \chi_1', 1) \leq e^{\lambda_f a_\tau \delta_1} W(x_1, z, \chi_1, 1) \leq e^{\lambda_f a_\tau \delta_1} \frac{a_p^2 a_{\rm inv}^2}{\underline{c}(\ell)} v_1$$

Then, the mismatch of jump times Δ_τ makes W increase to

$$|g(\Psi_f(x_1', \Delta_\tau)) - \Psi_f\left(g(\chi_1'), \Delta_\tau\right)|^2 \quad \text{or} \quad |\Psi_f(g(x_1'), \Delta_\tau) - g(\Psi_f(\chi_1', \Delta_\tau))|^2$$

depending which of x or χ jumps first. Since $x_1' \in \mathcal{X}$ and $\Psi_f(x_1', \Delta_\tau) \in \mathcal{X}$ by Assumption 2.1, and $\chi_1' \in \mathcal{X}_m'$ according to (P1), we have

$$\begin{aligned}
&|g(\Psi_f(x_1', \Delta_\tau)) - \Psi_f\left(g(\chi_1'), \Delta_\tau\right)| \\
&\quad \leq |g(x_1') - g(\chi_1')| + |g(\Psi_f(x_1', \Delta_\tau)) - g(x_1')| \\
&\qquad\qquad + |\Psi_f(g(\chi_1'), \Delta_\tau) - g(\chi_1')| \\
&\quad \leq a_g|x_1' - \chi_1'| + a_g|\Psi_f(x_1', \Delta_\tau) - x_1'| \\
&\qquad\qquad + \left|\int_0^{\Delta t} f(\chi(s))ds\right| \\
&\quad \leq a_g|x_1' - \chi_1'| + a_g\left|\int_0^{\Delta t} f(x(s))ds\right| + c_f\Delta_\tau \\
&\quad \leq a_g|x_1' - \chi_1'| + a_g c_f'\Delta_\tau + c_f\Delta_\tau \\
&\quad \leq a_g|x_1' - \chi_1'| + (a_g c_f' + c_f)a_\tau|x_1 - \chi_1|
\end{aligned}$$

where a_g is the Lipschitz constant of g on $\mathcal{X}_m'$, $s \mapsto x(s)$ denotes the solution flowing with f from x_1', $\chi(s)$ the solution flowing with f from $g(\chi_1')$, and c_f, c_f' are bounds of f around $g(D \cap \mathcal{X}_m')$ and $D \cap \mathcal{X}_m'$ respectively. We obtain in the same way the same bound for $|\Psi_f(g(x_1'), \Delta_\tau) - g(\Psi_f(\chi_1', \Delta_\tau))|$. Actually, during this mismatch period, assuming χ jumps first, i.e. $\chi_1' \in D$, we have similarly

$$\begin{aligned}
|x|_D &\leq |x - \chi_1'| \leq |x_1' - \chi_1'| + c_f'\Delta_\tau \\
|\chi|_{g(D)} &\leq |\chi - g(\chi_1')| \leq c_f\Delta_\tau \\
|\chi - g(x)| &\leq a_g|x_1' - \chi_1'| + a_g c_f'\Delta_\tau + c_f\Delta_\tau
\end{aligned}$$

which gives $U(x, \chi, q) \leq a_{01}|x_1 - \chi_1|$ using $|x_1' - \chi_1'| \leq a_1|x_1 - \chi_1|$ and $\Delta_\tau \leq a_\tau|x_1 - \chi_1|$.

Finally, in the last period, both x and χ have jumped once, so necessarily $q = 0$, $\tau \in [0, \Delta]$, and both system and observer flow according to f during at most Δ units of time, until the timer τ reaches Δ. When that happens ϕ reaches $\mathcal{D}_0$, at jump time t_{j+3} and the solutions have thus reached $x(t_{j+3}, j' + 1)$ and $\phi(t_{j+3}, j + 2)$ respectively. During this period, $|\chi - x|$ thus grows by at most $e^{\frac{1}{2}\lambda_f\Delta}$ which gives a_0. More precisely, denoting $(\chi_2, x_2) := (\chi(t_{j+3}, j + 2), x(t_{j+3}, j' + 1))$ at that time, we thus have

$$\begin{aligned}
&|x_2 - \chi_2|^2 \\
&\quad \leq e^{\lambda_f\Delta}\left(a_g|x_1' - \chi_1'| + (a_g c_f' + c_f)a_\tau|x_1 - \chi_1|\right)^2 \\
&\quad \leq \frac{1}{2}e^{\lambda_f\Delta}\left(a_g^2|x_1' - \chi_1'|^2 + (a_g c_f' + c_f)^2 a_\tau^2|x_1 - \chi_1|^2\right) \\
&\quad \leq \frac{1}{2}e^{\lambda_f\Delta}\left(a_g^2 e^{\lambda_f a_\tau\delta_1} + (a_g c_f' + c_f)^2 a_\tau^2\right)\frac{a_p^2 a_{\text{inv}}^2}{\underline{c}(\ell)}v_1
\end{aligned}$$

