Transport Speed Estimation for a 1-D Hyperbolic PDE Based on Output Flow Measurement

Eduardo B.R.F. Paiva, Olivier Lepreux, Delphine Bresch-Pietri

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Eduardo B. R. F. Paiva[1,2], Olivier Lepreux[1], and Delphine Bresch-Pietri[2]

***Abstract*— We present a method for estimating the transport speed of a class of systems modeled by a 1-D hyperbolic Partial Differential Equation (PDE) where the quantity governed by the PDE is a speed deficit and the measurement fed to the estimator is the transport speed minus this deficit at a given location. Such problem arises when estimating the free flow wind speed using measurements taken inside the wake zone of a wind turbine. We present a simple and easy to implement method and provide sufficient conditions for its convergence.**

***Index Terms*— Hyperbolic systems, parameter estimation, time-delay systems, distributed parameter systems, wind speed estimation**

I. INTRODUCTION

In this paper, we present a method for estimating the transport speed for a 1-D first-order hyperbolic Partial Differential Equation (PDE) based on a special type of point-wise output measurement. This type of system appears in modeling the wake of a wind turbine [1], [2], for which the speed deficit $u(x,t)$ caused by the turbine is governed by such a PDE. The transport speed of this hyperbolic PDE is the free-flow wind speed that one is interested in estimating for, e.g., control and monitoring purposes. Unfortunately, one who only has access to a measurement device (such as one mounted on a second turbine) located downwind inside the wake zone cannot measure the free-flow wind speed, but only the speed information inside the wake. This measured output flow is the difference between the free-flow wind speed and the speed deficit governed by the PDE.

In presence of uncertain parameters, adaptive control [3] has proven to be an efficient tool for controller and observer design, both for parabolic [4] and hyperbolic [5] PDEs. Yet, while a large number of works, e.g. [6], [7], [8], consider the case of uncertain source terms or uncertain boundary coefficients, the transport speed itself is usually assumed to be known. To the best of our knowledge, one of the few studies on this challenging problem is [9], but it requires boundary sensing at both extremities which is difficult to obtain in practice.

The problem under consideration in this paper differs from those usually considered in the literature of estimation for systems described by hyperbolic PDEs since we do not directly measure the output boundary value of the transport PDE. Besides, another challenge of the problem at stake is that the transport speed is time-varying. This notably complicates the estimation task, as adaptive designs deal with constant parameters.

We propose a simple approach based on an analytical solution of the transport PDE expressed in terms of a transport delay [10], [11]. Feeding this expression with an estimate of the transport speed, we build a virtual measurement. The estimate of the transport speed is then updated by integrating the difference between the actual measurement and this virtual one in a PI design manner [12]. We provide sufficient conditions for the estimate of the transport speed to converge to an arbitrarily small neighborhood of the true transport speed using this method. This is the main contribution of this paper.

The rest of this paper is organized as follows. In Section II, we present a mathematical statement of the problem under consideration. In Section III, we present the proposed transport speed estimation method as well as the main result of this paper that gives sufficient conditions to ensure its convergence. In Section IV, we present numerical results of the application of the proposed method to the aforementioned problem of estimation of the free-flow wind speed for an array of wind turbines. In Section V, we present some concluding remarks.

II. PROBLEM STATEMENT

Consider the PDE

$$u_t + \lambda(t)u_x = -w(x)\lambda(t)u(x,t) + S(x,\lambda(t)), \qquad (1)$$

with the boundary condition (BC) and initial condition (IC)

$$u(0,t) = 0, \quad u(x,-T) = b(x), \qquad (2)$$

where, $u \in \mathcal{L}_2([0,L],\mathbb{R})$ for a given $L > 0$, $\lambda \in C^1(\mathbb{R},[\lambda_m,\infty))$ with $\lambda_m > 0$, and w and S are continuous functions. We take the convention that $-T < 0$ is sufficiently far in the past so that the IC does not affect the solution u for positive time[1].

Our goal is to estimate the transport speed λ measuring only the following quantity, which we refer to as output flow,

$$y(t) = \lambda(t) - u(L,t). \qquad (3)$$

Furthermore, we make the following assumptions:

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[1]More precisely, T is larger than the maximum possible transport delay. Such a delay is defined in Section III and more details on the solution are provided in Appendix I.

Assumption 1: There is a constant $\zeta \geq 0$ such that, for all $t \in \mathbb{R}$, $|\dot{\lambda}(t)| \leq \zeta$.