Then, the observer jumps back to mode $q = 2$ and z is reset to $z_2 := z(t_{j+3}, j + 3) = T(\Pi(\chi_2))$ with $\chi_2 \in \mathcal{X}_m$ according to (P3), and therefore

$$\begin{aligned}
&W(x_2, z_2, \chi_2, 2) = V_\ell(x_2, z_2) \leq \overline{c}(\ell)|z_2 - T(x_2)|^2 \\
&\quad \leq \overline{c}(\ell)|T(\Pi(\chi_2)) - T(x_2)|^2 \\
&\quad \leq \overline{c}(\ell)a_T^2|\,\Pi(\chi_2) - x_2|^2 \leq \overline{c}(\ell)a_T^2 a_p^2|\chi_2 - x_2|^2 \\
&\quad \leq \frac{1}{2}\frac{\overline{c}(\ell)}{\underline{c}(\ell)}a_T^2 a_p^4 e^{\lambda_f\Delta}\left(a_g^2 e^{\lambda_f a_\tau\delta_1} + (a_g c_f' + c_f)^2 a_\tau^2\right)a_{\text{inv}}^2 v_1 \\
&\quad \leq a\frac{\overline{c}(\ell)}{\underline{c}(\ell)}v_1
\end{aligned}$$

for $a > 0$ independent from ℓ, with a_T the Lipschitz constant of T on $\mathcal{X}_m$.

After that, ϕ flows with $q = 2$ and V_ℓ decreases exponentially. Therefore, if additionally $v_1 < \frac{1}{a}\frac{\underline{c}(\ell)}{\overline{c}(\ell)}v_\ell$, namely (22) holds, V_ℓ remains smaller than v_ℓ and x does not jump while ϕ flows in $\mathcal{C}_2$ by item (a). It follows that ϕ flows either for all times, or until reaching t_{j+4} where another transition to open-loop phase occurs. Let us evaluate the minimal length of this high-gain phase. z flows with the high-gain observer at least till $|\,\Pi(T_{\text{inv}}(z))|_D = \delta_1$, so by item (a), x flows according to f during this period and when $|\,\Pi(T_{\text{inv}}(z))|_D = \delta_1$, by item (b), $x \in D_{\delta_0}$. So x must have had time to flow with f from $g(D \cap \mathcal{X})$ to D_{δ_0} while staying in $\mathcal{X}$ where $\Pi = \text{Id}$, namely at least τ_m^0 units of time have elapsed since the previous jump of x according to (P3). The time between the previous jump of x and the beginning of the high-gain phase with $q = 2$ being at most $\Delta_\tau + \Delta$, it follows that the high-gain phase lasts at least $\tau_m' > 0$ verifying

$$\tau_m' \geq \tau_m^0 - \Delta_\tau - \Delta \geq \tau_m^0 - 2\Delta > 0$$

since v_1 was chosen to ensure $\Delta_\tau < \Delta$. Therefore, after a full cycle, V_ℓ grows at most by $ae^{-\ell\lambda\tau_m'}\frac{\overline{c}(\ell)}{\underline{c}(\ell)}$.

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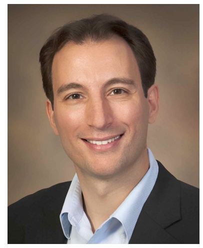
Pauline Bernard graduated in Applied Mathematics from Mines Paris in 2014 (formerly MINES ParisTech). She joined the Centre Automatique et Sytèmes of Mines Paris and obtained her Ph.D. in Mathematics and Control from Université PSL in 2017. For her work on observer design for nonlinear systems, she obtained the European Ph.D. award on Control for Complex and Heterogeneous Systems 2018. As a post-doctoral scholar, she visited the Hybrid Systems Lab at the University California Santa Cruz, USA, and the Center for Research on Complex Automated Systems at the University of Bologna, Italy. Since 2019, she has been an associate professor at the Centre Automatique et Sytèmes of Mines Paris, Université PSL, France. Her research interests cover the observation and output regulation of nonlinear and hybrid systems.



Ricardo Sanfelice received the B.S. degree in Electronics Engineering from the Universidad de Mar del Plata, Buenos Aires, Argentina, in 2001, and the M.S. and Ph.D. degrees in Electrical and Computer Engineering from the University of California, Santa Barbara, CA, USA, in 2004 and 2007, respectively. In 2007 and 2008, he held postdoctoral positions at the Laboratory for Information and Decision Systems at the Massachusetts Institute of Technology and at the Centre Automatique et Systèmes at the École de Mines de Paris. In 2009, he joined the faculty of the Department of Aerospace and Mechanical Engineering at the University of Arizona, Tucson, AZ, USA, where he was an Assistant Professor. In 2014, he joined the University of California, Santa Cruz, CA, USA, where he is currently Professor in the Department of Electrical and Computer Engineering. Prof. Sanfelice is the recipient of the 2013 SIAM Control and Systems Theory Prize, the National Science Foundation CAREER award, the Air Force Young Investigator Research Award, the 2010 IEEE Control Systems Magazine Outstanding Paper Award, and the 2020 Test-of-Time Award from the Hybrid Systems: Computation and Control Conference. He is Associate Editor for Automatica and a Fellow of the IEEE. His research interests are in modeling, stability, robust control, observer design, and simulation of nonlinear and hybrid systems with applications to power systems, aerospace, and biology.