Assumption 2: The function $F : (x, \lambda) \in [0, L] \times \mathbb{R}_+ \mapsto S(x, \lambda)/\lambda$ is globally Lipschitz with respect to its second argument, i.e., for every $x \in [0, L]$, there exists $M(x) \geq 0$ such that, for all $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$,

$$|F(x, \lambda_1) - F(x, \lambda_2)| \leq M(x)|\lambda_1 - \lambda_2|. \tag{4}$$

Assumption 3: The constants

$$\alpha = \frac{1}{W(L)} \int_0^L W(\xi)M(\xi)d\xi, \tag{5}$$

$$\beta = \frac{1}{W(L)} \int_0^L W(\xi)(L - \xi)M(\xi)d\xi, \tag{6}$$

where F and M are as defined in Assumption 2 and

$$W(x) = \exp\left(\int_0^x w(r)dr\right), \tag{7}$$

are such that

$$\alpha + \frac{\beta\zeta}{\lambda_m^2} < 1, \tag{8}$$

with ζ defined in Assumption 1 and λ_m the lower bound of λ.

III. TRANSPORT SPEED ESTIMATION

Here, we present the proposed transport speed estimation approach. We start by presenting the solution to the PDE and the update law we propose, then proceed to the proof of convergence of the method and to its implementation.

A. Proposed Estimation Method

Our method grounds on the explicit solution to (1) with conditions (2), which is

$$u(x, t) = \frac{1}{W(x)} \int_0^x \frac{W(\xi)S(\xi, \lambda(t - \tau(x - \xi, t)))}{\lambda(t - \tau(x - \xi, t))}d\xi, \tag{9}$$

for $(x, t) \in [0, L] \times \mathbb{R}_+$, where W is defined in (7) and τ is a transport delay defined by the relation

$$\int_{t-\tau(x,t)}^t \lambda(r)dr = x. \tag{10}$$

Further details are provided in Appendix I.

Let $\hat{\lambda} \in C^1([-T, \infty), \mathbb{R})$ denote the transport speed estimate. Using this estimate and the certainty equivalence principle, we define, for $t \geq 0$,

$$\hat{u}(t) = \frac{1}{W(L)} \int_0^L \frac{W(\xi)S(\xi, \hat{\lambda}(t - \hat{\tau}(L - \xi, t)))}{\hat{\lambda}(t - \hat{\tau}(L - \xi, t))}d\xi, \tag{11}$$

$$\hat{y}(t) = \hat{\lambda}(t) - \hat{u}(t), \tag{12}$$

with $\hat{\tau}(x, t)$ given by

$$\int_{t-\hat{\tau}(x,t)}^t \hat{\lambda}(r)dr = x. \tag{13}$$

The transport speed estimate is chosen as $\hat{\lambda}(t) = \hat{\lambda}(0) \in [\lambda_m, \infty)$ for $t \in [-T, 0]$ and satisfies the following update law for $t \geq 0$

$$\dot{\hat{\lambda}}(t) = k(y(t) - \hat{y}(t)), \tag{14}$$

where $k > 0$ is a constant gain to be chosen.

B. Sufficient Conditions for Convergence

Here, we present sufficient conditions to guarantee the convergence of the proposed method. With this aim in view, we define the error variables

$$\tilde{\lambda}(t) = \lambda(t) - \hat{\lambda}(t), \tag{15}$$

$$\tilde{u}(t) = u(L, t) - \hat{u}(t). \tag{16}$$

Theorem 1: Consider the system (1)-(2) satisfying Assumptions 1, 2, and 3 and the estimated transport speed $\hat{\lambda}$ defined through (11)-(14). There exist $\hat{\lambda}_m \in (0, \lambda_m)$ and $k_* > 0$ such that, if $k \geq k_*$ and $\sup_{s\in[-\bar{\tau},0]} |\tilde{\lambda}(s)| \leq \lambda_m - \hat{\lambda}_m$, then $\hat{\lambda}(t) \geq \hat{\lambda}_m$ for $t \geq 0$ and the dynamics of the estimation error $\tilde{\lambda}$ is practically exponentially stable in the sense that there exists a constant $\sigma > 0$ such that

$$|\tilde{\lambda}(t)| \leq \sup_{s\in[-\bar{\tau},0]} |\tilde{\lambda}(s)|e^{-\sigma t} + \frac{\zeta}{k\left(1 - \alpha - \frac{\beta\zeta}{\hat{\lambda}_m^2}\right)}, \tag{17}$$

where $\bar{\tau} = L/\hat{\lambda}_m$.

Proof: Consider the Lyapunov function candidate $V(t) = \tilde{\lambda}^2(t)/2$. We have

$$\dot{V}(t) = \tilde{\lambda}(t)\dot{\tilde{\lambda}}(t) = \tilde{\lambda}(t)\left(\dot{\lambda}(t) - \dot{\hat{\lambda}}(t)\right) \tag{18}$$

$$= \tilde{\lambda}(t)\left(\dot{\lambda}(t) - k\tilde{\lambda}(t) + k\tilde{u}(t)\right) \tag{19}$$

$$\leq |\tilde{\lambda}(t)|\zeta - k\tilde{\lambda}^2(t) + k|\tilde{\lambda}(t)||\tilde{u}(t)|, \tag{20}$$

where we used Assumption 1 ($|\dot{\lambda}(t)| \leq \zeta$).

Notice that, from (9) and (11)

$$\tilde{u}(t) = \frac{1}{W(L)} \int_0^L \frac{W(\xi)S(\xi, \lambda(t - \tau(L - \xi, t)))}{\lambda(t - \tau(L - \xi, t))}d\xi - \frac{1}{W(L)} \int_0^L \frac{W(\xi)S(\xi, \hat{\lambda}(t - \hat{\tau}(L - \xi, t)))}{\hat{\lambda}(t - \hat{\tau}(L - \xi, t))}d\xi \tag{21}$$

$$= \frac{1}{W(L)} \int_0^L W(\xi)\left[F(\xi, \lambda(t - \tau(L - \xi, t))) - F(\xi, \hat{\lambda}(t - \hat{\tau}(L - \xi, t)))\right]d\xi, \tag{22}$$

where $F(x, \lambda(t)) = S(x, \lambda(t))/\lambda(t)$. Now, notice that we can add and subtract $W(\xi)F(\xi, \lambda(t-\hat{\tau}(L-\xi, t)))$ under the integral sign to yield

$$\tilde{u}(t) = \left\{\frac{1}{W(L)} \int_0^L W(\xi)\left[F(\xi, \lambda(t - \hat{\tau}(L - \xi, t))) - F(\xi, \hat{\lambda}(t - \hat{\tau}(L - \xi, t)))\right]d\xi\right\} + \left\{\frac{1}{W(L)} \int_0^L W(\xi)\left[F(\xi, \lambda(t - \tau(L - \xi, t))) - F(\xi, \lambda(t - \hat{\tau}(L - \xi, t)))\right]d\xi\right\} \tag{23}$$

$$\triangleq I_1 + I_2, \tag{24}$$

with I_1 the first pair of curly brackets on the RHS of (23) and I_2 the second one.

Since, from Assumption 3, $\alpha + \beta\zeta/\lambda_m^2 < 1$, there exists $\hat{\lambda}_m \in (0, \lambda_m)$ such that $\alpha + \beta\zeta/\hat{\lambda}_m^2 < 1$. Assume for now that $\hat{\lambda}(t) \geq \hat{\lambda}_m$ for $t \geq 0$. Notice this, together with $\lambda(t) \geq \lambda_m > \hat{\lambda}_m$, implies that both τ and $\hat{\tau}$ are upper bounded by $\bar{\tau} = L/\hat{\lambda}_m$. With Assumption 2, we can upper bound I_1 as:

$$I_1 \leq \frac{1}{W(L)} \int_0^L W(\xi) M(\xi) \left| \tilde{\lambda}(t - \hat{\tau}(L-\xi, t)) \right| d\xi \quad (25)$$

$$\leq \alpha \sup_{s \in [-\bar{\tau}, 0]} |\tilde{\lambda}(t+s)|, \quad (26)$$

where α is as in (5).

For I_2, we can first use the Lipschitz property of F to write

$$|F(\xi, \lambda(t - \tau(L-\xi,t))) - F(\xi, \lambda(t - \hat{\tau}(L-\xi,t)))| \leq M(\xi) \left| \lambda(t - \tau(L-\xi,t)) - \lambda(t - \hat{\tau}(L-\xi,t)) \right|. \quad (27)$$

According to the Mean Value Theorem, there exists $\tau_*(L-\xi, t)$ such that

$$\begin{aligned} &|\lambda(t - \tau(L-\xi,t)) - \lambda(t - \hat{\tau}(L-\xi,t))| \\ &= \left| (\tau(L-\xi,t) - \hat{\tau}(L-\xi,t)) \times \dot{\lambda}(t - \tau_*(L-\xi,t)) \right| \\ &\leq |(\tau(L-\xi,t) - \hat{\tau}(L-\xi,t))| \, \zeta. \end{aligned} \quad (28)$$

Furthermore, we can bound $|\tau - \hat{\tau}|$ noticing that, by definition of τ and $\hat{\tau}$ in (10) and (13), respectively,

$$\int_{t-\hat{\tau}(x,t)}^t \hat{\lambda}(r) dr = x = \int_{t-\tau(x,t)}^t \lambda(r) dr. \quad (29)$$

Then

$$-\int_{t-\hat{\tau}(x,t)}^t \tilde{\lambda}(r) dr = \int_{t-\tau(x,t)}^t \lambda(r) dr - \int_{t-\hat{\tau}(x,t)}^t \lambda(r) dr, \quad (30)$$

which implies

$$-\int_{t-\hat{\tau}(x,t)}^t \tilde{\lambda}(r) dr = \int_{t-\tau(x,t)}^{t-\hat{\tau}(x,t)} \lambda(r) dr, \quad (31)$$

and thus

$$\left| \int_{t-\hat{\tau}(x,t)}^t \tilde{\lambda}(r) dr \right| = \left| \int_{t-\tau(x,t)}^{t-\hat{\tau}(x,t)} \lambda(r) dr \right|. \quad (32)$$

It is clear that

$$\left| \int_{t-\hat{\tau}(x,t)}^t \tilde{\lambda}(r) dr \right| \leq \hat{\tau}(x,t) \sup_{s \in [-\bar{\tau}, 0]} |\tilde{\lambda}(t+s)|, \quad (33)$$

$$\left| \int_{t-\tau(x,t)}^{t-\hat{\tau}(x,t)} \lambda(r) dr \right| \geq |\tau(x,t) - \hat{\tau}(x,t)| \hat{\lambda}_m. \quad (34)$$

Therefore, we have

$$|\tau(x,t) - \hat{\tau}(x,t)| \leq \frac{\hat{\tau}(x,t)}{\hat{\lambda}_m} \sup_{s \in [-\bar{\tau}, 0]} |\tilde{\lambda}(t+s)|. \quad (35)$$

Hence, combining (27), (28), and (35), we can bound I_2 as:

$$I_2 \leq \frac{1}{W(L)} \int_0^L W(\xi) |\tau(L-\xi,t) - \hat{\tau}(L-\xi,t)| M(\xi) \zeta d\xi \quad (36)$$

$$\leq \frac{\zeta}{W(L)\hat{\lambda}_m} \sup_{s \in [-\bar{\tau}, 0]} |\tilde{\lambda}(t+s)| \times \int_0^L W(\xi) \hat{\tau}(L-\xi,t) M(\xi) d\xi. \quad (37)$$

Finally, using the fact that $\hat{\lambda}(t) \geq \hat{\lambda}_m$ for all t implies $\hat{\tau}(L-\xi, t) \leq (L-\xi)/\hat{\lambda}_m$ for all t, we have

$$\begin{aligned} I_2 &\leq \frac{\zeta}{W(L)\hat{\lambda}_m^2} \sup_{s \in [-\bar{\tau}, 0]} |\tilde{\lambda}(t+s)| \int_0^L W(\xi)(L-\xi) M(\xi) d\xi \\ &= \frac{\beta\zeta}{\hat{\lambda}_m^2} \sup_{s \in [-\bar{\tau}, 0]} |\tilde{\lambda}(t+s)|. \end{aligned} \quad (38)$$

Then, gathering (20), (24), (26), and (38), we have

$$\dot{V}(t) \leq |\tilde{\lambda}(t)|\zeta - k\tilde{\lambda}^2(t) + k \left[\alpha + \frac{\beta\zeta}{\hat{\lambda}_m^2} \right] \sup_{s \in [-\bar{\tau}, 0]} \tilde{\lambda}^2(t+s)$$

$$\leq \zeta\sqrt{2V(t)} - 2kV(t) + 2k \left[\alpha + \frac{\beta\zeta}{\hat{\lambda}_m^2} \right] \sup_{s \in [-\bar{\tau}, 0]} V(t+s).$$

From Young's inequality, it holds that $\zeta\sqrt{2V(t)} \leq \epsilon\zeta^2/2 + V(t)/\epsilon$ for every $\epsilon > 0$. Defining $\rho = 2k\epsilon > 0$, we obtain

$$\begin{aligned} \dot{V}(t) \leq \frac{\rho\zeta^2}{4k} - 2k \left[1 - \frac{1}{\rho} \right] V(t) \\ + 2k \left[\alpha + \frac{\beta\zeta}{\hat{\lambda}_m^2} \right] \sup_{s \in [-\bar{\tau}, 0]} V(t+s). \end{aligned} \quad (39)$$

Let us choose a certain $\rho > 1$ such that

$$\alpha + \frac{\beta\zeta}{\hat{\lambda}_m^2} < 1 - \frac{1}{\rho}, \quad (40)$$

which exists due to Assumption 3. Then, from Theorem 2 in Appendix II, there exists a constant $\sigma_1 > 0$ such that

$$V(t) \leq \sup_{s \in [-\bar{\tau}, 0]} V(s) e^{-\sigma_1 t} + \frac{\rho\zeta^2}{8k^2 \left[1 - \frac{1}{\rho} - \alpha - \frac{\beta\zeta}{\hat{\lambda}_m^2} \right]}. \quad (41)$$

Recall that $|\tilde{\lambda}(t)| = \sqrt{2V}$, thus

$$|\tilde{\lambda}(t)| \leq \sqrt{2 \sup_{s \in [-\bar{\tau}, 0]} V(s) e^{-\sigma_1 t} + \frac{\rho\zeta^2}{4k^2 \left[1 - \frac{1}{\rho} - \alpha - \frac{\beta\zeta}{\hat{\lambda}_m^2} \right]}} \quad (42)$$

$$\leq \sup_{s \in [-\bar{\tau}, 0]} |\tilde{\lambda}(s)| e^{-\sigma t} + \frac{\zeta\sqrt{\rho}}{2k\sqrt{1 - \frac{1}{\rho} - \alpha - \frac{\beta\zeta}{\hat{\lambda}_m^2}}}, \quad (43)$$

where $\sigma = \sigma_1/2 > 0$ and we used the fact that $\sqrt{a^2 + b^2} \leq |a| + |b|$ for any $a, b \in \mathbb{R}$. Notice that, if we choose $\rho = 2/(1 - \alpha - \beta\zeta/\hat{\lambda}_m^2) > 1$, the RHS of (43) is minimized and becomes the RHS of (17). Also, notice that this value of ρ satisfies (40) in virtue of Assumption 3.

The only thing left is to justify our claim that $\hat{\lambda}$ does not go below $\hat{\lambda}_m$. Recall that we assume in the theorem statement that $\sup_{s\in[-\bar{\tau},0]} |\tilde{\lambda}(s)| < \lambda_m - \hat{\lambda}_m$ and consider

$$k \geq k_* \triangleq \frac{\zeta}{\left(\lambda_m - \hat{\lambda}_m - \sup_{s\in[-\bar{\tau},0]} |\tilde{\lambda}(s)|\right)\left(1 - \alpha - \frac{\beta\zeta}{\hat{\lambda}_m^2}\right)}. \tag{44}$$

Notice this implies that

$$\sup_{s\in[-\bar{\tau},0]} |\tilde{\lambda}(s)| \leq \Omega \triangleq \lambda_m - \hat{\lambda}_m - \frac{\zeta}{k\left(1 - \alpha - \frac{\beta\zeta}{\hat{\lambda}_m^2}\right)}. \tag{45}$$

Then

$$\hat{\lambda}(0) = \lambda(0) - \tilde{\lambda}(0) \geq \lambda_m - \sup_{s\in[-\bar{\tau},0]} |\tilde{\lambda}(s)| > \hat{\lambda}_m. \tag{46}$$

By way of contradiction, let us assume there exists $t_1 > 0$ such that $\hat{\lambda}(t_1) < \hat{\lambda}_m$. By continuity of $\hat{\lambda}$, this implies the existence of $t_2 \in (0, t_1)$ such that $\hat{\lambda}(t) > \hat{\lambda}_m$ for all $t \in [0, t_2)$ and $\hat{\lambda}(t_2) = \hat{\lambda}_m$. Then, from (17),

$$|\tilde{\lambda}(t_2)| \leq \sup_{s\in[-\bar{\tau},0]} |\tilde{\lambda}(s)| e^{-\sigma t_2} + \frac{\zeta}{k\left(1 - \alpha - \frac{\beta\zeta}{\hat{\lambda}_m^2}\right)}. \tag{47}$$

Hence, it follows that

$$\hat{\lambda}(t_2) = \lambda(t_2) - \tilde{\lambda}(t_2) \tag{48}$$

$$\geq \lambda_m - \sup_{s\in[-\bar{\tau},0]} |\tilde{\lambda}(s)| e^{-\sigma t_2} + \frac{\zeta}{k\left(1 - \alpha - \frac{\beta\zeta}{\hat{\lambda}_m^2}\right)} \tag{49}$$

$$\geq \Omega\left(1 - e^{-\sigma t_2}\right) + \hat{\lambda}_m \tag{50}$$

$$> \hat{\lambda}_m \tag{51}$$

which contradicts the definition of t_2. Thence, we cannot have $t_1 > 0$ such that $\hat{\lambda}(t_1) < \hat{\lambda}_m$. This concludes the proof. ■

C. Implementation

Here, we present the numerical implementation we used. We opted for a simple implementation with a first-order discretization and a constant time-step Δt.

As shown in Appendix I, alternatively to (9)-(10), we can express the solution to (1) with conditions (2) as

$$u(x,t) = \frac{1}{W(x)} \int_{t-\tau(x,t)}^{t} W(x - \Lambda(s,t)) S(x - \Lambda(s,t), \lambda(s)) ds, \tag{52}$$

for $(x,t) \in [0, L] \times \mathbb{R}_+$, where

$$\Lambda(s,t) = \int_s^t \lambda(r) dr. \tag{53}$$

We compute $\hat{u}$ using this time-integral formulation as it is easier to implement than (9).

Algorithm 1 presents the proposed implementation. We use $\hat{\lambda}_i$ to denote $\hat{\lambda}(i\Delta t)$ and similar notation for the other variables. If j is a negative index, we use $\hat{\lambda}_j = \hat{\lambda}(0)$. Notice we start the integration at $s = t$ and keep integrating backwards while $\hat{\Lambda} < L$. This is because, by the definition of the delay, $\Lambda(t - \tau(L,t), t) = L$. This is done to avoid explicitly calculating the delay in advance. Notice we use a saturation in the update law, which is for guaranteeing that the estimate remains lower bounded regardless of the errors caused by discretization.

Algorithm 1 Implementation of the proposed method

for $i = 0, 1, 2, \ldots$ **do**
  $j \leftarrow i$, $\hat{u}_i \leftarrow 0$, $\hat{\Lambda} \leftarrow 0$
  while $\hat{\Lambda} < L$ **do**
    $\hat{u}_i \leftarrow \hat{u}_i + \frac{1}{W(L)} W(L - \hat{\Lambda}) S(L - \hat{\Lambda}, \hat{\lambda}_j) \Delta t$ (cf. (52))
    $\hat{\Lambda} \leftarrow \hat{\Lambda} + \hat{\lambda}_j \Delta t$ (cf. (53))
    $j \leftarrow j - 1$
  end while
  $\hat{y}_i \leftarrow \hat{\lambda}_i - \hat{u}_i$
  $\hat{\lambda}_{i+1} \leftarrow \max\left\{\hat{\lambda}_i + k(y_i - \hat{y}_i)\Delta t, \hat{\lambda}_m\right\}$ (cf. (14))
end for

IV. APPLICATION AND NUMERICAL RESULTS

Here, we apply the proposed method to estimate the free-flow wind speed to which a wind turbine is subjected given a measurement of the wind speed inside its wake zone. We present some numerical results to illustrate the effectiveness of the method.

We consider the model proposed in [1], [2], where an equation of the form of (1) is used to model the wake of a wind turbine. In this model, $\lambda(t)$ is the free-flow wind speed, $u(x,t)$ is the speed deficit, $S(x, \lambda(t))$ represents the impact of the turbine on the flow, and $w(x)$ is related to the wake expansion. More precisely:

$$S(x, \lambda(t)) = \frac{2a_{turb}}{d_w^2(x)} G(x) \lambda^2(t), \tag{54}$$

$$w(x) = \frac{2}{d_w(x)} \frac{d}{dx}(d_w(x)), \tag{55}$$

where

$$G(x) = \frac{1}{\frac{D}{2}\sqrt{2\pi}} \exp\left(-\frac{(x - x_{turb})^2}{2(D/2)^2}\right), \tag{56}$$

$$d_w(x) = 1 + k_w \ln\left(1 + \exp\left(\frac{x - x_{turb} - D}{D/2}\right)\right), \tag{57}$$

the coordinate x is measured along the free-flow direction, $a_{turb} > 0$ is the induction factor of the turbine (that we consider constant), $D > 0$ is the rotor diameter, $x_{turb} \in (0, L)$ is the turbine's x position, and $k_w > 0$ is the growth rate of the wake diameter.

Notice the function F defined in Assumption 2 is linear with respect to λ in this example. Hence, Assumption 2 is trivially satisfied with $M(x) = 2a_{turb}G(x)/d_w^2(x)$. In our simulations, we fixed the parameters $a_{turb} = 0.25$, $D = 126$ m, $x_{turb} = 5D$, $k_w = 0.05$, and the position of the measurement $L = 7D$. This yields $\alpha \approx 0.4085$ and $\beta \approx 102.9412$ m. Fig. 1 represents the pairs of λ_m and ζ that satisfy (8) for these values of α and β.

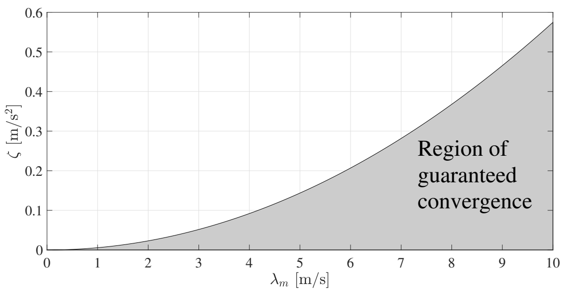

Fig. 1. Region where the inequality condition of Theorem 1 is satisfied for different pairs of $\hat{\lambda}_m$ (lower bound of the transport speed $\lambda(t)$) and ζ (upper bound of $|\dot{\lambda}(t)|$).

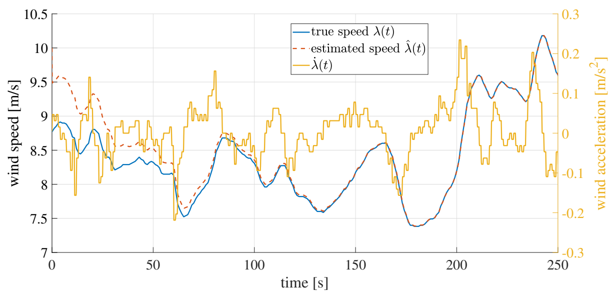

Fig. 2. Simulation using experimental data for the wind speed time-series. We use $k = 10$ and $\hat{\lambda}(0) = 10$ m/s in this case.

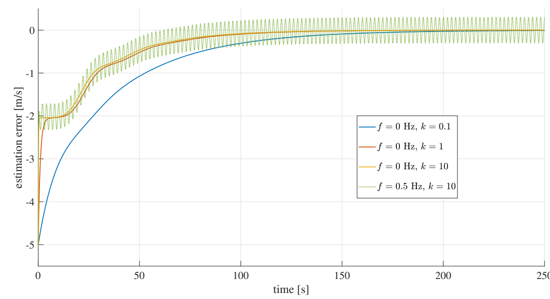

Fig. 3. Time evolution of the estimation error $\tilde{\lambda}(t)$ using $\lambda(t) = 10 + \sin(2\pi f t)$ m/s with different frequencies f and applying different gains k. We use $\hat{\lambda}(0) = 15$ m/s in each case.

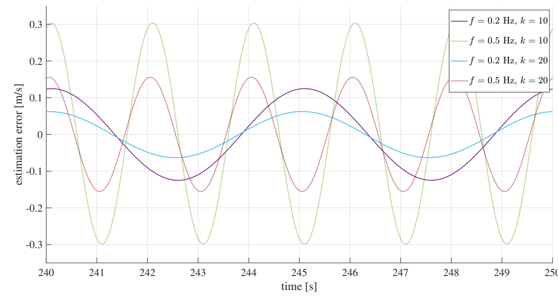

Fig. 4. Steady-state behavior of the estimation error when $\lambda(t) = 10 + \sin(2\pi f t)$ m/s with different frequencies f and applying different gains k.

Fig. 2 presents results for a simulation using experimental data for the wind speed time evolution. These wind speed values were obtained from LiDAR data provided by Leosphere within the framework of the ANR (French National Research Agency) project SmartEole. Notice that, in this case, we could set $\lambda_m = 7$ m/s and $\zeta = 0.25$ m/s^2, which gives a point inside the shaded region of Fig. 1.

To further assess the performance of the method, we simulated the system with inputs of the form $\lambda(t) = 10 + \sin(2\pi f t)$ for different frequencies f and using different gains k for the estimation. Notice we have $\zeta = 2\pi f$. Fig. 3 displays the time evolution of the estimation error $\tilde{\lambda}(t)$ in each case. Fig. 4 is similar, but its results are zoomed in the steady-state behavior. We notice that we have exponential convergence with constant λ and in the other cases the steady-state error remains confined in a region whose size depends on k and ζ. Notice that ζ is too large in some of these cases to satisfy the conditions required by Assumption 3, but the method works nevertheless. This illustrates that the presented method could still be useful beyond the limitations imposed in this paper. Future works should focus on this direction.

V. CONCLUSIONS

We presented an easy-to-implement method for estimating the time-varying transport using only measurements of the output flow for systems governed by a class of 1-D hyperbolic PDE. Both mathematical proofs with sufficient conditions for the convergence of the estimation procedure and an example of a possible practical application were provided.

Some possible paths for future works are using higher-order filters instead of the simple integrator proposed here for updating the estimate and proving convergence under less conservative conditions. One particularly interesting development would be dealing with the requirement that the estimate of the transport speed is lower bounded without requiring the initial error to be sufficiently small, such as formally showing that an update law with a saturation of the estimate still achieves convergence.

APPENDIX I
SOLUTION TO THE PDE

Here, we consider a more general version of (1) where $S(x, \lambda(t))$ is replaced with $S(x, t)$. We keep the same BC and IC displayed in (2).

First, we introduce the new variable

$$v(x, t) = W(x)u(x, t), \tag{58}$$

with W as in (7). It is easy to show that v satisfies the PDE

$$v_t + \lambda(t)v_x = h(x, t), \tag{59}$$

where $h(x, t) = W(x)S(x, t)$, with BC $v(0, t) = 0$ and IC $v(x, -T) = b_1(x) = W(x)b(x)$.

Next, we apply the Laplace transform to change from the variable x to a variable p, which yields

$$v_t^*(p, t) + p\lambda(t)v^*(p, t) = h^*(p, t), \tag{60}$$

with IC $v^*(p,-T) = b_1^*(p)$, where $*$ indicates the transformed functions. Then, we have

$$v^*(p,t) = e^{-\int_{-T}^{t} p\lambda(r)dr} \times \left[b_1^*(p) + \int_{-T}^{t} e^{\int_{-T}^{s} p\lambda(r)dr} h^*(p,s)ds \right] \quad (61)$$
$$= e^{-\int_{-T}^{t} p\lambda(r)dr} b_1^*(p) + \int_{-T}^{t} e^{-\int_{s}^{t} p\lambda(r)dr} h^*(p,s)ds. \quad (62)$$

Applying the inverse Laplace transform, we have

$$v(x,t) = b_1(x - \Lambda(-T,t))\theta(x - \Lambda(-T,t)) + \int_{-T}^{t} h(x - \Lambda(s,t), s)\theta(x - \Lambda(s,t))ds, \quad (63)$$

with $\theta(s) = 1$ if $s \geq 0$ and $\theta(s) = 0$ otherwise and Λ as in (53).

Recall that $\lambda(t) \geq \lambda_m > 0$ for all t. Hence, if $(x,t) \in [0,L] \times \mathbb{R}_+$ and $T > L/\lambda_m$, we have $x - \Lambda(-T,t) < 0$ and $t - \tau(x,t) > -T$, with τ as defined in (10). Thus, for $(x,t) \in [0,L] \times \mathbb{R}_+$,

$$v(x,t) = \int_{t-\tau(x,t)}^{t} h(x - \Lambda(s,t), s)ds. \quad (64)$$

Therefore, the final solution is

$$u(x,t) = \frac{v(x,t)}{W(x)} = \frac{1}{W(x)} \int_{t-\tau(x,t)}^{t} h(x - \Lambda(s,t), s)ds$$
$$= \frac{1}{W(x)} \int_{t-\tau(x,t)}^{t} W(x - \Lambda(s,t))S(x - \Lambda(s,t), s)ds. \quad (65)$$

We can also express the solution in terms of an integral over space. Consider the new variable $\xi = x - \Lambda(s,t)$. We have $d\xi = \lambda(s)ds$ and $s = t - \tau(x - \xi, t)$, this can be seen by noticing that $\Lambda(t - \tau(x - \xi, t), t) = x - \xi$, which follows from the definition of τ. Also, $\xi = 0$ when $s = t - \tau(x,t)$, and $\xi = x$ when $s = t$. Hence,

$$u(x,t) = \frac{1}{W(x)} \int_{0}^{x} \frac{W(\xi)S(\xi, t - \tau(x - \xi, t))}{\lambda(t - \tau(x - \xi, t))} d\xi. \quad (66)$$

Appendix II
Generalized Halanay's Inequality

Theorem 2, presented next, is a particular case of [13, Theorem 2.1], which is a generalization of Halanay's inequality [14].

Theorem 2: Let V be a positive function defined on $[t_0 - \bar{\tau}, \infty)$ with derivative $\dot{V}$ on $[t_0, \infty)$ for some constant $\bar{\tau} \geq 0$. If there are constants $a > b > 0$ and $c \geq 0$ such that

$$\dot{V}(t) \leq c - aV(t) + b \sup_{s \in [-\bar{\tau}, 0]} V(t+s), \text{ for all } t \geq t_0, \quad (67)$$

then there is a constant $\sigma > 0$ such that

$$V(t) \leq \sup_{s \in [-\bar{\tau}, 0]} V(t_0 + s)e^{-\sigma(t-t_0)} + \frac{c}{a-b}, \text{ for all } t \geq t_0. \quad (68)$$

References

[1] C. R. Shapiro, P. Bauweraerts, J. Meyers, C. Meneveau, and D. F. Gayme, "Model-based receding horizon control of wind farms for secondary frequency regulation," *Wind Energy*, vol. 20, no. 7, pp. 1261–1275, 2017.

[2] C. R. Shapiro, J. Meyers, C. Meneveau, and D. F. Gayme, "Dynamic wake modeling and state estimation for improved model-based receding horizon control of wind farms," in *2017 American Control Conference (ACC)*. IEEE, 2017, pp. 709–716.

[3] P. A. Ioannou and J. Sun, *Robust adaptive control*. Courier Corporation, 2012.

[4] A. Smyshlyaev and M. Krstic, *Adaptive control of parabolic PDEs*. Princeton University Press, 2010.

[5] H. Anfinsen and O. M. Aamo, *Adaptive control of hyperbolic PDEs*. Springer, 2019.

[6] F. Di Meglio, D. Bresch-Pietri, and U. J. F. Aarsnes, "An adaptive observer for hyperbolic systems with application to underbalanced drilling," *IFAC Proceedings Volumes*, vol. 47, no. 3, pp. 11 391–11 397, 2014.

[7] M. Bin and F. Di Meglio, "Boundary estimation of boundary parameters for linear hyperbolic pdes," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3890–3904, 2016.

[8] A. Hasan, O. M. Aamo, and M. Krstic, "Boundary observer design for hyperbolic PDE–ODE cascade systems," *Automatica*, vol. 68, pp. 75–86, 2016.

[9] H. Anfinsen, H. Holta, and O. M. Aamo, "Adaptive control of a scalar 1-D linear hyperbolic PDE with uncertain transport speed using boundary sensing," in *2020 American Control Conference (ACC)*. IEEE, 2020, pp. 1575–1581.

[10] D. Bresch-Pietri and N. Petit, "Implicit integral equations for modeling systems with a transport delay," in *Recent Results on Time-Delay Systems*. Springer, 2016, pp. 3–21.

[11] K. Zenger and A. J. Niemi, "Modelling and control of a class of time-varying continuous flow processes," *Journal of Process Control*, vol. 19, no. 9, pp. 1511–1518, 2009.

[12] N.-T. Trinh, V. Andrieu, and C.-Z. Xu, "Output regulation for a cascaded network of 2× 2 hyperbolic systems with PI controller," *Automatica*, vol. 91, pp. 270–278, 2018.

[13] H. Wen, S. Shu, and L. Wen, "A new generalization of Halanay-type inequality and its applications," *Journal of Inequalities and Applications*, vol. 2018, no. 1, pp. 1–12, 2018.

[14] A. Halanay, *Differential equations: Stability, oscillations, time lags*. Academic press, 1966, vol. 23.